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Markov chains with marked transitions[1]

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Abstract

Several useful point processes such as the Markovian arrival process, the input and departure processes of finite-capacity Markovian queues, and various models for counters and biological phenomena are obtained by considering Markov chains with marked transitions. This point of view yields many examples of interesting dependent point processes and provides a unified formalism for their study. This paper presents some characterizations of Markov chains with marked transitions. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The considerations in this paper apply to Markov chains with a discrete or a continuous time parameter. Although the analysis to be presented can be carried out for chains with infinite state spaces, all but a few cases of applied interest involve finitely many states. We therefore restrict our attention to *point processes generated by marked transitions of an m-state, irreducible, continuous parameter Markov chain with a generator D.*

Before proceeding with the main discussion, we remove the inessential restriction – transitions from a state to itself are not allowed – that is commonly imposed on continuous parameter Markov chains. The easiest way to remove that restriction is to consider the m-state Markov renewal process with an irreducible embedded Markov chain with transition probability matrix P and exponential sojourn time distributions given by

$$F_{ij}(x) = 1 - \exp\{-\sigma_i x\} \tag{1}$$

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for $1 \leqslant i,\ j \leqslant m$. The m-state Markov chain with generator D is then none other than the semi-Markov process corresponding to that Markov renewal process. The generator D and the parameters P and $\sigma_i,\ 1 \leqslant i \leqslant m$, of the Markov renewal process are related by

$$D_{ii} = -(1 - P_{ii})\sigma_i \quad \text{for } 1 \leqslant i \leqslant m \tag{2}$$

and

$$D_{ij} = P_{ij}\sigma_i \quad \text{for } 1 \leqslant i,\ j \leqslant m,\ i \neq j. \tag{3}$$

We denote the stationary probability vector of D by θ.

We consider a finite or countable set C^0 of nonzero K-tuples of nonnegative integers, where K is a fixed finite positive integer. A generic element of C^0 is denoted by $\boldsymbol{h} = (h_1, \ldots, h_K)$, where h_k is a nonnegative integer, $1 \leqslant k \leqslant K$, and at least one of $\{h_k,\ 1 \leqslant k \leqslant K\}$ is positive. The set C^0 may have different interpretations according to circumstances. In many instances, the set C^0 is used in cases where transitions may bear multiple labels. For example, K can be the number of possible types of arriving customers to a queue with batch arrivals and h_k the number of type k customers in a batch.

We consider a Markov renewal process $\{(J_n, L_n, \tau_n),\ n \geqslant 0\}$ on the state space

$$[\{1, \ldots, m\} \times C^0] \times [0, \infty)$$

with the transition probability matrix

$$P\{J_n = j,\ L_n = \boldsymbol{h}, \tau_n \leqslant x \mid J_{n-1} = i\} = \left[\int_0^x \exp(D_0 u)\, \mathrm{d}u D_{\boldsymbol{h}} \right]_{ij} \tag{4}$$

with $1 \leqslant i, j \leqslant m$, $\boldsymbol{h} \in C^0$, $x \geqslant 0$. For convenience, we set $\tau_0 = 0$. The matrices $D_{\boldsymbol{h}}$ are nonnegative. The matrix D_0 has negative diagonal elements and is assumed to be nonsingular. That requirement serves to ensure that, with probability one, infinitely many transition epochs are marked. In addition, we have that

$$D = D_0 + \sum_{\boldsymbol{h} \in C^0} D_{\boldsymbol{h}}. \tag{5}$$

The point process defined by matrices $\{D_0, D_{\boldsymbol{h}}, \boldsymbol{h} \in C^0\}$ is called a *Markovian arrival process with marked arrivals* which is abbreviated as MMAP.

Example a (*The Markovian arrival process*). Let the set $C^0 = \{C\}$ be the set of nonnegative integers. D_h is interpreted as the matrix of transition rates which involve the “arrival” of a batch of size h. The point process generated by the epochs and the batch sizes of successive arrivals is the *batch Markovian arrival process* (BMAP). That process is extensively used in modeling arrival processes to queues, see Asmussen and Ramaswami (1990), Lucantoni (1991), Lucantoni et al. (1990), and Neuts (1981). The BMAP, which has many useful particular cases, often leads to algorithmically tractable models. An extension using multiple labels can be used to model Markovian arrival processes of batches with several types of customers.

