DIFFERENTIABILITY OF THE MATRICES R AND G IN THE MATRIX-ANALYTIC METHOD

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ABSTRACT

The differentiability, with respect to a parameter of the model, of the matrices R and G that arise in the matrix-analytic method is studied. Some conditions for the differentiability of R and G are given.

Key Words: Differentiability, matrix-analytic method, queueing theory.

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1. Introduction

Let Ω be an open set of real numbers. For $x \in \Omega$, let $\{A_n(x),\ n \geq 0\}$ be a sequence of nonnegative square matrices of order m such that the matrix $A(x) = \sum_{n=0}^{\infty} A_n(x)$ is an irreducible stochastic or substochastic matrix. $R(x)$ is the minimal nonnegative solution to the equation

$$X = \sum_{n=0}^{\infty} X^n A_n(x), \tag{1}$$

and $G(x)$, the minimal nonnegative solution to

$$X = \sum_{n=0}^{\infty} A_n(x) X^n. \tag{2}$$

We study the differentiability of $R(x)$ and $G(x)$ under certain conditions on the sequence $\{A_n(x), n \geq 0\}$, especially when $\{A_n(x), n \geq 0\}$ have only finitely many nonzero matrices.

$R(x)$ and $G(x)$ are two important matrices in the matrix-analytic method (see Neuts [3], [4]; Ramaswami [5], [6]). They are useful in the computation of the distributions of queue lengths, waiting times and busy periods of *GI/MAP/1* and *MAP/G/1* queues. $R(x)$ arises in the study of *GI/MAP/1* queues and $G(x)$ of *MAP/G/1* queues. $\{A_n(x)\}$ are blocks in the transition matrices of the embedded Markov chains, either at arrivals or departures, and x is a system parameter. In He and Neuts [2], the differentiability of $R(x)$ and $G(x)$ is discussed for a special quasi birth-and-death process. The present results are used there to derive second order expansions of some system descriptors.

In Section 2, we study the differentiability of $R(x)$ in irreducible cases. In Section 3, the differentiability of $G(x)$ is discussed. In Section 4, the technical details associated with reducible cases and cases related to continuous parameter Markov processes are discussed. In Section 5, we summarize the results obtained.

2. Differentiability of the Matrix R

Let $\pi(x)$ be the left eigenvector corresponding to the maximal eigenvalue, $sp(A(x))$, of $A(x)$. $\pi(x)$ is nonnegative and normalized by $\pi(x)\mathbf{e} = 1$, where $\mathbf{e}$ is the column vector with all components one. Clearly, $sp(A(x)) \leq 1$. Let

$\beta(x) = \sum_{n=0}^{\infty} nA_n(x)\mathbf{e}$. We define the matrix

$$M_r(x_1, x_2) = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} [R^i(x_1)]^T \otimes [R^{n-i-1}(x_2)A_n(x_2)], \quad x_1, x_2 \in \Omega, \qquad (3)$$

where the superscript T denotes transpose and $\otimes$ is the *Kronecker product* of matrices (see Graham [1]). The matrix $M_r(x_1, x_2)$ exists since $R(x_i)$, $i = 1, 2$, and $\{A_n(x_2)\}$ are nonnegative matrices. Furthermore, the next lemma shows that $M_r(x_1, x_2)$ is finite and its maximum eigenvalue is less than 1.

Lemma 2.1. Assume that $A(x)$ is an irreducible stochastic or substochastic matrix. If $A(x_i)\mathbf{e} < \mathbf{e}$, or if $A(x_i)\mathbf{e} = \mathbf{e}$ and $\pi(x_i)\beta(x_i) > 1$, $i = 1, 2$, we have that

$$sp(M_r(x_1, x_2)) < 1, \quad x_1, x_2 \in \Omega.$$

Proof. By Corollary 1.3.1, Lemmas 1.3.4 and 1.3.5 in Neuts [3], $sp(R(x_i)) < 1$, $i = 1, 2$. We rewrite $M_r(x_1, x_2)$ as

$$M_r(x_1, x_2) = \sum_{n=0}^{\infty} [R^n(x_1)]^T \otimes B_n(x_2),$$

where $B_n(x_2) = \sum_{i=0}^{\infty} R^i(x_2)A_{n+i+1}(x_2)$, $n \geq 0$. Since $A(x_2)\mathbf{e} \leq \mathbf{e}$,

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(x_2)\mathbf{e} &= \sum_{n=1}^{\infty} [\sum_{i=0}^{n-1} R^i(x_2)] A_n(x_2)\mathbf{e} \\ &= (I - R(x_2))^{-1}(A(x_2) - R(x_2))\mathbf{e} \leq \mathbf{e},\end{aligned}$$

so that $sp(\sum_{n=0}^{\infty} B_n(x_2)) \leq 1$.

