PRODUCTION SYSTEMS WITH RIGID DEMANDS, RANDOM YIELDS, AND INSPECTIONS

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ABSTRACT

When production yield is random and demand needs to be satisfied in full, several production runs may need to be attempted until the number of usable products is sufficient. The tradeoff is then between using small lots possibly incurring setup cost many times and large lots which may result in costly overproduction. We generalize this problem to include inspection costs and show how to compute the optimal lot size and the expected number of inspections. Four useful models with binomial, discrete uniform, all-or-nothing, and interrupted geometric yields are examined in detail. Numerical results are presented to obtain insights into the lotsizing in production systems with random yields.

Key words: Random Yield, Inspection, Rigid Demand, Dynamic Programming

1. Introduction

We consider a manufacturing facility (machine or work center) where production is in lots or runs involving a fixed setup cost and a variable per-unit processing cost. When such a facility has quality problems and demands need to be satisfied in their entirety, several production runs may be needed until the total number of usable units is sufficient. We assume that defective units and usable units in excess of the demand have no value and must be scrapped. This situation often arises when demand is for small quantities and products are custom made as, for example, in some high-tech industries (see more applications in Yano and Lee [6] (1995)). The tradeoff is then between using small lots, possibly incurring setup cost many times, and large lots, which may result in costly overproduction.

This problem dates back to 60-70's, when it was often referred to as "reject allowance" (e.g. Klein [4] (1966); Beja [2] (1977)). Many authors considered binomial yields - a situation where successes of units within a lot are independent of each other, with the same success probability (see, for example, Sepheri, Silver and New [5] (1986) and references therein). Yano and Lee [6] (1995) provides a comprehensive literature review concerning manufacturing with random yields. They consider both rigid demand and non-rigid demand (non-rigid demand is a situation where there is only one production run and a penalty for shortage). Grosfeld-Nir and Gerchak [3] (1992) establish certain general properties related to lotsizing problems and show examples where intuitions could fail.

Clearly, inspection is an intrinsic part of this model; products must be inspected and their quality be determined in order to decide when to cease production. Yet, in the context of rigid demand, authors have ignored inspection cost, i.e., implicitly assumed that inspection is free (or that the whole lot is inspected - in which case inspection cost is simply added to the variable cost). We formulate the problem with costly inspection and provide recursions for optimal lotsizing and for computing the expected number of inspections. Our assumptions are:

- Inspection does not require setup; units exiting the manufacturing facility are inspected one-at-a-time until the demand is satisfied, or all units have been exhausted.
- Units to be inspected are selected at random.

The rest of the paper is organized as follows. In Section 2, the model of interest is introduced. A recursive formula which is convenient for computing the optimal lot size and its corresponding expected total costs is derived. Section 3 deals with the Binomial yield model. It shall be proved that the optimal lot size is independent of the inspection cost in this model. Section 4 briefly examines models with discrete uniform, all-or-nothing, and interrupted geometric yields. Several numerical examples are presented in Sections 3 and 4. Finally, Section 5 summarizes the results obtained in this paper.

2. The Model

A machine produces in lots or runs. The cost associated with a lot of size N is $\alpha + \beta N$, where the parameters α and β are referred to as the "setup cost" and the "variable cost" (per-unit processing cost), respectively. As a batch exits the machine, units are inspected one-at-a-time until the demand is satisfied or all units are exhausted. Inspection cost is γ per unit. If inspection reveals that the number of usable units, in a batch exiting the machine, is short of the outstanding demand, the problem repeats itself with the demand remaining. Defective units and usable units in excess of the demand have no value and are scrapped. The objective is to determine, for demand D, the optimal lot size N_D which minimizes the expected total of setup, variable, and inspection costs.

We define $I_D(y, N)$, $D \leq y \leq N$, to be the expected number of inspections, if the remaining demand is D and there are $y \geq D$ good units among N units waiting for inspection. Then

$$I_D(y, N) = \sum_{x=D}^{D+N-y} x \frac{\binom{y}{D-1}\binom{N-y}{x-D}}{\binom{N}{x-1}} \frac{(y-D+1)}{(N-x+1)}. \quad (1)$$

We denote by $p(y,N)$, $0 \leq y \leq N$, the probability that the yield is y when the lot size is N, and define the following cost functions:

U_D is the optimal (minimal) expected cost to satisfy demand D.