Example b (*The input and output processes of Markovian queues*). Markovian queues are described by continuous parameter Markov chains. For finite capacity queues, we may label the transitions corresponding to an arrival by $\boldsymbol{a}$; those consisting of a departure by $\boldsymbol{d}$. The set C^0 now consists of the elements $\boldsymbol{a}$ and $\boldsymbol{d}$ and the transition rate matrix D is accordingly written as the sum of D_0, $D_{\boldsymbol{a}}$ and $D_{\boldsymbol{d}}$. As a specific example, for the elementary loss system with c channels and exponential inter arrival and service times, the nonzero elements of these matrices are given by

$$(D_0)_{ii} = -\lambda - i\mu \quad \text{for } 0 \leqslant i \leqslant c-1,$$

$$(D_0)_{cc} = -c\mu,$$

$$(D_{\boldsymbol{a}})_{i,i+1} = \lambda \quad \text{for } 0 \leqslant i \leqslant c-1,$$

$$(D_{\boldsymbol{d}})_{i,i-1} = i\mu \quad \text{for } 1 \leqslant i \leqslant c.$$

That example can readily be adapted to focus, for example, on the departure process only when at least $c' \leqslant c$ channels are busy or, by adding one more state, to include as differently marked transition epochs when calls are lost (the overflow process). Neuts (1991) discusses an unbounded queueing model for which the joint distribution of the numbers of arrivals and departures is studied essentially by considering marked transitions.

Example c (*A thinning operation*). Several models for excitatory and inhibitory impulse streams in neurons as well as certain labeling operations for packet streams in communications models may be viewed as selective thinning operations whereby a given point process is thinned by interactions with a second point process (see Neuts, 1993). To consider a specific example, we use a MAP with single arrivals and parameter matrices C_0 and C_1 of order m. There is a second, independent Poisson process of rate α. Events in the MAP are labeled 1 if in the superposition of both processes, they are immediately preceded by an event in the Poisson process and they are labeled 2, otherwise. The process with labeled transitions is a MAP with parameter matrices

$$D_0 = \begin{pmatrix} C_0 - \alpha I & \alpha I \\ 0 & C_0 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 0 & 0 \\ C_1 & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

If the rate α is very high, most points are labeled 1 and, as α decreases, more and more receive the label 2. In addition, it is to be expected that the attrition of points labeled 1 is most pronounced where the original points are clustered. For a fixed value of α, we can so separate the arrival stream into two dependent thinned point processes, the first reflecting the unclustered behavior, the second representative of the clusters. That labeling procedure, which is but one of many, can be used to quantify the sensitivity of a queue to clustering in the input stream (see Liu and Neuts, 1994).

The Markov arrival process with marked transitions (MMAP) is useful in modeling input processes of stochastic systems with several types of items (e.g., customers or orders). In telecommunications, for instance, a multi-service network provides voice, data, and video services (see Roberts, 1991). The nature of the transmitted information from those sources is dramatically different. A uniform formulation to describe the input process while each source is still distinguishable will be useful in analyzing the influence of information from each source on the system. In manufacturing, multi-item orders are common in practice. The MMAP can be used to model the demand process at the item level so that an inventory control analysis can be carried out in terms of individual items while the correlation of different items is considered as well.

The remainder of the paper is organized as follows. In Section 2, the counting process associated with the MAP with marked transitions is studied in detail. In Section 3, some characterizations of the MMAP, such as the types of arrivals, Palm measures, peakedness functions, and first passage times, are discussed. These descriptors give some basic characteristics of the MMAP. In Section 4, we show some closure properties of the MMAP under several kinds of transformations and operations.

