# An Efficient Algorithm for Computing the Optimal Replenishment Policy for an Integrated Inventory-Production System

Qi-Ming HE[1], E.M. Jewkes[2], and J. Buzacott[3]
[1]Department of Industrial Engineering, DalTech, Dalhousie University, Halifax, Nova Scotia, Canada B3J 2X4; *qmhe@newton.ccs.tuns.ca*
[2]Department of Management Sciences, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1
[3]Schulich School of Business, York University, North York, Ontario, Canada, M3J 1P3

**Abstract**: This paper develops an efficient algorithm for computing the optimal replenishment policy in an inventory-production system consisting of a warehouse and a workshop. The inventory process is formulated into a Markov decision process and a quasi-birth-and-death Markov process respectively. An interesting relationship between performance measures of interest for the two formulations is found. As a result, an algorithm for computing the optimal replenishment policy is developed. Using the algorithm developed, a numerical example is analyzed so as to gain insights into the inventory-production system of interest.

## 1. Introduction

This paper deals with a simple supply chain which is modelled as an *inventory-production system* consisting of a warehouse and a workshop (see Figure 1.1). Demands from customers are accepted at the workshop. Products are produced in the workshop to satisfy customer demands. Raw materials used in production are supplied by an outside supplier through the warehouse to the workshop.

One of the most important problems associated with the inventory-production system is to reduce inventory costs in the warehouse. In order to do so, a "good" inventory control policy must be found and used in inventory control in the warehouse (see HE [5] for more discussion). The objective of this paper is to develop an algorithm to find the optimal replenishment policy.

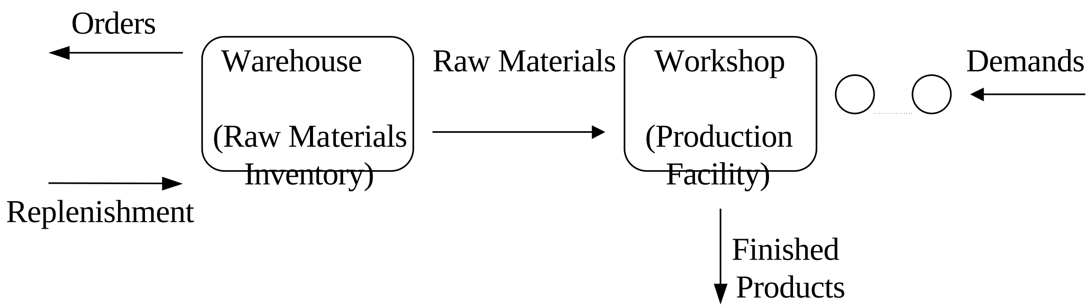


**Figure 1.1** The Inventory Production System

The inventory-production model of interest is a special two echelon system (see Federgruen [4]). However, inventory control in such a special echelon system has not been addressed yet. The inventory-production model also has much to do with the *M/M/1* queue (see Cohen [3]), the stochastic EOQ model (see Bartmann and Beckman [1]), and quasi-birth-and-death (QBD) Markov processes since it is a combined queueing and inventory model. Although the *M/M/1* queue and the stochastic EOQ model are simple and well studied, their combined model is complicated and has not been investigated, especially the optimal inventory control in such a combined model. This paper is the first attempt to study the combined model.

The main mathematical tools used in this paper are matrix analytic methods (MAM) and Markov decision processes (MDP). By using matrix analytic methods, numerically tractable solutions can usually be obtained. By using Markov decision processes, efficient algorithms for computing the optimal policy can usually be developed. This explains why matrix analytic methods and Markov decision processes are utilized in this paper. Readers are referred to Neuts [7] and [8] for more about matrix analytic methods and Chakravarthy and Alfa [2] for more recent development of matrix analytic methods. Readers are referred to Puterman [9] and Tijms [10] for more details about Markov decision processes.

The rest of the paper is organized as follows. Section 2 defines the inventory-production of interest explicitly and introduces several useful concepts. In Section 3, the model of interest is formulated into a Markov decision process and some functional equations are established. In Section 4, the model of interest is formulated into QBD Markov process. Some results about the fundamental periods, costs incurred in fundamental periods are presented. In Section 5, an interesting relationship between a set of Markov decision process measures and a set of matrix analytic method measures is derived. In Section 6, an algorithm is developed for computing the optimal replenishment policy using the relationship obtained in Section 5 and the policy iteration method. A numerical example is presented in Section 7 where a detailed analysis is conducted to gain insights into inventory control of the inventory-production system. Section 8 summarizes this paper and discusses some future research directions.

## 2. Modelling of the Inventory-Production System

The inventory-production system of interest is defined explicitly as follows. In the workshop, raw materials are processed into products according to customer demands. Before the workshop begins to process each customer demand, a call for a unit of raw materials is sent to the warehouse. Raw materials are sent from the warehouse to the workshop if the warehouse is not empty when the call arrives; otherwise, production in the workshop is delayed until the call is filled when new raw materials arrive to the warehouse. As soon as raw materials are available, production begins in the workshop. Finished products are delivered to customers immediately. Orders for raw materials are issued according to some replenishment policy from the warehouse to the outside supplier. Ordered raw materials are transported from the supplier to the warehouse. Once stored in the warehouse, raw materials await calls from the workshop. The warehouse and the workshop are close to each other so that the transportation time between them is negligible.

This paper considers an inventory-production model with a Poisson demand process with parameter $\lambda$. Production times of products have a common exponential distribution with parameter $\mu$ ($>\lambda$). The production times and the demand process are independent. The leadtimes of raw materials are zero. There is a fixed ordering cost $K$ associated with each order, regardless of the order size. The holding cost is $C_h$ per unit raw materials held per unit time. The inventory-production system is reviewed continuously so that replenishment decisions can be made anytime. Production occurs in the workshop whenever there are demands and raw materials in the system. No shortage of raw materials is allowed.

According to its definition, the inventory-production system of interest can be decomposed into two subsystems: an *M/M/1* queue (the workshop) and an inventory system (the warehouse) as depicted in Figure 2.1. The workshop can be modeled as an *M/M/1* queue since no shortage of raw materials is allowed so that the queueing process is not influenced by the raw material replenishment process. The warehouse is modeled as an inventory system with zero leadtimes and demands from the workshop (which occurs every time when the workshop begins to produce a new product). The status of the *M/M/1* queue at time $t$ is represented by the number of customers (or unfilled demands) in the workshop, denote by $q(t)$, i.e., the queue length. Assuming that $\rho=\lambda/\mu<1$ so that the *M/M/1* queue can reach its steady state. The status of the inventory system at time $t$ is represented by the number of units of raw materials in the warehouse, denote by $I(t)$, i.e., the inventory level. Thus, the status of the inventory-production system can be represented by $(q(t), I(t))$ at time $t$.