If $sp(R(x_1)) = 0$, $R(x_1)$ has the Jordan canonical decomposition

$$R(x_1) = P \begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix} P^{-1},$$

where P is an invertible matrix. Hence, we have

$$\sum_{n=0}^{\infty} [R^n(x_1)]^T \otimes B_n(x_2) = [(P^{-1})^T \otimes I] \begin{pmatrix} 0 & & & \\ * & \ddots & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & 0 \end{pmatrix} (P^T \otimes I).$$

Therefore, $sp(M_r(x_1, x_2)) = 0$.

If $sp(R(x_1)) > 0$, since $A(x_1)$ is irreducible, then by Lemma 1.3.2 in Neuts [3], the left eigenvector $\mathbf{u}$ of $R(x_1)$ corresponding to $sp(R(x_1))$ may be chosen to be positive. Hence, we have

$$\begin{aligned}[\sum_{n=0}^{\infty}(R^n(x_1))^T \otimes B_n(x_2)](\mathbf{u}^T \otimes \mathbf{e}) &= \sum_{n=0}^{\infty}[(sp(R(x_1)))^n \mathbf{u}^T] \otimes (B_n(x_2)\mathbf{e}) \\ &< \mathbf{u}^T \otimes [(\sum_{n=0}^{\infty} B_n(x_2))\mathbf{e}] \\ &\leq \mathbf{u}^T \otimes \mathbf{e}.\end{aligned}$$

Therefore, since $\mathbf{u}$ is positive, we have $sp(M_r(x_1, x_2)) < 1$. □

Set

$$A^*(z, x) = \sum_{n=0}^{\infty} z^n A_n(x). \quad 0 < z < 1, \quad x \in \Omega.$$

It is obvious that $A(x) = \lim_{z\to 1} A^*(z, x)$.

Lemma 2.2. If the sequence $\{A_n(x)\}$ satisfy the conditions of Lemma 2.1 for all x in Ω and its terms are continuous in Ω, then $R(x)$ is continuous in Ω.

Proof. Let $\rho(x) = sp(R(x))$. First, we claim that for $x_0 \in \Omega$, there exists a neighborhood $S(x_0)$ of x_0 such that $\rho(x) < \delta(x_0) < 1$, for $x \in S(x_0)$.

Suppose that the claim is false, then there exists a nondecreasing subsequence $\rho(x_n) \to 1$, with $\lim_{n\to\infty} x_n = x_0$. Let $\mathbf{u}(x)$ be the left eigenvector of $R(x)$ corresponding to $\rho(x)$ with $\mathbf{u}(x)\mathbf{e} = 1$. Since $R(x_n)$ satisfies equation (1), we have that

$$\mathbf{u}(x_n)R(x_n) = \mathbf{u}(x_n)\rho(x_n) = \mathbf{u}(x_n)A^*(\rho(x_n), x_n). \tag{4}$$

Furthermore, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{\mathbf{u}(x_{n_i})\}$ converges to a nonnegative vector $\mathbf{u}$ with $\mathbf{ue} = 1$. Therefore $\mathbf{u} = \mathbf{u}A(x_0)$. Since $A(x_0)$ is irreducible, we know that $\mathbf{u} = \pi(x_0)$. For $\{R(x_n)\}$, there exists a subsequence $\{R(x_{n_i})\}$ such that $\{R(x_{n_i})\}$ converges to a nonnegative matrix $\hat{R}$ with $\mathbf{u}\hat{R} = \mathbf{u}$.