$U_D(N)$ is the expected cost to satisfy demand D, if the lot size is N whenever the demand is D and an optimal lot size is chosen whenever the remaining demand is less then D.

Obviously, $U_D = \min_N \{U_D(N)\}$ and

$$U_D(N) = \alpha + \beta N + p(0, N)[\gamma N + U_D(N)]$$
$$+ \sum_{y=1}^{\min\{D-1,N\}} p(y, N)[\gamma N + U_{D-y}] \quad (2)$$
$$+ \gamma \sum_{y=D}^{N} p(y, N) I_D(y, N).$$

This leads to

$$U_D(N) = [\alpha + (\beta + \gamma)N$$
$$+ \sum_{y=1}^{\min\{D-1,N\}} p(y, N)(\gamma N + U_{D-y})$$
$$+ \gamma \sum_{y=D}^{N} p(y, N) I_D(y, N)] \quad (3)$$
$$\cdot \frac{1}{1 - p(0, N)} - \gamma N.$$

Therefore, the optimal lot size and the expected cost can be computed recursively in D.

Next, we show that there is a surprisingly simple way to compute $I_D(y,N)$. First note that $I_D(N,N)=D$, $D \leq N$ and for $D \leq y < N$,

$$I_D(y, N) = 1 + \frac{y}{N} I_{D-1}(y-1, N-1)$$
$$+ (1 - \frac{y}{N}) I_D(y, N-1). \quad (4)$$

In particular, for $1 \leq y < N$,

$$I_1(y, N) = 1 + (1 - \frac{y}{N}) I_1(y, N - 1). \qquad (5)$$

Proposition 2.1

$$I_D(y, N) = \frac{N + 1}{y + 1} D, \; D \le y \le N. \qquad (6)$$

Proof. We will use induction over D (and, induction over N within the induction over D). Consider $D=1$; we must prove that

$$I_1(y, N) = \frac{N + 1}{y + 1}, \; 1 \le y \le N. \qquad (7)$$

Clearly equation (7) holds true for $N=1$ (which implies that $y=1$ and recall that $I_D(N, N) = D, D \le N$). Suppose that equation (7) holds true for N, then (using equation (5))

$$I_1(y, N + 1) = 1 + (1 - \frac{y}{N + 1}) \frac{N + 1}{y + 1}$$
$$= \frac{N + 2}{y + 1}. \qquad (8)$$

Therefore, equation (6) is true for $D=1$, $1 \le y \le N$. Suppose that equation (6) is true for D. For $D+1$, we must prove that, for $D+1 \le y \le N$.

$$I_{D+1}(y, N) = \frac{N + 1}{y + 1} (D + 1). \qquad (9)$$

Clearly equation (9) holds true for $N=D+1$ (which implies $y=D+1$, then $I_{D+1}(D+1,D+1)=D+1$). Suppose that equation (9) holds true for N; then, for $N+1$, we have (using equation (4))

$$I_{D+1}(y, N + 1)$$
$$= 1 + \frac{y}{N + 1} I_D(y - 1, N) + (1 - \frac{y}{N + 1}) I_{D+1}(y, N)$$
$$= 1 + \frac{y}{(N + 1)} \frac{(N + 1)}{y} D$$
$$+ (1 - \frac{y}{N + 1}) \frac{(N + 1)}{(y + 1)} (D + 1)$$

$$= \frac{(N + 2)}{(y + 1)} (D + 1).$$

This completes the proof.

Corollary 2.2 As a consequence of equations (3) and (6), we have

$$U_D(N) = [\alpha + (\beta + \gamma)N$$
$$+ \sum_{y=1}^{\min\{D-1,N\}} p(y, N)(\gamma N + U_{D-y})$$
$$+ \gamma(N + 1)D \sum_{y=D}^{N} \frac{p(y, N)}{y + 1}] \qquad (10)$$
$$\cdot \frac{1}{1 - p(0, N)} - \gamma N.$$

An efficient algorithm for computing the optimal lot size and its corresponding expected total costs (including setup, processing, and inspection costs) can be developed using formulas obtained in this section. Numerical examples are shown in Sections 3 and 4 where several special cases are examined in more detail.