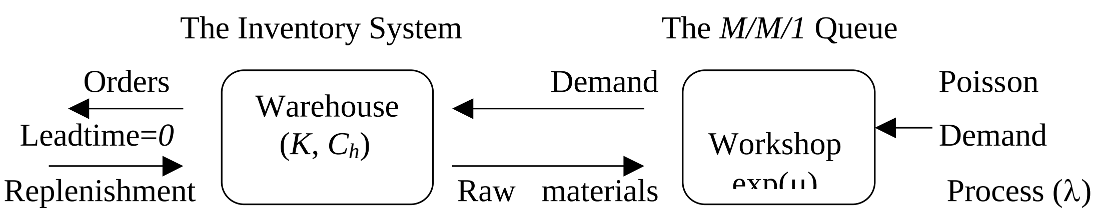


**Figure 2.1** The Inventory-Production System with Zero Leadtimes

Raw materials inventory in the warehouse is controlled according to a replenishment policy, which determines when and how much to order raw materials from the outside supplier. In this paper, only replenishment policies based on system status $(q(t), I(t))$ are considered. Thus, a replenishment policy $\pi$ is a function of $(q(t), I(t))$. At an arbitrary epoch $t$, if $\pi(q(t), I(t))>0$, an order of size $\pi(q(t), I(t))$ is issued and filled; otherwise, no action is taken. Since the leadtimes are zero, it makes no sense to order raw materials when $I(t)$ is positive. Thus, $\pi(q(t), I(t))=0$ when $I(t)>0$. This observation implies that a replenishment policy can be represented by a vector $\pi = (\pi(0), \pi(1), \pi(2), ...)$, where $\pi(q)$ is the order size when the inventory level is zero and the number of unfilled demands is $q$.

Inventory control under the above definition depends on information about the queue length. However, complete information about the queue length may not be available to the decision maker in the warehouse. Thus, the concept of information level is introduced and feasible replenishment policies with certain level of information are introduced next.

Information of the level $l$ ($\geq -1$) is defined explicitly as follows. With information of the level $l$, if the queue length $q(t)\leq l$ at time $t$, $q(t)$ is known to the inventory decision maker. If $q(t)>l$, only the fact "$q(t)>l$" is known but not the exact value of $q(t)$. For instances, *information of the level -1* means that no information about the queue length is available. When *information of the level zero* is available, the decision maker in the warehouse can find out whether or not the queue is empty, or equivalently, whether the workshop is busy or idle.

When information of the level $l$ is available, the order size can be adjusted according to the queue length up to $l$. When the queue length is larger than $l$, the order sizes are the same since the exact queue length is not available. Thus, the set of the feasible replenishment policies under consideration when information of the level $l$ is available is defined as

$$\prod[l] \equiv \{\pi: \pi(q) \leq 2(K\mu/C_h)^{0.5} + 2,\ q \geq 0,\ \pi(q)=\pi(\infty) \text{ for } q > l \text{ and } \pi(\infty)>0\}. \quad (2.1)$$

Note that all the policies in $\prod[l]$ have a fixed tail. The order size for large queue length is called the tail order size. The upper bound of the order size is introduced since it is never optimal to order more than $2(K\mu/C_h)^{0.5} +2$ units of raw materials at a time (see Chapter 4 in HE [5]). The upper bound $2(K\mu/C_h)^{0.5} +2$ can be proved by splitting any order with an order size larger than $2(K\mu/C_h)^{0.5} +2$ into two smaller orders with equal order sizes. It can be shown that the total inventory costs (ordering costs plus holding costs) are reduced.

The objective of this paper is to develop an algorithm for computing the optimal replenishment policy in $\prod[l]$ - *the optimal replenishment policy of the level l* - which minimizes the average total cost per product.

## 3. The Average Total Costs per Product - an MDP Approach

Based on system assumptions, it is easy to see that, for any replenishment policy $\pi\in\prod[l]$, the corresponding stochastic process $(q(t), I(t))$ is a Markov process. It is clear that the queueing system can reach its steady state when $\lambda<\mu$ and so does the inventory-production system. In this section, functional equations are established for computing the average total cost per product for any $\pi\in\prod[l]$, following an MDP approach. Suppose that $\pi\in\prod[l]$ is applied in inventory control. For $0\leq q$, $i\leq n$ and $n\geq 1$, define

$V^{\pi}(q, i, n)$ = The average total cost to produce $n$ products, given that there are $q$ demands and $i$ units of raw materials in the system initially.

$V^{\pi}(q, n)$ = The average total cost to produce $n$ products, given that there are $q$ demands and zero units of raw materials in the system initially.

In steady state (when $\rho=\lambda/\mu<1$), the average total cost per product is defined as

$$g(\pi) \equiv \lim_{n\to\infty} \frac{V^\pi(q,i,n)}{n} \left(= \lim_{n\to\infty} \frac{V^\pi(q,n)}{n}\right). \tag{3.1}$$

Clearly, $g(\pi)$ is independent of the initial state and is finite. Define the relative cost function

$$h^\pi(q, i, n) = V^\pi(q, i, n) - V^\pi(1, 1, n), \quad q\geq 0,\ i\geq 0,\ n\geq 1. \tag{3.2}$$

By Theorem 3.1 in Tijms [10], the limit of $\{h^\pi(q, i, n), n\geq 1\}$ exists and is finite for each state $(q, i)$. Denote by, for $q\geq 0$ and $i\geq 0$,

$$h^\pi(q, i) = \lim_{\{n\to\infty\}} h^\pi(q, i, n) = \lim_{\{n\to\infty\}} [V^\pi(q, i, n) - V^\pi(1, 1, n)]. \tag{3.3}$$

Then, for $q\geq 0$, $i\geq 0$, $n\geq 0$,

$$V^\pi(q,i,n) \approx ng(\pi) + h^\pi(q,i) + \varepsilon(q,i,n), \tag{3.4}$$

where $\varepsilon(q, i, n)\to 0$ as $n\to\infty$. It can be proved that $\{g(\pi), h^\pi(q, i)\}$ satisfy

$$\begin{aligned}
& g(\pi) + h^\pi(1,1) = \frac{C_h}{\mu} + \omega h^\pi(0) + \sum_{j=1}^{\infty} \omega(1-\omega)^j h^\pi(j);\\
q \geq 2, \quad & g(\pi) + h^\pi(q,1) = \frac{C_h}{\mu} + \sum_{j=0}^{\infty} \omega(1-\omega)^j h^\pi(q-1+j);\\
& g(\pi) + h^\pi(1,i) = \frac{iC_h}{\mu} + \omega\left[\frac{(i-1)C_h}{\lambda} + h^\pi(1,i-1)\right]\\
i \geq 2, \quad & \qquad + \sum_{j=1}^{\infty} \omega(1-\omega)^j h^\pi(j,i-1);\\
i,q \geq 2, \quad & g(\pi) + h^\pi(q,i) = \frac{iC_h}{\mu} + \sum_{j=0}^{\infty} \omega(1-\omega)^j h^\pi(q-1+j,i-1),
\end{aligned} \tag{3.5}$$

where $\omega=\mu/(\lambda+\mu)$. Let $\mathbf{I}\{\cdot\}$ be the $0$ - $1$ function. Then

$$h^\pi(0) = K\mathbf{I}\{\pi(0) > 0\} + \pi(0)C_h/\lambda + h^\pi(1,\pi(0)) \text{ and}$$
$$h^\pi(q) = K + h^\pi(q,\pi(q)), \quad q \geq 1.$$