By Lemma 2.1, $\sum_{i=0}^{\infty} B_i(x_n)\mathbf{e} \leq \mathbf{e}$ so that $\mathbf{u}(x_n)\sum_{i=0}^{\infty} B_i(x_n)\mathbf{e} \leq \mathbf{u}(x_n)\mathbf{e} = 1$. Therefore,

$$\sup_{n\to\infty} \mathbf{u}(x_n)\sum_{i=0}^{\infty} B_i(x_n)\mathbf{e} \leq 1. \tag{5}$$

On the other hand,

$$\mathbf{u}(x_n)\sum_{i=0}^{\infty} B_i(x_n) = \mathbf{u}(x_n)\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\rho^j(x_n)A_i(x_n).$$

Since $\lim_{x_n\to x_0}\rho(x_n) = 1$, $\lim_{x_n\to x_0}\mathbf{u}(x_n) = \mathbf{u}$ and $\lim_{x_n\to x_0} A_i(x_n) = A_i(x_0)$, it follows that

$$\sup_{n\to\infty} \mathbf{u}(x_n)\sum_{i=0}^{\infty} B_i(x_n)\mathbf{e} \geq \mathbf{u}\sum_{i=1}^{\infty} iA_i(x_0)\mathbf{e} = \pi(x_0)\beta(x_0) > 1.$$

This contradicts equation (5). Therefore, the claim is true.

For any subsequence $\{x_m\}$ of $\{x_n\}$ such that $\lim_{x_m\to x_0} R(x_m) = \hat{R}$, there exists a subsequence $\{x_{m_i}\}$ such that $\{\rho(x_{m_i})\}$ converges to a nonnegative number $\hat{\rho}$ ($< \delta(x_0) < 1$). Since $\rho(x_n) < \delta(x_0) < 1$, $\{R^i(x_n)\}$ converges to the zero matrix uniformly in n. Hence, it can be proved that $\hat{R} = \sum_{i=0}^{\infty} \hat{R}^i A_i(x_0)$. By (4), we have $\mathbf{u}A^*(\hat{\rho}, x_0) = \mathbf{u}\hat{\rho}$. By Theorem 1.3.3 in Neuts [3], we know that the solution X to equation (1) with $sp(X) < 1$ at x_0 is unique. Therefore, we have $\hat{R} = R(x_0)$. This implies that $\lim_{x_n\to x_0} R(x_n) = R(x_0)$. □

For a matrix C of dimensions $m_1 \times m_2$, $\phi(C)$ is the m_1m_2-vector obtained by forming the *direct sum* of the rows of C. Call ϕ the *direct sum transform* of matrices. If X, Y and Z are matrices and their product XYZ is well defined, then $\phi(XYZ) = \phi(Y)X^T \otimes Z$. This property is used in the proof of Theorem 2.3.

Theorem 2.3. Suppose that, for all x in Ω, $A_n(x) = 0$ for $n > N > 0$ and $\{A_n(x),\ n \leq N\}$ are differentiable up to order $K(\geq 1)$ and satisfy the conditions of Lemma 2.1. If $\{A_n^{(K)}(x)\}$ are continuous, the minimal nonnegative solution, $R(x)$, of (1) is differentiable in Ω up to order K.

Proof. From (1), we have, for $x_0 \in \Omega$,

$$\begin{aligned} R(x) - R(x_0) &= \sum_{n=1}^{N}\sum_{i=0}^{n-1} R^i(x_0)[R(x) - R(x_0)]R^{n-i-1}(x)A_n(x) \\ &\quad + \sum_{n=0}^{N} R^n(x_0)[A_n(x) - A_n(x_0)]. \end{aligned}$$

Taking direct sum transforms of both sides, we have, by the property of the transform ϕ, that

$$\phi(R(x_0) - R(x)) = \phi(\sum_{n=0}^{N} R^n(x_0)(A_n(x) - A_n(x_0))) \cdot [I - M_r(x_0, x)]^{-1}. \quad (6)$$

By virtue of the proofs of Lemmas 2.1 and 2.2, $[I - M_r(x_0, x)]^{-1}$ exists and is continuous at x_0 in x. Dividing both sides of (4) by $x - x_0$ and letting $x \to x_0$, it follows that

$$\phi(R'(x_0)) = \phi(\sum_{n=0}^{N} R^n(x_0)A'_n(x_0)) \cdot [I - M_r(x_0, x_0)]^{-1}. \tag{7}$$

Therefore, $R(x)$ is differentiable at x_0.