3. Binomial Yield

In this section, we assume that yields are binomial, i.e.

$$p(y, N) = \binom{N}{y} \theta^y (1 - \theta)^{N-y}, \qquad (11)$$

where θ is the success probability. We shall prove that the optimal lot size with costly inspection is the same as the optimal lot size when inspection is free. This will follow from showing that the expected total inspection cost does not depend upon the production policy (the lots to enter the machine). Let

I_D be the expected number of inspections - until the end of production, if the demand is D and all lot sizes are optimal.

$I_D(N)$ be the expected number of inspections - until the end of production, if the lot size is N whenever the demand is D, and lot

sizes are optimal whenever the remaining demand is less than D.

Then $I_D = \min_N\{I_D(N)\}$ and, by incorporating equation (6),

$$I_D(N) = [Np(0, N)$$
$$+ \sum_{y=1}^{\min\{D-1,N\}} p(y, N)(N + I_{D-y})$$
$$+ \sum_{y=D}^{N} p(y, N)\frac{(N + 1)}{(y + 1)} D] \qquad (12)$$
$$\cdot \frac{1}{1 - p(0, N)}.$$

The expected inspection number can be calculated using equation (12) for any yield distribution.

Next, we shall show that for binomial yield $I_D(N)$ does not depend upon N. The next lemma is helpful in proving that the expected number of inspections does not depend upon the lot size (Theorem 3.2).

Lemma 3.1 For the binomial yield case,

(a)
$$Np(0, N) + \sum_{y=1}^{N} p(y, N)\frac{N + 1}{y + 1}$$
$$= \frac{1 - p(0, N)}{\theta} \qquad 6.$$

Also, let

$$S_{D,N}(i) \equiv \sum_{y=D}^{i} p(y, N)[\frac{(N + 1)}{(y + 1)} D + \frac{y - D}{\theta} - N],$$

for7 $D < i < N$, and note that

$$S_{D,N}(i) = S_{D,N}(i - 1) + p(i, N)[\frac{(N + 1)}{(i + 1)} D$$
$$+ \frac{i - D}{\theta} - N], \quad D < i < N.$$

Then

(b) for $i=D, D+1, ..., N-1$,

$$S_{D,N}(i) = -p(i + 1, N)(i + 1 - D)\frac{1 - \theta}{\theta}; 9 \text{ and}$$

(c) $S_{D,N}(N) = 0$.

Proof. Using equation (11), the left-hand side of (a) becomes

$$N(1 - \theta)^N + \sum_{y=1}^{N} \binom{N}{y}\theta^y(1 - \theta)^{N-y}\frac{N + 1}{y + 1}$$
$$= N(1 - \theta)^N + \frac{1}{\theta}\sum_{y=1}^{N} p(y + 1, N + 1)$$
$$= \frac{1}{\theta}\{N\theta(1 - \theta)^N + [1 - (1 - \theta)^{N+1}$$
$$- (N + 1)\theta(1 - \theta)^N]\}$$
$$= \frac{1 - (1 - \theta)^N}{\theta},$$

which proves (a). To prove (b), we use induction on i. Consider $i=D$; we must show that

$$S_{D,N}(D) = -p(D + 1, N)\frac{1 - \theta}{\theta} \text{ 10.}$$

Now

$$S_{(D,N)}(D) = -p(D, N)\frac{N - D}{D + 1}$$
$$= -\frac{N!}{D!(N - D)!}\theta^D(1 - \theta)^{N-D}\frac{N - D}{D + 1}$$
$$= \frac{N!}{(D + 1)!(N - (D + 1))!}\theta^{D+1}(1 - \theta)^{N-(D+1)}\frac{1 - \theta}{\theta}.$$

Suppose that (b) holds for i ($i<N-1$). For $i+1$, we have

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$$S_{D,N}(i+1) = S_{D,N}(i) + p(i+1, N)$$
$$\cdot \left(\frac{N+1}{i+2}D + \frac{i+1-D}{\theta} - N\right)$$
$$= p(i+1, N)\frac{(N-i-1)(D-i-2)}{i+2}$$
$$= -p(i+2, N)(i+2-D)\frac{1-\theta}{\theta}.$$