Theoretically, when $\lambda<\mu$, an algorithm can be developed for computing the average total cost per product $g(\pi)$ using equation (3.5) and the idea of value iteration method (see Puterman [9]). Furthermore, an algorithm for computing optimal replenishment policy can be developed by using equation (3.5) and the policy iteration method (also see Puterman [9]). However, there are two technical difficulties.

The first one is related to the tail order size $\pi(\infty)$ when only information of the level $l$ is available. The Markov decision process defined by equation (3.5) cannot be solved as an ordinary Markov decision process since it is not clear how to determine the optimal tail order size. For Markov decision processes with this type of constraints, to our knowledge, little research has been done except Kulkarni and Serin [6]. An enumeration method is proposed to get around of this problem in Section 6.

The second difficulty has much to do with the infinite state space of $(q(t), I(t))$. There is no direct way to evaluate the summations with infinite items in equation (3.5). Fortunately, since the tail order size is fixed, the problem can be transformed into a finite semi-Markov decision process with a finite state space using matrix analytic methods. Subsequently, an algorithm for computing the optimal replenishment policy can be developed (see Sections 4, 5, and 6).

## 4 The Average Total Cost per Product - an MAM approach

In this section, the Markov process $(q(t), I(t))$ corresponding to a particular replenishment policy $\pi\in\prod[l]$ is constructed. Inventory costs incurred in a busy period are analyzed and an algorithm is developed for computing the average total cost per product. For brevity, details are omitted. Readers are referred to HE [5] and Neuts [7] for derivations and proofs of the results given in this section.

### 4.1 The QBD Markov Process

For a replenishment policy $\pi$ in $\prod[l]$, the stochastic process $(q(t), I(t))$ is a two-dimensional Markov process. The state space and the infinitesimal generator of the Markov process $(q(t), I(t))$ are constructed as follows. Denote by

$$\pi_{max} = \max_{\{q\geq 0\}}\{\pi(q)\} \quad \text{and} \quad q_{max} = \min_{\{q\geq 0\}}\{q\text{: } \pi(q) = \pi_{max}\}. \tag{4.1}$$

$\pi_{max}$ is the maximum order size of $\pi$. $q_{max}$ is the smallest queue length where the order size is $\pi_{max}$. The envelope function $\pi^e$ is defined recursively as

$$\pi^{e}(q)=\begin{cases}\pi_{\max}, & q \ge q_{\max};\\ \max\{\pi(1),\cdots,\pi(q),\pi^{e}(q+1)-1\}, & q < q_{\max}.\end{cases} \tag{4.2}$$

While $\pi$ may fluctuate dramatically, $\pi^e$ is a nondecreasing function and is no less than $\pi$. The state space of ($q(t)$, $I(t)$) is then given by $S = \cup^{\infty}_{q=0} S_q$, where

$S_0 = \{(0, \max\{\mathbf{I}\{\pi(0)\neq 0\}, 0\}), ..., (0, \pi^e(0))\}$ and
$S_q = \{(q, 1), ..., (q, \pi^e(q))\}$, $q\geq 1$.

The subset $S_q$ in which all states have a queue length $q$ is called the *level* $q$. Then ($q(t)$, $I(t)$) is an irreducible QBD Markov process with an infinitesimal generator

$$Q=\begin{pmatrix} A_{0,0} & A_{0,1} & & & & & & \\ A_{1,0} & A_{1,1} & A_{1,2} & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & A_{l,l-1} & A_{l,l} & A_{l,l+1} & & & \\ & & & A_2 & A_1 & A_0 & & \\ & & & & A_2 & A_1 & A_0 & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \ddots \end{pmatrix}, \tag{4.3}$$

where all the matrix blocks are given explicitly in Appendix and all other elements are zero. It can be proved that he Markov process ($q(t)$, $I(t)$) is positive recurrent when $\lambda<\mu$.

## 4.2 Fundamental Periods

A fundamental period is defined as the first passage time period during which the Markov process ($q(t)$, $I(t)$) reaches the level $q$, given that the Markov process starts in the level $q+1$ (see Neuts [7]). It is easy to see that the busy period is a special type of fundamental period where the Markov process goes from level $1$ to level $0$ for the first time.

First, consider state transitions during fundamental periods. For $q>l$, the fundamental periods from level $q+1$ to $q$ are probabilistically equivalent since they have a common infinitesimal generator. Define, for $1\leq i, j\leq\pi_{\max}$,

$G^{\pi}_{i,j}(k)$ = The probability that the Markov process ($q(t)$, $I(t)$) reaches, for the first time, the level $q$ in ($q$, $j$) and there are a total of $k$ products produced during this period of time, given that the Markov process started in state ($q+1$, $i$) at time $0$.

Define

$$G^{*,\pi}_{i,j}(z) = \sum_{k=0}^{\infty} z^k G^{\pi}_{i,j}(k), \quad 0 \le z \le 1. \tag{4.4}$$

Let $G^{*,\pi}(z)$ be a $\pi_{\max}\times\pi_{\max}$ matrix with elements defined in equation (4.4). Then $G^{*,\pi}(z)$ is the minimum nonnegative solution to the equation

$$G^{*,\pi}(z) = z(-A_1)^{-1}A_2 + (-A_1)^{-1}A_0[G^{*,\pi}(z)]^2. \tag{4.5}$$

Denote by $G^{\pi} = \lim_{\{z\to 1\}}G^{*,\pi}(z)$, this limit exists and $G^{\pi}$ is the minimal nonnegative solution to the equation

$$A_2 + A_1G^{\pi} + A_0(G^{\pi})^2 = 0. \tag{4.6}$$

In fact, $G^{\pi}_{i,j}$ is the probability that the Markov process ($q(t)$, $I(t)$) reaches, for the first time, the level $q$ in state ($q$, $j$), given that the Markov process starts in state ($q+1$, $i$). It can be proved that $G^{\pi}$ is a stochastic matrix when the Markov chain is positive recurrent.