2. The counting process

Without loss of generality, we assume that an arrival marked by $\boldsymbol{h} \in C^0$, $\boldsymbol{h} = (h_1, \dots, h_K)$, carries h_k *items* of type k, $1 \leqslant k \leqslant K$. Therefore, there are total of K types of items. We denote by $N_k(t)$ the number of items of type k arriving in $(0, t)$, $1 \leqslant k \leqslant K$. $\boldsymbol{N}(t) = (N_1(t), \dots, N_K(t))$ is called the counting process of the MMAP, where $\{N_k(t),\ 1 \leqslant k \leqslant K\}$ are Markov additive processes.

2.1. Basic properties

Let $\boldsymbol{z} = (z_1, \dots, z_K)$. We define the matrix of generating functions

$$D^*(\boldsymbol{z}) = D_0 + \sum_{\boldsymbol{h} \in C^0} \boldsymbol{z}^{\boldsymbol{h}} D_{\boldsymbol{h}}, \quad 0 \leqslant z_k < 1,\ 1 \leqslant k \leqslant K, \tag{6}$$

where $\boldsymbol{z}^{\boldsymbol{h}} = z_1^{h_1} \cdots z_K^{h_K}$. Let $\boldsymbol{n} = (n_1, \dots, n_K)$, where $n_1, \dots, n_K$ are nonnegative integers. Let $J(t)$ be the phase of the Markov renewal process $\{J_n, L_n, \tau_n\}$ at time t. We define

$$p_{ij}(\boldsymbol{n}, t) = \boldsymbol{P}(\boldsymbol{N}(t) = \boldsymbol{n}, J(t) = j \mid J(0) = i), \quad 1 \leqslant i,\ j \leqslant m.$$

The matrix with these elements is denoted by $P(\boldsymbol{n}, t)$. $\{P(\boldsymbol{n}, t),\ \boldsymbol{n} \geqslant 0\}$ satisfy the Kolmogorov differential equations

$$P'(\boldsymbol{n}, t) = P(\boldsymbol{n}, t) D_0 + \sum_{\boldsymbol{h} \leqslant \boldsymbol{n}, \boldsymbol{h} \in C^0} P(\boldsymbol{n} - \boldsymbol{h}, t) D_{\boldsymbol{h}}, \quad \boldsymbol{n} \geqslant \boldsymbol{0}. \tag{7}$$

Using Eq. (7), it can be shown by routine calculations that the generating function of $\boldsymbol{N}(t)$, $P^*(\boldsymbol{z}, t) = \sum_{\boldsymbol{n} \geqslant 0} \boldsymbol{z}^{\boldsymbol{n}} P(\boldsymbol{n}, t)$, is given by

$$P^*(\boldsymbol{z}, t) = \exp\{t D^*(\boldsymbol{z})\}. \tag{8}$$

The stationary version of the MMAP is obtained by choosing the initial phase according to the probability vector θ (which is the stationary probability vector of the underlying Markov process D).

When we consider only the point process formed by items of type k, we obtain a BMAP with coefficient matrices

$$D_0(k) = D_0 + \sum_{\boldsymbol{h}\in C^0,\, h_k=0} D_{\boldsymbol{h}}, \qquad D_n(k) = \sum_{\boldsymbol{h}\in C^0,\, h_k=n} D_{\boldsymbol{h}}, \quad n \geqslant 1$$

and the counting process $N_k(t)$, $1 \leqslant k \leqslant K$. Similarly, the point process consisting of all items regardless of type is a BMAP with coefficient matrices

$$D_n = \sum_{\boldsymbol{h}\in C^0,\, h_1+\cdots+h_K=n} D_{\boldsymbol{h}}, \quad n \geqslant 1$$

and with counting process $N_{\text{all}}(t) = \sum_{k=1}^{K} N_k(t)$.

For details about the counting process of the BMAP, we refer to Lucantoni (1991), Narayana and Neuts (1992) and Neuts (1979). Some results from those papers are also used here.