To prove (c), simply note that

$$S_{D,N}(N) = S_{D,N}(N-1) + p(N, N)$$
$$\cdot \left(D - \frac{D}{\theta} + \frac{N}{\theta} - N\right)$$
$$= p(N, N)\left(-\frac{(N-D)}{\theta} + N - D + D - \frac{D}{\theta}\right.$$
$$\left. + \frac{N}{\theta} - N\right)$$
$$= 0.$$

This completes the proof.

Theorem 3.2. For binomial yields, we have

$$I_D(N) = \frac{D}{\theta}, \quad D \geq 1, \ N \geq 1. \tag{13}$$

Proof. We will use induction. Consider $D=1$; then

$$I_1(N) = \frac{Np(0, N) + \sum_{y=1}^{N} p(y, N)\frac{N+1}{y+1}}{1 - p(0, N)}$$
$$= \frac{\dfrac{1 - p(0, N)}{\theta}}{1 - p(0, N)} = \frac{1}{\theta}.$$

The first equality follows by equation (11); the second by Lemma 3.1 (a).

Suppose that equation (13) holds for demand $1, 2, ..., D-1$, i.e., $I_y = y/q$, $1 \leq y < D$. Then, for demand D and $N < D$, we have (using equation (12))

$$I_D(N) = \frac{\displaystyle\sum_{y=0}^{N} Np(y, N) + \sum_{y=1}^{N} p(y, N)\frac{D-y}{\theta}}{1 - p(0, N)}$$

$$= \frac{N + (1 - p(0, N))\dfrac{D}{\theta} - \dfrac{1}{\theta}\displaystyle\sum_{y=1}^{N} yp(y, N)}{1 - p(0, N)}$$

$$= \frac{N + (1 - p(0, N))\dfrac{D}{\theta} - \dfrac{N\theta}{\theta}}{1 - p(0, N)}$$

$$= \frac{D}{\theta},$$

where the third equality follows from

$$\sum_{y=1}^{N} yP(y, N) = N\theta. \tag{12}$$

For demand D and $N \geq D$, note that

$$\sum_{y=0}^{D-1} p(y, N) = 1 - \sum_{y=D}^{N} p(y, N) \tag{13}$$

and

$$\sum_{y=1}^{D-1} p(y, N) I_{D-y}$$
$$= \frac{D}{\theta}[1 - p(0, N)] - N - \sum_{y=D}^{N} p(y, N)\frac{(D-y)}{\theta}.$$

Therefore (using equation (11))

$$I_D(N) = [N + \frac{D}{\theta}(1 - p(0, N)) - N$$

$$+ \sum_{y=D}^{N} p(y, D)(\frac{(N + 1)}{y + 1}D - \frac{D - y}{\theta} - N)]$$

$$\cdot \frac{1}{1 - p(0, N)}$$

$$= \frac{D}{\theta} + \frac{\sum_{y=D}^{N} p(y, N)(\frac{(N + 1)}{y + 1}D + \frac{y}{\theta} - \frac{D}{\theta} - N)}{1 - p(0, N)}$$

$$= \frac{D}{\theta} + \frac{S_{(D,N)}(N)}{1 - p(0, N)}.$$

Incorporating part (c) of Lemma 3.1, we obtain $I_D(N) = D/\theta$. This completes the proof.

On average, one out of $1/\theta$ products is good when yields are binomial. Thus, in average, it will take $1/\theta$ inspections to identify a good units regardless of the lot size. Therefore, the average number of inspections is independent of the lot size. This result implies that the optimal lot size can be found without considering the inspection cost - a model which has been considered by many authors (see Sepheri, Silver, and New [5] and references therein).