For boundary levels $0<q\le l$, define the matrix $G^{*,\pi}(q, z)$ ($G^{\pi}(q)$) analogous to $G^{*,\pi}(z)$ ($G^{\pi}$) for the fundamental period from the level $q$ to the level $q-1$. The matrix $G^{*,\pi}(q, z)$ and $G^{\pi}(q)$ are $\pi^e(q+1)\times\pi^e(q)$ matrices and they satisfy equations

$$G^{*,\pi}(q,z) = z(-A_{q,q})^{-1}A_{q,q-1} + (-A_{q,q})^{-1}A_{q,q+1}G^{*,\pi}(q+1,z)G^{*,\pi}(q,z); \tag{4.7}$$

$$G^{\pi}(q) = -(A_{q,q} + A_{q,q+1}G^{\pi}(q+1))^{-1}A_{q,q-1}, \quad q \ge 1. \tag{4.8}$$

For the level zero, define $G_0^{*,\pi}(z)$ as the transform of the total number of products produced during the first passage time from the level zero to the level zero. This time period is equivalent to a busy cycle. It consists of the time the Markov process stays in the level zero (and then left the level zero) and the time period it takes to move from the level one to the level zero for the first time. It is then easy to see that

$$G^{*,\pi}(0,z) = (-A_{00})^{-1} A_{01} G^{*,\pi}(1,z) \quad \text{and} \quad G^{\pi}(0) = (-A_{00}^{-1} A_{01}) G^{\pi}(1). \tag{4.9}$$

where $G^{\pi}(0) = \lim_{\{z\to 1\}} G^{*,\pi}(0, z)$.

### 4.3 The Average Costs Incurred in Fundamental Periods

For $q>l$, let

$\Phi^{\pi}_{i,j}$ = The average total cost incurred during the fundamental period during which the Markov process reaches the level $q\text{-}1$ for the first time in state $(q\text{-}1, j)$, given that it started in state $(q, i)$.

Let $\Phi^{\pi}$ be a $\pi_{\max}\times\pi_{\max}$ matrix with elements defined above. It can be proved that

$$\Phi^{\pi} = \Phi^{\pi}_{0} + (-A_{1}^{-1} A_{0})(G^{\pi}\Phi^{\pi} + \Phi^{\pi} G^{\pi}), \tag{4.10}$$

where

$$\Phi^{\pi}_{0} = -A_{1}^{-1}[C_{h} diag(1,\cdots,\pi_{\max})G^{\pi} + diag(K,0,\cdots,0)A_{2}]. \tag{4.11}$$

Apparently, the elements of the matrix $\Phi_0^{\pi}$ are the average costs incurred before the next transition. The second part on the right hand side of equation (4.10) is the average cost after the first transition.

For boundary levels $0<q\le l$, define the matrix $\Phi^{\pi}(q)$ analogous to $\Phi^{\pi}$ for the fundamental period from the level $q$ to the level $q\text{-}1$. Then

$$\Phi^{\pi}(q) = \Phi^{\pi}_{0}(q) + (-A_{q,q}^{-1} A_{q,q+1})[G(q+1)\Phi^{\pi}(q) + \Phi^{\pi}(q+1)G(q)], \tag{4.12}$$

where

$$\Phi^{\pi}_{0}(q) = -A_{q,q}^{-1}[C_{h} diag(1,\cdots,\pi^{e}(q))G(q) + diag(K,0,\cdots,0)A_{q,q-1}]. \tag{4.13}$$

When $q=0$ and $\pi(0)=0$,

$$\Phi^{\pi}(0) = \Phi^{\pi}_{0}(0) + (-A_{0,0}^{-1} A_{0,1})\Phi^{\pi}(1), \tag{4.14}$$

where

$$\Phi^{\pi}_{0}(0) = -A_{0,0}^{-1}[C_{h} diag(0,\cdots,\pi^{e}(0)-1)G^{\pi}(0) + diag(K,0,\cdots,0)A_{0,1}]. \tag{4.15}$$

When $q=0$ and $\pi(0)>0$, $\Phi^{\pi}_{0}$ is obtained by removing the second term on the right hand side of equation (4.15). In addition, the inventory level at state $(0, i)$ is $i$ instead of $i-1$. The corresponding correction should be made in equation (4.15) as well.

### 4.4 The Average Total Cost per Product

Finally, an algorithm for computing the average total cost per product $g(\pi)$ can be developed. To find the average total cost per product, consider the embedded Markov chain at the beginning epochs of busy cycles. By definition, it is known that this embedded Markov chain has a transition matrix $G^{\pi}(0)$ which is given in equation (4.9). Denote by $\beta$ the left invariant vector of $G^{\pi}(0)$, i.e., $\beta G^{\pi}(0) = \beta$, $\beta \geq \mathbf{0}$, and $\beta\mathbf{e} = 1$. Conditioning on the initial state of a busy cycle, the average total cost incurred in a busy cycle can be obtained as

$$\sum_{i=1}^{\pi(0)} \beta_i (\Phi^{\pi}(0)\mathbf{e})_i, \tag{4.16}$$

i.e., the product of vectors $\beta$ and $\Phi^{\pi}(0)\mathbf{e}$: $\beta\Phi^{\pi}(0)\mathbf{e}$.

According to Cohen [3], the average number of customers served in a busy cycle is $\mu/(\mu-\lambda)$ for the $M/M/1$ queue. Since the total number of products produced in a busy cycle equals the total number of customers served in a busy cycle for the $M/M/1$ queue, the average total cost per product is given by

$$g(\pi) = \beta\Phi^{\pi}(0)\mathbf{e}(\mu-\lambda)/\mu. \tag{4.17}$$

In summary, an algorithm for computing the average total cost per product corresponding to a replenishment policy $\pi$ in $\prod[l]$ by using equations given in this section.

**Note**: For more general QBD Markov processes, the average number of customers served in a busy cycle may not be $\mu/(\mu-\lambda)$. For those cases, equations (4.5) and (4.7) can be used to derive necessary formulas.

## 5. A Key Relationship

This section proves a relationship between the functions $\{g(\pi)\ h^{\pi}(q, i), q\geq 0, i\geq 0\}$ defined in Section 3 and the matrices $\{G^{\pi}, \Phi^{\pi}, G^{\pi}(q), \Phi^{\pi}(q), 0\leq q\leq l\}$ defined in Section 4 for $\pi$ in $\prod[l]$. The relationship between the two sets of measures is interesting since it brings

Markov decision processes and matrix analytic methods closer. This relationship shall be used in developing an algorithm for computing the optimal replenishment policy in Section 6. For $q \geq 0$, $n \geq 1$, denote by

$$\mathbf{V}^{\pi}(q, n) = (V^{\pi}(q, 1, n), ..., V^{\pi}(q, \pi^{e}(q), n))^{T} \quad \text{and}$$
$$\mathbf{h}^{\pi}(q) = (h^{\pi}(q, 1), ..., h^{\pi}(q, \pi^{e}(q)))^{T},$$

where "T" represents matrix transpose.