Suppose that $R(x)$ is differentiable up to order $i < K$. We shall show that $R(x)$ is differentiable up to order $i + 1$. By induction, $R^{(i)}(x)$ satisfies

$$R^{(i)}(x) = \sum_{n=1}^{N} \sum_{j=0}^{n-1} [R(x)]^j R^{(i)}(x)[R(x)]^{n-j-1} A_n(x) + f_i(x), \tag{8}$$

where

$$f_i(x) = \sum_{n=0}^{N} [\sum_{\substack{j_1+\cdots+j_{n+1}=i \\ 0 \le j_1,\cdots,j_n < i}} \frac{i!}{j_1! \cdots j_{n+1}!} R^{(j_1)}(x) \cdots R^{(j_n)}(x) A_n^{(j_{n+1})}(x)], \quad i \ge 1,$$

Then, similar to (6), we have

$$\begin{aligned}
&\phi(R^{(i)}(x_0) - R^{(i)}(x)) \\
&= \phi(\sum_{n=0}^{N} \sum_{j=0}^{n-1} [R(x_0)]^j R^{(i)}(x_0)[(R(x_0))^{n-j-1} A_n(x_0) - [R(x)]^{n-j-1} A_n(x)] \\
&\quad + \sum_{n=0}^{N} \sum_{j=0}^{n-1} [(R(x_0))^j - (R(x))^j] R^{(i)}(x)(R(x))^{n-j-1} A_n(x) \\
&\quad + f_i(x_0) - f_i(x)) \cdot [I - M_r(x_0, x)]^{-1}.
\end{aligned}$$

Dividing both sides by $x - x_0$ and letting $x \to x_0$, we obtain

$$\phi(R^{(i+1)}(x_0)) = \phi(f_{i+1}(x_0)) \cdot [I - M_r(x_0, x_0)]^{-1}, \tag{9}$$

and, since the $\{A_n(x)\}$ are differentiable at x_0 up to order K, so is $R(x)$. □

Note 2.1. Theorem 2.3 also holds for $N = \infty$ if $f_i(x)$ (defined in the proof of Theorem 2.3) exists, $1 \le i \le K$. However, we have

$$f_1(x) = \sum_{n=0}^{\infty} R^n(x)A'_n(x). \tag{10}$$

and $f_i(x)$ is much more complicated for $i \geq 2$. So, it is next to impossible to verify the existence of $\{f_i(x)\}$ when $N = \infty$.

Note 2.2. We consider an *M/M/1* queue with an arrival rate $\lambda(x)$ and a service rate 1. Routine calculations give that

$$R(x) = \begin{cases} \lambda(x), & \text{if } \lambda(x) \leq 1; \\ \\ 1, & \text{if } \lambda(x) > 1. \end{cases} \tag{11}$$

Equation (11) shows that for any x_0 with $\lambda(x_0) = 1$ and $\lambda'(x_0) \neq 0$, $R(x)$ may not be differentiable at x_0. Therefore, Ω is assumed to be an open set.

To compute the derivatives of $R(x)$, we solve equation (8). We can use an iterative process to obtain approximation solutions. Alternately, we use the direct sum transformation (9), which gives the explicit solution for $\{R^{(i)}(x), i \geq 1\}$, provided that $R(x)$ is known.

3. Differentiability of the Matrix G

We define

$$M_g(x_1, x_2) = \sum_{n=1}^{\infty} (\sum_{i=0}^{n-1} A_n(x_1) G^i(x_1))^T \otimes G^{n-1-i}(x_2), \quad x_1, x_2 \in \Omega. \tag{12}$$

As in the case with the matrix $M_r(x_1, x_2)$, the invertibility of the matrix $M_g(x_1, x_2)$ is the key to the differentiability of the matrix $G(x)$.