Example 3.1. Consider a system with a setup cost $\alpha=40$, a unit-processing cost $\beta=1$, and a binomial yield with parameter $\theta=0.9$. The optimal lot sizes are given in Figure 3.1 for any nonnegative inspection cost.
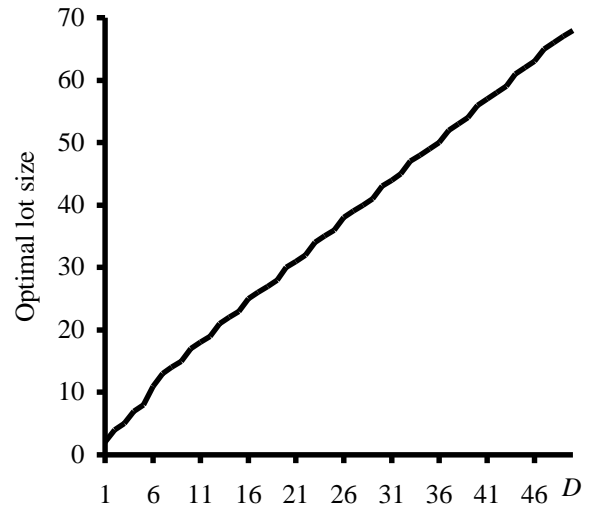

Figure 3.1 The optimal lot sizes for demands from zero to *50*

The optimal order size increases when the demand increases. The reason is that the expected number of usable units in a lot - θN - increases in the lot size N. Thus, a large lot means more usable units, which implies a large optimal lot size when the demand is large.

4. Other Yields

In this section, we briefly discuss manufacturing models with discrete uniform ($p(y, N)=1/(N+1)$, $y=0,1,...,N$), all-or-nothing ($p(N, N)=1-p(0, N)=\theta$, $0<\theta<1$), and interrupted geometric yields.

4.1 Discrete Uniform Yield

This type of yield has been studied by Anily [1] (1992). She proved that (when inspection is free) the optimal lot size strictly increases in the demand, which (for discrete uniform yield) also implies that the optimal lot size is no less than the demand. She also proved certain robustness properties: a small deviation in lot sizes will result in little extra production cost.

We have studied the consequence of costly inspection on a manufacturing facility with discrete uniform yields. We found out that the optimal lot size does not exceed that of free inspection. However, the proof turns to be extremely tedious and we choose to not include it here. Instead, we provide numerical results (Figure 4.1) which are of interest.

Example 4.1 (Example 3.1 continued) Consider a system with a setup cost $\alpha=40$, a unit-processing cost $\beta=1$, and a discrete uniform yield. The optimal lot sizes are given in Figure 4.1 for inspection cost from *0* to *75*. The horizontal axis represents the inspection cost and the vertical axis represents the optimal lot size.
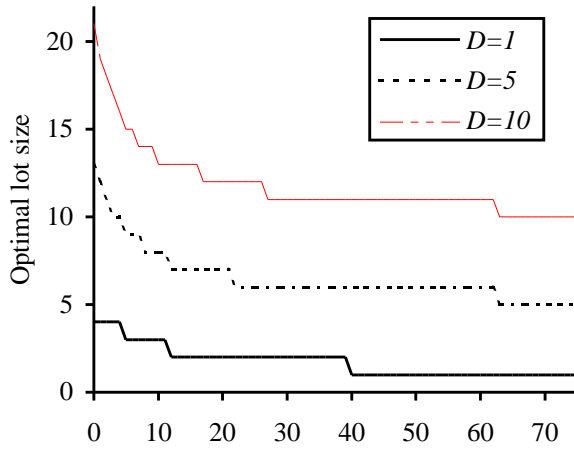

Figure 4.1 The optimal lot sizes for discrete uniform yields

Figure 4.1 shows that the optimal lot size varies as a function of the inspection cost. This reveals a difference between the binomial yield and the discrete uniform yield. Furthermore, Figure 4.1 shows that the optimal lot size decreases when the inspection cost increases. The reason is that the inspection cost is equivalent to the processing cost in certain sense. Thus, an increase in inspection cost means an increase in the processing cost, which implies a smaller optimal lot size. Nonetheless, the optimal lot size is always as large as the demand.