**Theorem 5.1** When the inventory-production system with replenishment policy $\pi$ in $\prod[l]$ can reach its steady state, it has, for $q>l$,

$$g(\pi)\mathbf{u}^{\pi} + \mathbf{h}^{\pi}(q+1) = \Phi^{\pi}\mathbf{e} + G^{\pi}\mathbf{h}^{\pi}(q); \tag{5.1}$$

for $1<q\leq l$,

$$g(\pi)\mathbf{u}^{\pi}(q) + \mathbf{h}^{\pi}(q) = \Phi^{\pi}(q)\mathbf{e} + G^{\pi}(q)\mathbf{h}^{\pi}(q-1); \tag{5.2}$$

for $q=1$,

$$g(\pi)\mathbf{u}^{\pi}(1) + \mathbf{h}^{\pi}(1) = \Phi^{\pi}(1)\mathbf{e} + G^{\pi}(1)[\Phi_{0}^{\pi}(0) - A_{0,0}^{-1}A_{0,1}\mathbf{h}^{\pi}(1)], \tag{5.3}$$

where

$$\mathbf{u}^{\pi} = \sum_{k=1}^{\infty} kG^{\pi}(\infty, k)\,\mathbf{e},\quad \mathbf{u}^{\pi}(q) = \sum_{k=1}^{\infty} kG^{\pi}(q,k)\,\mathbf{e} \quad \text{and} \quad G^{\pi}(q) = \sum_{k=1}^{\infty} G^{\pi}(q,k). \tag{5.4}$$

where $\mathbf{u}^{\pi}(q)$ (and $\mathbf{u}^{\pi}$) is the average number of products produced during the first passage from the level $q$ to the level $q-1$, and $G^{\pi}(q, k)$ was defined in Section 4.2.

For the inventory-production system of interest, since the queueing system is not affected by inventory control, the number of products produced in a fundamental period is the same as that of the *M/M/1* queue, which is given explicitly as $\mu/(\mu-\lambda)$, regardless of the inventory levels. Then, for $q>0$,

$$\mathbf{u}^{\pi}(q) = \mu/(\mu-\lambda)\mathbf{e} \quad \text{and} \quad \mathbf{u}^{\pi} = \mu/(\mu-\lambda)\mathbf{e}. \tag{5.5}$$

**Proof**. Similar to the cost function $\Phi^{\pi}_{i,j}$ defined in Section 3, define

$\Phi^{\pi}_{i,j}(q, n)$ the average total costs incurred before or when $n$ products are completed during the first passage time from level $q$ to the level $q$-1 in the state ($q$-1, $j$), given that the process started in the state ($q$, $i$).

$\Phi^{\pi}(q, n)$ is an $\pi^e(q)\times\pi^e(q\text{-}1)$ matrix with elements $\Phi_{i,j}(q, n)$. Clearly, $\Phi^{\pi}(q, n)\leq\Phi^{\pi}(q)$, $\{\Phi^{\pi}(q, n), n>0\}$ is a nondecreasing sequence and $\lim_{\{n\to\infty\}}\Phi^{\pi}(q, n)=\Phi^{\pi}(q)$.

Consider the average total cost vector $\mathbf{V}^{\pi}(q, n)$. It is clear that $\mathbf{V}^{\pi}(q, n)$ can be decomposed into two parts: the total cost incurred during the first passage from the level $q$ to the level $q$-1 and the total cost incurred thereafter. Conditioning on the initial inventory level and the queue length at the first transition from level $q$ to level $q$-1, it has, for $q>0$,

$$V^{\pi}(i,q,n) = \sum_{j=1}^{\max\{\pi(q),\pi(q-1)\}}\Phi^{\pi}_{i,j}(q,n) + \sum_{j=1}^{\pi(q-1)}\sum_{k=0}^{n-1}G^{\pi}_{i,j}(q,k)V^{\pi}(j,q-1,n-k). \tag{5.6}$$

Note that when $k\geq n$, product completes before the first passage ends. Writing the above equation into a matrix form, yields

$$\mathbf{V}^{\pi}(q,n) = \Phi^{\pi}(q,n)\mathbf{e} + \sum_{k=0}^{n-1}G^{\pi}(q,k)\mathbf{V}^{\pi}(q-1,n-k). \tag{5.7}$$

Using equation (3.4), yields, for $q>0$,

$$\begin{aligned} ng(\pi)\mathbf{e} + \mathbf{h}^{\pi}(q) &= \Phi^{\pi}(q,n)\mathbf{e} + \sum_{k=0}^{n-1}G^{\pi}(q,k)[(n-k)g(\pi)\mathbf{e} + \mathbf{h}^{\pi}(q-1)] + o(1) \\ &= \Phi^{\pi}(q,n)\mathbf{e} + g(\pi)n\sum_{k=0}^{n-1}G^{\pi}(q,k)\mathbf{e} - g(\pi)\sum_{k=0}^{n-1}kG^{\pi}(q,k)\mathbf{e} \\ &\quad + \sum_{k=0}^{n-1}G^{\pi}(q,k)\mathbf{h}^{\pi}(q-1) + o(1). \end{aligned} \tag{5.8}$$

Thus, to prove equation (5.1) or (5.2), one only needs to prove $\lim_{n\to\infty} n(\mathbf{e} - \sum_{k=0}^{n-1}G^{\pi}(q,k)\mathbf{e}) = 0$. Note that $G^{\pi}(q) = \sum_{k=0}^{n-1}G^{\pi}(q,k) + \sum_{k=n}^{\infty}G^{\pi}(q,k)$. Then

$$0 \leq n(\mathbf{e} - \sum_{k=0}^{n-1}G^{\pi}(q,k)\mathbf{e}) = \sum_{k=n}^{\infty}nG^{\pi}(q,k)\mathbf{e} \leq \sum_{k=n}^{\infty}kG^{\pi}(q,k)\mathbf{e} \xrightarrow{n\to\infty} \mathbf{0},$$

since the mean number of customers served in a fundamental period $\mathbf{u}^{\pi}(q)$ is finite, for $q>0$.

To prove equation (5.3), the level $0$ is considered. Note that (see equation (4.14)):

$$\mathbf{V}^{\pi}(0,n)=\Phi_0^{\pi}(0)\mathbf{e}+(-A_{0,0}^{-1}A_{0,1})\mathbf{V}^{\pi}(1,n)\,. \tag{5.9}$$

This leads to

$$\mathbf{h}^{\pi}(0)=\Phi_0^{\pi}(0)\mathbf{e}+(-A_{0,0}^{-1}A_{0,1})\mathbf{h}^{\pi}(1)\,, \tag{5.10}$$

which proves equation (5.3). This completes the proof.