Lemma 3.1. Assume that $A(x)$ is an irreducible stochastic or substochastic matrix. If $A(x_i)\mathbf{e} < \mathbf{e}$, or if $A(x_i)\mathbf{e} = \mathbf{e}$ and $\pi(x_i)\beta(x_i) < 1$, $i = 1, 2$, we have that

$$sp(M_g(x_1, x_2)) < 1, \quad x_1, x_2 \in \Omega.$$

Proof. (12) may be rewritten as

$$M_g(x_1, x_2) = \sum_{n=0}^{\infty} [\bar{B}_n(x_1)]^T \otimes G^n(x_2),$$

where $\bar{B}_n(x_1) = \sum_{i=0}^{\infty} A_{n+i+1}(x_1) G^i(x_1)$, $n \geq 0$.

When $A(x_1)\mathbf{e} < \mathbf{e}$, $sp(G(x_1)) < 1$. By routine calculations, we have

$$\sum_{n=0}^{\infty} \bar{B}_n(x_1) = [A(x_1) - G(x_1)][I - G(x_1)]^{-1}.$$

Therefore, $\pi(x_1)\sum_{n=0}^{\infty}\bar{B}_n(x_1) < \pi(x_1)$, so that $sp(\sum_{n=0}^{\infty}\bar{B}_n(x_1)) < 1$.

When $A(x_1)\mathbf{e} = \mathbf{e}$, we have $sp(G(x_1)) = 1$. Denote by $\mathbf{g}$, the left invariant vector of $G(x_1)$. By routine calculations,

$$\sum_{n=0}^{\infty}\bar{B}_n(x_1) = (A(x_1) - G(x_1) + \beta(x_1)\mathbf{g})(I - G(x_1) + \mathbf{eg})^{-1}.$$

The matrix $I - G(x_1) + \mathbf{eg}$ is invertible since $\pi(x_1)\beta(x_1) < 1$. Hence,

$$\pi(x_1)(\sum_{n=0}^{\infty}\bar{B}_n(x_1)) = \pi(x_1) + (\pi(x_1)\beta(x_1) - 1)\mathbf{g} < \pi(x_1).$$

Therefore, since every component of $\pi(x_1)$ is positive, $sp(\sum_{n=0}^{\infty}\bar{B}_n(x_1)) < 1$.

Combining the results of the two cases, we have $sp(\sum_{n=0}^{\infty}\bar{B}_n(x_1)) < 1$. By applying the T_c transform (see Appendix) to the matrix $M_g(x_1, x_2)$,

$$T_c(M_g(x_1, x_2)) = \sum_{n=0}^{\infty} T_c((\bar{B}_n(x_1))^T \otimes G^n(x_2)) = \sum_{n=0}^{\infty} G^n(x_2) \otimes [\bar{B}_n(x_1)]^T.$$

The proof that $sp(T_c(M_g(x_1, x_2))) < 1$ is similar to that of Lemma 2.1. Since $sp(T_c(M_g(x_1, x_2))) = sp(M_g(x_1, x_2))$, we have $sp(M_g(x_1, x_2)) < 1$. □

As in Lemma 2.2 and Theorem 2.3, the following results hold for $G(x)$.

Lemma 3.2. If $\{A_n(x)\}$ satisfy the conditions given in Lemma 3.1 for all x in Ω and are continuous in Ω, then $G(x)$ is continuous in Ω. □

Theorem 3.3. Suppose that, for all x in Ω, $A_n(x) = 0$ for $n > N > 0$ and $\{A_n(x)\}$ are differentiable up to order $K(\geq 1)$ and satisfy the conditions given in Lemma 3.1. If $\{A_n^{(K)}(x)\}$ are continuous, the minimal nonnegative solution, $G(x)$, to equation (2) is differentiable up to order K. □

4. Discussion

The results given by Theorems 2.3 and 3.3 can be generalized to the cases where $A(x)$ is reducible. Suppose that $A(x)$ has the structure

$$A(x) = \begin{pmatrix} A(1,x) & \cdots & * \\ & \ddots & \vdots \\ & & A(k,x) \end{pmatrix},$$

where the diagonal blocks $A(i,x)$, $1 \le i \le k$, are irreducible matrices. Then $\{A_n(x)\}$ have the same structure as $A(x)$ and we denote their corresponding diagonal blocks by $\{A_n(i,x), 1 \le i \le k\}$. The minimal nonnegative solution $R(x)$ to the matrix equation (1) has the same structure as $A(x)$. The diagonal blocks of $R(x)$, $R(i,x)$, $1 \le i \le k$, satisfy