4.2 All-Or-Nothing Yield

When the yields are all-or-nothing with ($p(N, N)=1-p(0,N)=\theta$, $0<\theta<1$), expression (10) becomes

$$U_D(N) = \frac{\alpha + [\beta + (1-\theta)\gamma]N}{\theta} \qquad (14)$$
$$+ U_{D-N} + \min(D, N)\gamma.$$

Clearly, for this yield, the optimal lot size $N_D=D$ and $U_D = (\alpha+\beta D+\gamma D)/\theta$. Furthermore, there is no point for inspecting the whole lot when the yield distribution is known as all-or-nothing; all we need to do is inspect one unit. Therefore, the optimal expected cost is given by $U_D = (\alpha+\beta D+\gamma)/\theta$. All-or-nothing yield illustrates that inspecting the whole lot is not always optimal.

4.3 Interrupted Geometric Yield

Suppose that while processing a unit there is a constant probability θ that the machine gets out of control, then, this unit and all units processed afterwards are defective, while all items produced prior to this event are good. In such a case,

$$p(y, N) = (1-\theta)\theta^y, \ y = 0,1,...,N-1, \ p(N, N) = \theta^N.$$

It is interesting to observe that for interrupted geometric yield a major concern is the order in which items are selected for inspection. The inspection scheduling rule should be chosen to minimize the expected number of inspections corresponding to a specific lot size. At the present time the optimal lotsizing problem is still open. For the random inspection scheme considered in this paper, the optimal lot size problem is illustrated by the following example.

Example 4.2 (example 3.1 continued) Consider a system with a setup cost $\alpha=40$, a unit-processing cost $\beta=1$, and an interrupted geometric yield with parameter $\theta=0.9$. The optimal lot sizes are given in Figure 4.2 for inspection cost from *0* to *100*. The horizontal axis represents the inspection cost and the vertical axis represents the optimal lot size.
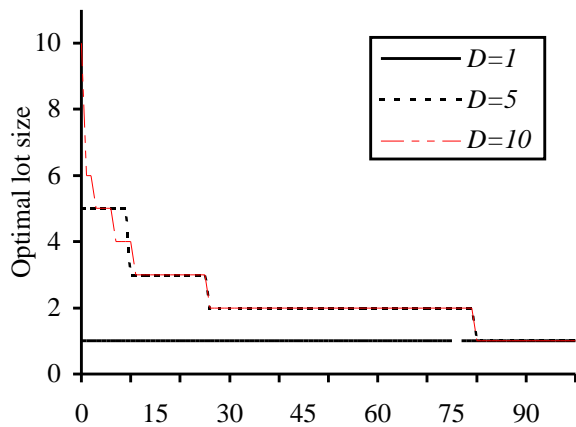
Figure 4.2 The optimal lot sizes for interrupted geometric yields

It is interesting to see that the optimal lot size converges to *1* when the inspection cost goes to infinity. This observation has much to do with the property that any units following a defective one are defective. Thus, when the inspection cost is larger, it is better to avoid inspecting defective units. This characteristic also makes the interrupted geometric yield quite different from other yields.

5. Conclusion

In this paper we have shown how to incorporate inspection cost in "rigid demand" models. An efficient algorithm has been developed for computing the optimal lot size as well as the minimal mean total production costs. Several models with special yield distributions are discussed as well. As there is growing awareness of the importance of quality in manufacturing, it is clear that more manufacturers choose to carefully inspect their products, even though inspection may be quite expensive. For this reason the study of optimal lotsizing with costly inspection becomes more important.

By exploring a few special cases, we have learned that optimal lots might be dramatically different when yield distributions are different (see Figures 3.1, 4.1, and 4.2, and Section 4.2). Knowing the type of production yield can be useful in making lotsizing decisions, even without exact calculations.

For binomial yields units may be selected at random for inspection, and all units should be inspected - as long as there are demands remaining. This is not always true for other type of yields as we have demonstrated in the last section. In particular, the interrupted geometric yield reveals that the optimal lot size depends on the inspection procedure. This area calls for additional research.

We also learned that the influence of the inspection cost on the optimal lot size may change dramatically when the yield distributions are different. For binomial and all-or-nothing yields, the inspection cost have no influence at all. For discrete uniform and interrupted geometric yields, the influence is significant.

In general, numerical results demonstrate that the optimal lot size decreases when the inspection cost increases (see Figures 4.1 and 4.2).

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