Using equations (5.1), (5.2), and (5.3), it is possible to calculate $\{g(\pi), \mathbf{h}^{\pi}(q), q\geq 1\}$ from $\{G^{\pi}, \Phi^{\pi}, G^{\pi}(q), \Phi^{\pi}(q), 0\leq q\leq l\}$ for any replenishment policy $\pi$ in $\prod[l]$. This leads to an algorithm for computing the optimal replenishment policy of the level $l$.

**Note**: It is clear from the proof of Theorem 5.1 that the relationship between $\{g(\pi), \mathbf{h}^{\pi}(q), q\geq 1\}$ from $\{G^{\pi}, \Phi^{\pi}, G^{\pi}(q), \Phi^{\pi}(q), 0\leq q\leq l\}$ may hold for much more general stochastic models where the cost structure is defined appropriately and the model can be represented by a QBD Markov process (with level dependent transitions). Equations (5.1), (5.2), and (5.3) show that the relative cost functions are determined by the difference of the total costs incurred during fundamental periods with different initial (inventory) states.

## 6. An Algorithm for Computing the Optimal Replenishment Policy

This section presents an algorithm for computing the optimal replenishment policy based on Theorem 5.1 and the policy iteration method in the theory of dynamic programming.

As was indicated in Section 3, any algorithm directly developed from equation (3.5) may have two difficulties: 1) how to determine the optimal tail order size; 2) how to do summations with infinite items. To overcome these difficulties, the following algorithm which completes the search for the optimal replenishment policy in two steps is proposed. The following algorithm is developed in a top-down manner.

Step 1) For a fixed $\pi(\infty)$ ($1\leq\pi(\infty)\leq 2(K\mu/C_h)^{0.5}+2$), find the suboptimal replenishment policy $\pi^*_1(\pi(\infty))$ which minimizes the average total cost per product in $\prod[l, \pi(\infty)]$, where

$$\prod[l, \pi(\infty)] \equiv \{\pi: \pi(q) \le 2(K\mu/C_h)^{0.5} + 2,\ q \ge 0,\ \pi(q) = \pi(\infty) \text{ for } q > l\}; \tag{6.1}$$

$$\pi_l^*(\pi(\infty)) = \arg \min_{\{\pi \in \Pi[l,\, \pi(\infty)]\}} \{g(\pi)\}\,. \tag{6.2}$$

Step 2) Compare all the suboptimal replenishment policies when $\pi(\infty)$ going from $1$ to $2(K\mu/C_h)^{0.5} + 2$ so as to find the optimal replenishment policy, i.e.,

$$\pi_l^* = \pi_l^*(\pi^*(\infty)) \quad \text{and} \quad g(\pi_l^*(\pi^*(\infty))) = \min_{1 \le i \le 2\sqrt{K\mu/C_h} + 2} \{g(\pi_l^*(i))\}. \tag{6.3}$$

When $l = -1$, i.e., no information about the queue length is available, Step 1) is completed in one iteration. The search for the optimal replenishment policy is completed by considering only $\pi = (\pi(\infty), \pi(\infty), ...)$ for $\pi(\infty)$ going from $1$ to $2(K\mu/C_h)^{0.5} + 2$. In fact, an explicit solution has been found (see HE [5]).

When $l > -1$, Step 1) of the algorithm can be carried out using the policy iteration method. The basic idea is, starting with a carefully chosen replenishment policy, to calculate $\{\mathbf{h}^\pi(q),\ 1<q<l\}$ for $\pi$ in $\prod[l, \pi(\infty)]$ using equations (5.1), (5.2), and (5.3). Then an new replenishment policy $\pi'$ is obtained as

$$\begin{aligned} &\pi'(q) = \arg \min_{1 \le i \le \pi^\wedge e(q)} \{h^\pi(q,i)\},\ 1 \le q \le l; \\ &\pi'(0) = 0, \quad \pi'(q) = \pi(\infty), \quad \text{for } q > l. \end{aligned} \tag{6.4}$$

$\pi'(0) = 0$ since the leadtimes are zero (Information of $\{q(t)=0\}$is available when $l > -1$) and $\pi'(q) = \pi(\infty)$, for $q>l$, since the tail order size is fixed. Then Step 1) can be expanded to

Step 1.1) Initialize the policy iteration process by choosing $\pi = (0, \pi_{\max}, ..., \pi_{\max}, \pi(\infty), ...)$;

Step 1.2) Calculate $\{g(\pi), \mathbf{u}^\pi(q), \Phi^\pi(q), G^\pi(q), \mathbf{u}^\pi, \Phi^\pi, G^\pi\}$ using formulas presented in Section 4;

Step 1.3) Calculate $\{\mathbf{h}^\pi(q),\ 1 \le q \le l\}$ using Theorem 5.1;

Step 1.4) Determine a new policy $\pi'$ using equation (6.4);

Step 1.5) If $\pi = \pi'$, stop; otherwise, repeat steps 1.2) to 1.5) with $\pi'$.

While steps 1.1), 1.2), 1.4), and 1.5) are clear, more details about Step 1.3) are needed. The problem is that one of $\{\mathbf{h}^\pi(q),\ 1 \le q \le l\}$ must be determined first so that equations (5.1), (5.2), and (5.3) can then be used to determine the rest of the vectors in the set. From equation (5.3), it can be proved that

$$\begin{aligned}\mathbf{h}^{\pi}(1) &= [\Phi^{\pi}(1) + G^{\pi}(1)\Phi_0^{\pi}(0) - g(\pi)\mathbf{u}^{\pi}(1)][\mathbf{I} - G^{\pi}(1)(-A_{00}^{-1}A_{01}) + \mathbf{e}\theta]^{-1} \\ &\quad + (\mathbf{h}^{\pi}(1)\mathbf{e})\theta \\ &\equiv \mathbf{c} + (\mathbf{h}^{\pi}(1)\mathbf{e})\theta,\end{aligned} \tag{6.5}$$

where $\theta$ is the left invariant vector of the matrix $G^{\pi}(1)(-A_{00})^{-1}A_{01}$. Thus, it is possible to determine $\mathbf{h}^{\pi}(1)$ first, which is equivalent to determine the product $\mathbf{h}^{\pi}(1)\mathbf{e}$. This can be done by using the original definition of $\{\mathbf{h}^{\pi}(q),\ 1 \le q \le l\}$. Since the state $(q=1,\ i=1)$ was chosen as the base state to determine the relative cost functions, then $h^{\pi}(1,\ 1) = 0$ (see equation (3.3)). The product $\mathbf{h}^{\pi}(1)\mathbf{e}$ is determined by setting $h^{\pi}(1,\ 1) = 0$ in equation (6.5), which yields

$$\mathbf{h}^{\pi}(1)\mathbf{e} = -(\mathbf{c})_1/(\theta)_1. \tag{6.6}$$

This completes Step 1.3) and so does the whole algorithm.