2.2. The first and second moments and the correlations

Let

$$\overline{D} = \sum_{n=1}^{\infty} nD_n, \qquad \overline{\overline{D}} = \sum_{n=2}^{\infty} n^2 D_n, \tag{9}$$

$$\overline{D}(k) = \sum_{n=1}^{\infty} nD_n(k), \qquad \overline{\overline{D}}(k) = \sum_{n=2}^{\infty} n^2 D_n(k), \quad 1 \leqslant k \leqslant K. \tag{10}$$

We define factorial moment functions

$$M_{(k_1,\ldots,k_i)}(t) = \left.\frac{\partial^i P^*(\boldsymbol{z},t)}{\partial z_{k_1}\cdots\partial z_{k_i}}\right|_{z_1=\cdots=z_K=1}, \quad 1 \leqslant k_1,\ldots,k_i \leqslant K, \ i \geqslant 1$$

for the MMAP and

$$M_i(t) = \left.\frac{\partial^i (P^*(\boldsymbol{z},t)|_{z_1=\cdots=z_K=z})}{\partial z^i}\right|_{z=1}, \quad i \geqslant 0,$$

for the BMAP which consists of items of all types.

For the mean matrix, it was proved (see Eq. (16) in Narayana and Neuts, 1992) that

$$M_{(k)}(t) = \int_0^t \exp\{Du\}\overline{D}(k)\exp\{D(t-u)\}\,\mathrm{d}u, \quad 1 \leqslant k \leqslant K. \tag{11}$$

Therefore, in the stationary version, the mean number of items of type k arriving in $(0,t)$ is given by

$$\boldsymbol{E}_\theta N_k(t) = \theta M_{(k)}(t)\boldsymbol{e} = \lambda_k^* t,$$

where $\lambda_k^* = \theta\overline{D}(k)\boldsymbol{e}$, the arrival rate of items of type k, $1 \leqslant k \leqslant K$, and $\boldsymbol{e}$ is the column vector with all components one. For $N_{\text{all}}(t)$, the arrival rate is $\lambda^* = \theta\overline{D}\boldsymbol{e}$. Apparently,

$$M_1(t) = \sum_{k=1}^{K} M_{(k)}(t) \quad \text{and} \quad \lambda^* = \sum_{k=1}^{K} \lambda_k^*. \tag{12}$$

For the second moment, it was proved (see Eq. (17) in Narayana and Neuts, 1992) that

$$M_{(k,k)}(t) = 2\int_0^t \int_0^u \exp\{Dv\}\overline{D}(k)\exp\{D(u-v)\}\,\mathrm{d}v\,\overline{D}(k)\exp\{D(t-u)\}\,\mathrm{d}u$$
$$+\int_0^t \exp\{Du\}[\overline{\overline{D}}(k)-\overline{D}(k)]\exp\{D(t-u)\}\,\mathrm{d}u. \tag{13}$$

Next, we consider the correlation of the numbers of items of any two types at time t. Without loss of generality, we assume that $K=2$.

Theorem 2.1.

$$M_{(1,2)}(t) = \tfrac{1}{2}[M_2(t) - M_{(1,1)}(t) - M_{(2,2)}(t)]. \tag{14}$$

Proof. Eq. (14) is a matrix generalization of the elementary identity satisfied by any two real numbers v_1 and v_2:

$$v_1v_2 = \tfrac{1}{2}[(v_1+v_2)(v_1+v_2-1) - v_1(v_1-1) - v_2(v_2-1)].$$

Theorem 2.1 can be proved by setting $v_1 = N_1(t)I\{J(t)=j\}$, and $v_2 = N_2(t)I\{J(t)=j\}$, then taking expectation of both sides of the above equation given $J(0)=i$. The function $I\{J(t)=j\}$ is 1 if $J(t)=1$ is true and 0, otherwise. Details of the proof are omitted. □

We introduce the following vectors and constants:

$$\boldsymbol{c} = \theta\overline{D}(\boldsymbol{e}\theta - D)^{-1}, \qquad \boldsymbol{d} = (\boldsymbol{e}\theta - D)^{-1}\overline{D}\boldsymbol{e}, \qquad \lambda^{**} = \theta\overline{\overline{D}}\boldsymbol{e},$$
$$\boldsymbol{c}(k) = \theta\overline{D}(k)(\boldsymbol{e}\theta - D)^{-1}, \qquad \boldsymbol{d}(k) = (\boldsymbol{e}\theta - D)^{-1}\overline{D}(k)\boldsymbol{e}, \qquad \lambda_k^{**} = \theta\overline{\overline{D}}(k)\boldsymbol{e}$$