$$R(i,x) = \sum_{n=0}^{\infty} R^n(i,x) A_n(i,x), \quad 1 \le i \le k.$$

By routine calculations, we have

$$M_r(x_1,x_2) = \begin{pmatrix} \sum_{n=0}^{\infty} (R^n(1,x_1))^T \otimes B_n(x_2)) & & \\ \vdots & \ddots & \\ * & \cdots & \sum_{n=0}^{\infty} (R^n(k,x_1))^T \otimes B_n(x_2) \end{pmatrix}.$$

If $\{A_n(i,x)\}$, $1 \le i \le k$, satisfy the conditions in Lemma 2.1 and Theorem 2.3, we obtain the results in Theorem 2.3 for $R(x)$.

There is an entirely analogous theory for the cases where $A_n(x)$, $n \ge 0$ and $n \ne 1$, are nonnegative matrices, $A_1(x)$ has nonnegative off-diagonal elements and negative diagonal elements, and $A(x)\mathbf{e} \le 0$. The matrix $R(x)$ is here defined as the minimal nonnegative solution to the equation

$$0 = \sum_{n=0}^{\infty} X^n A_n(x). \tag{13}$$

If there is an analytic function $\tau(x)$ satisfying

$$\tau(x) > 1 + \max_{1 \le j \le m} \{-(A_1(x))_{jj}\},$$

for $x \in \Omega$, then (13) may be rewritten as

$$X = \sum_{n=0}^{\infty} X^n \tilde{A}_n(x), \tag{14}$$

where $\tilde{A}_n(x) = A_n(x)/\tau(x)$, $n \ge 0$ and $n \ne 1$, and $\tilde{A}_1(x) = I + A_1(x)/\tau(x)$. Equation (14) is the same as equation (1), so Theorem 2.3 is applicable. Similar results hold for the matrix $G(x)$.

5. Conclusions

We have proved that, under certain conditions on $\{A_n(x)\}$, the matrices $R(x)$ and $G(x)$ are differentiable. Let us use $R(x)$ as an example. For the first order differentiability, $R(x)$ is differentiable when $\{A'_n(x),\ n \geq 0\}$ exist and are continuous, and $\sum_{n=0}^{\infty} R^n(x)A'_n(x)$ exists (see Lemma 2.1, Theorem 2.3 and (10)). For $\{A_n(x)\}$ with only finitely many nonzero matrices, $R(x)$ is differentiable up to order K when $\{A_n(x)\}$ are differentiable up to order K and $\{A_n^{(K)}(x)\}$ are continuous (see Lemma 2.1 and Theorem 2.3).

Although we obtained some conditions for the differentiability of $R(x)$ and $G(x)$, this paper raises more questions than it has solved. For example, when $\{A_n(x)\}$ have infinitely many nonzero matrices, for higher order differentiability, the conditions imposed on $\{f_i(x)\}$ are too complicated to be explicitly checked. More tangible conditions in terms of $\{A_n(x)\}$ are required in applications.

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APPENDIX

We define a transform, T_c, on the Kronecker product of matrices A and B by

$$T_c(A \otimes B) = B \otimes A,$$

where A and B are square matrices of orders m_1 and m_2 respectively.

Lemma A. There exists a permutation matrix $P_{m_1 m_2}$, $P_{m_1 m_2}^2 = I$, satisfying

$$T_c(A \otimes B) = P_{m_1 m_2}(A \otimes B)P_{m_1 m_2}. \tag{15}$$

Hence, $T_c(A \otimes B)$ and $A \otimes B$ are similar.

Proof. See (2.14), Graham [1].

□

Using Lemma A, we can define a generalized T_c on sums of Kronecker products of matrices, i.e., the transform $T_c(A_1 \otimes B_1 + A_2 \otimes B_2)$ is well-defined and

$$T_c(A_1 \otimes B_1 + A_2 \otimes B_2) = T_c(A_1 \otimes B_1) + T_c(A_2 \otimes B_2),$$

where A_1 and A_2 are of the same dimension and so are B_1 and B_2. Lemma A still holds for the generalized transformation.