Now, the original problem of finding the optimal replenishment policy has transformed into a finite state semi-Markov decision problem for which the optimal solution can be found in finite steps. This algorithm is used in Section 7 to analyze an inventory-production system.

It is clear that the optimal replenishment policy with information of the level $l$ exists. However, it does not mean that the algorithm developed in this section will always find the optimal policy. The usual problems associated with dynamic programming may show up. For example, cyclic iterations may generate a diverging sequence of replenishment policies. Another problem with this algorithm is that it may take a lot of time to complete each iteration. Unfortunately, these issues are beyond the scope of this paper and are left as future research. Nonetheless, according to numerical experimentation, the algorithm finds the optimal replenishment policy most of the time.

## 7. A Numerical Example

This section presents a numerical example. The optimal policies of different information levels are obtained by using the algorithm developed in Section 6. Some insights into the inventory control process of the inventory-production system are gained.

**Example 7.1** Consider inventory-production systems with $C_h=0.2$, $K=10$, and $\mu=1$. The demand rate takes values $\lambda=0.1$, $0.618$, and $0.95$.

$\lambda=0.1$. The optimal replenishment policies with information levels *0, 1, 5, 10,* and full are given in Figure 7.1 respectively.

Figure 7.1 shows that the optimal replenishment policies for different information levels are dramatically different. When the information level increases, the optimal replenishment policy converges to the optimal replenishment policy with full information. The computation of the optimal replenishment policy requires more efforts as the level of information increases.

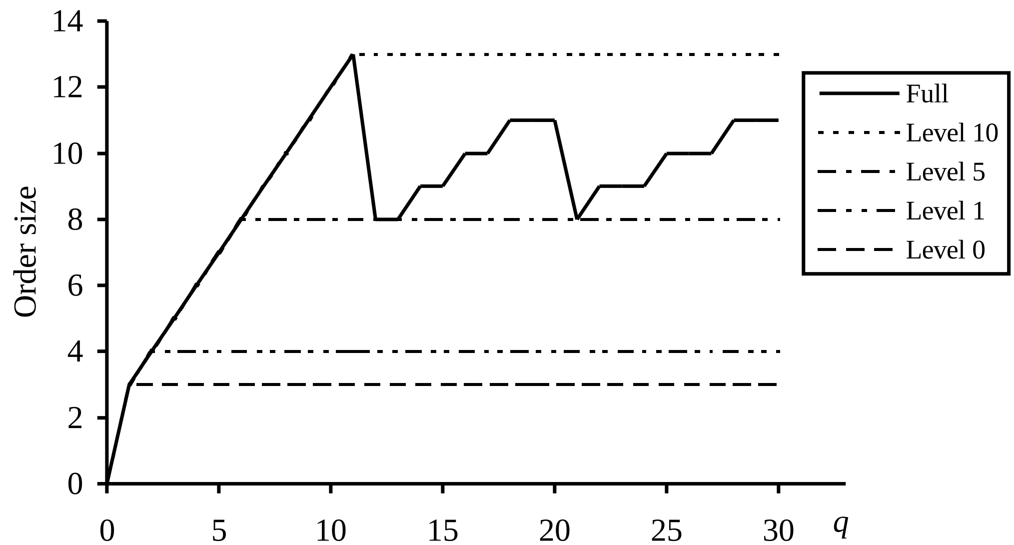


**Figure 7.1** The Optimal Replenishment Policies When $\lambda=0.1$

Figure 7.1 also shows that the optimal order size fluctuates when the queue length changes (e.g., the optimal replenishment policy with full information). The reason is that the order size is adjusted to reduce the holding cost during possible idle periods. It can be shown that if no holding cost is incurred during any idle period, the optimal order size is the same for all queue length. For more discussions on this and related issues, see Chapters 4 and 5 in HE [5].

$\lambda=0.618$. The optimal replenishment policies with information levels *-1*, *0*, *2*, *3*, and *4* are given in Table 7.2 respectively. In Table 7.2, each row represents an optimal replenishment policy with certain level of information. The last column in Table 7.2 gives that average total cost per unit time corresponding to each policy.

In this case, the optimal replenishment policies with information level *4* and higher are the same. This implies that only information of level *4*, not full information, is required to achieve the overall optimal inventory control. This also implies that information about the queue length larger than *4* has no value in terms of inventory control. In fact, it can be proved that the overall optimal inventory control is always achieved at a finite information level.

**Table 7.2** The optimal replenishment policies when $\lambda=0.618$

| Level | $q=0$ | $q=1$ | $Q=2$ | $q=3$ | $q=4$ | $q\geq5$ | $g(\pi)$ |
|---|---|---|---|---|---|---|---|
| *-1* | 8 | 8 | 8 | 8 | 8 | 8 | 2.706 |
| *0* | **0** | 8 | 8 | 8 | 8 | 8 | 2.582 |
| *1* | **0** | **7** | 9 | 9 | 9 | 9 | 2.571 |
| *2* | **0** | **7** | **8** | 9 | 9 | 9 | 2.569 |
| *3* | **0** | **7** | **8** | **9** | 10 | 10 | 2.568 |
| (≥)*4* | **0** | **7** | **8** | **9** | **9** | 10 | 2.568 |

In Table 7.2, the bolded cell means that the corresponding order size can be adjusted individually. For example, when the information level is *-1*, i.e., the warehouse places an order of size *8* whenever the inventory level becomes zero, regardless of the queue length (since there is absolutely no information about the queue length available to the decision maker). When the information level is zero, the order size at $q\geq1$ must be the same but the order size at $q=0$ can be different from others. As shown in Table 7.2, the order size at $q=0$ is adjusted to *0* when $l=0$. Compared to the $\lambda=0.1$ case, the fluctuation of the optimal replenishment policies is reduced, and information levels up to *4* are valuable in inventory control. The adjustment of the order size at the queue length from *0* to *4* is useful. Since the traffic intensity is neither large nor small, such adjustments in order sizes may decrease the holding cost during idle periods, even when the queue length is around *4*.

The last column of Table 7.2 shows that the major cost savings is achieved at lower information levels, especially the level zero. This conclusion is supported by all numerical examples.

$\lambda$*=0.95*. For this case, the optimal replenishment policies with information levels *-1*, *0*, *2*, and *3* are given in Table 7.3 with their corresponding average total costs per unit time, respectively.

**Table 7.3** The optimal replenishment policies when $\lambda=0.95$

| level | $q=0$ | $q=1$ | $q=2$ | $q=3$ | $q\geq 4$ | $g(\pi)$ |
|---|---|---|---|---|---|---|
| *-1* | 10 | 10 | 10 | 10 | 10 | 2.1578 |
| *0* | **0** | 10 | 10 | 10 | 10 | 2.1473 |
| *1* | **0** | **8** | 10 | 10 | 10 | 2.1443 |
| *2* | **0** | **9** | **9** | 10 | 10 | 2.1441 |
| (≥)*3* | **0** | **8** | **9** | **9** | 10 | 2.1438 |

Compared to $\lambda=0.1$ and $\lambda=0.618$, the optimal inventory control becomes much simpler. Thus, it concludes that information about the queue length becomes less valuable when production load is high. Nevertheless, numerical results show that information of level zero is important in terms of cost savings, even for high production load inventory-production system (see HE [5]).

In summary, numerical results show that lower level information is important in inventory control of the inventory-production system. It is useful for the workshop to inform the warehouse its number of products waiting to be produced in the near future, when such a number is small. It is particularly important to know when the number of products to be produced is zero.

**Remark**: Production-inventory systems with information of the level *-1*, *0*, or ∞ are of particular interest. While the analysis of systems with full information is complicated, the analysis of systems with information of the level *-1* or *0* is simple and explicit. More details about these special cases can be found in HE [5].

## 8. Summary and Future Research Directions

This paper formulated the inventory control process of an inventory-production system into a Markov decision process and developed an algorithm for the computing optimal replenishment policy. It also formulated the inventory process into a QBD Markov process. An interesting relationship between $\{g(\pi)\ h^{\pi}(q, i), q\geq 0, i\geq 0\}$ associated with the Markov decision processes and $\{G^{\pi}, \Phi^{\pi}, G^{\pi}(q), \Phi^{\pi}(q), 0\leq q\leq l\}$ associated with matrix analytic methods is established. The established relationship is utilized in developing an algorithm for computing the optimal replenishment policy when the information level is given.

This paper studied a stochastic model based on the specific context of an inventory-production system. The advantage of doing so is that more insights into the inventory control process can be gained and the usefulness of the methodology developed is shown. There are some drawbacks with such an approach as well. Nonetheless, the methodology developed and the main results obtained in this paper can be generalized to much more complicated stochastic models with similar features as that of $(q(t), I(t))$. For examples, using the same methodology, it is possible to study inventory-production systems which have exponential leadtimes of raw materials. It is also possible to study inventory-production models with general service time models. But the analysis becomes much more complicated. Interested readers are referred to HE [5] for more details.

## Appendix The Infinitesimal Generator of $(q(t), I(t))$

For the Markov process $(q(t), I(t))$, the states in $S_q$ are related to each other. The transitions become more complicated. In fact, the transitions between states are categorized into the following three types:

1) A transition from state $(q, i)$ to $(q+1, i)$ occurs when a demand arrives before the completion of the product in service, if any;
2) A transition from $(q, i)$ to $(q-1, i-1)$ occurs when a product is completed before the next arrival;
3) A transition from $(q, 1)$ to $(q-1, \pi(q-1))$ takes place when a product is completed before the next arrival and an order is issued and fulfilled.

The case $q=0$ is slightly different and has only transitions of types 1) and 3). A replenishment of raw materials may occur when the queue length goes from zero to one as well. Thus, the Markov process is a quasi-birth-and-death process with an infinitesimal generator given in equation (4.3). The transition blocks are specified as follows.

First, the dimensions of those matrices are given. $\{A_{q,q\text{-}1}\}$ are $\pi^e(q)\times\pi^e(q\text{-}1)$ matrices since the level $q$ (i.e., $S_q$) has $\pi^e(q)$ states and the level $q\text{-}1$ ($S_{q\text{-}1}$) has $\pi^e(q\text{-}1)$ states; Similarly, $\{A_{q,q}\}$ are $\pi^e(q)\times\pi^e(q)$ matrices; $\{A_{q,q+1}\}$ are $\pi^e(q)\times\pi^e(q+1)$ matrices; $A_0$, $A_1$ and $A_2$ are $\pi_{\max}\times\pi_{\max}$ matrices. Notice that the envelope function $\pi^e(q)$ represents the number of states in level $q$.

$A_{0,0} = -\lambda\mathbf{I}$: no transitions among the states of the level zero; the next transition epoch is the arrival epoch of a demand. $\mathbf{I}$ is the identity matrix.

$$A_{0,1} = \lambda(\mathbf{I},\mathbf{0}), \quad \text{for } \pi(0) > 0,\ \pi^e(0) = \pi^e(1) - 1;$$

$$A_{0,1} = \lambda\mathbf{I}, \qquad \text{for } \pi(0) > 0,\ \pi^e(0) = \pi^e(1);$$

$$A_{0,1} = \begin{pmatrix} 0 & 0 & \cdots & \lambda & \cdots & 0 & 0 \\ & \lambda & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \lambda & 0 \end{pmatrix}, \quad \text{for } \pi(0) = 0,\ \pi^e(0) = \pi^e(1) - 1.$$

When $\pi(0)>0$, no order is issued at the next arrival epoch when the queue length becomes one. This case is included for more general applications. For example, when no information about the queue length is available, an order with a positive order size must be issued at $q=0$ and $i=0$. When $\pi(0)=0$, an order of the size $\pi(1)$ is issued and filled when the Markov process moves from $(0, 0)$ to $(1, 0)$, which brings the inventory level to $\pi(1)$. For the matrix $A_{0,1}$ defined for the $\pi(0)=0$ case, the right hand side column (the zero column) is removed when $\pi^e(0)= \pi^e(1)$.

$$\text{For } q>0, \quad A_{q,q-1} = \begin{pmatrix} 0 & \cdots & \mu & \cdots & 0 \\ \mu & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \mu & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad \text{for} \quad \pi^e(q) > \pi^e(q-1).$$

When $\pi^e(q) = \pi^e(q-1)$, the last row is removed from $A_{q,q-1}$. When $\pi(0)=0$, the first row in $A_{1,0}$ becomes zeros. When $(q(t), I(t))$ makes the transition from $(q, 1)$ to level $q-1$, an order of the size $\pi(q-1)$ is issued and filled.

$A_{q,q} = -(\lambda+\mu)\mathbf{I}$. There is no transition among the states in the same level.

$A_{q,q+1} = (\lambda\mathbf{I}, \mathbf{0})$, when $\pi(q)>\pi(q+1)$, $q\geq 1$, and $A_{q,q+1} = \lambda\mathbf{I}$, when $\pi(q)=\pi(q+1)$.

$$A_0 = \lambda\mathbf{I},\ A_1 = -(\lambda+\mu)\mathbf{I},\ \text{and}\ \ A_2 = \begin{pmatrix} 0 & \cdots & \mu & \cdots & 0 \\ \mu & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \mu & 0 \end{pmatrix}.$$

In $A_2$, when the Markov process goes down one level from the state $(q, 1)$, an order of the size $\pi_\infty$ is issued and filled.

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