for $1\leqslant k\leqslant 2$. By pre-multiplying θ and post-multiplying $\boldsymbol{e}$ on both sides of Eq. (14), we obtain that

$$\lambda^{**} = \lambda_1^{**} + \lambda_2^{**} + 2v_{1,2}^{*},$$

where $v_{1,2}^{*} = \theta V_{1,2}\boldsymbol{e}$ and

$$V_{1,2} = \left.\frac{\partial^2 D^*(\boldsymbol{z})}{\partial z_1\partial z_2}\right|_{z_1=z_2=1} = \sum_{\boldsymbol{h}\in C^0,\boldsymbol{h}=(h_1,h_2)} h_1h_2D_{\boldsymbol{h}}.$$

The variance of $N_k(t)$ in the stationary version is given by

$$\mathrm{var}(N_k(t)) = [\lambda_k^{**} - 2(\lambda_k^{*})^2 + 2\boldsymbol{c}(k)\overline{D}(k)\boldsymbol{e}]t$$
$$-2\boldsymbol{c}(k)[I-\exp\{Dt\}]\boldsymbol{d}(k),\ \ 1\leqslant k\leqslant 2, \tag{15}$$

where I is the identity matrix. Dropping k from that formula yields the variance of $N_{\mathrm{all}}(t)$.

Theorem 2.2. *For the stationary version of the MMAP, the correlation of $N_1(t)$ and $N_2(t)$ is given by*

$$\begin{aligned}\mathrm{Cov}(N_1(t),N_2(t)) = \sigma_{12}t - [&\boldsymbol{c}(1)(I-\exp\{Dt\})\boldsymbol{d}(2)\\ &+\boldsymbol{c}(2)(I-\exp\{Dt\})\boldsymbol{d}(1)],\end{aligned} \tag{16}$$

where

$$\sigma_{12} = v_{1,2}^{*} - 2\lambda_1^{*}\lambda_2^{*} + \boldsymbol{c}(1)\overline{D}(2)\boldsymbol{e} + \boldsymbol{c}(2)\overline{D}(1)\boldsymbol{e}. \tag{17}$$

Proof. For the stationary version of the MMAP, we have

$$\mathrm{Cov}(N_1(t),N_2(t)) = \tfrac{1}{2}[\mathrm{var}(N_1(t)+N_2(t)) - \mathrm{var}(N_1(t)) - \mathrm{var}(N_2(t))].$$

Then simple calculations yield formulas (16) and (17). □

2.3. Asymptotic normality

Let

$$\bar{N}_k(t) = \frac{N_k(t)-\lambda_k^{*}t}{\sigma_k\sqrt{t}}, \quad 1 \leqslant k \leqslant K,$$

where σ_k is the coefficient of t in Eq. (15) for the variance of $N_k(t)$.

Theorem 2.3. *Let $\bar{\boldsymbol{N}}(t)$ be the row vector with elements $\bar{N}_k(t)$. Then we have*

$$\lim_{t\to\infty} \bar{\boldsymbol{N}}(t) = \mathrm{N}(0,\Sigma) \quad (\textit{in distribution}), \tag{18}$$

where $\mathrm{N}(0,\Sigma)$ *is the multivariate normal distribution and* Σ *is the covariance matrix with elements*

$$\frac{\sigma_{k_1k_2}}{\sigma_{k_1}\sigma_{k_2}}, \quad 1 \leqslant k_1,k_2 \leqslant K.$$

Proof. This result is standard because of the regenerative properties of the counting process $\{N(t)\}$. It can be proved in a standard manner (e.g., see pp. 136–137 of Asmussen, 1987; Keilson and Wishart, 1964). □

3. Some characteristics of the MMAP

The arrivals of an MMAP being of different types, it is useful to know how to distinguish them or to examine when arrivals of a certain type occur. Also, it is useful to see how the arriving process of arrivals of a special type affects the behavior of the process $N_{\mathrm{all}}(t)$. In this section, we study the types of arrivals and the peakedness functions of the arrival process. We also consider the first passage time to the arrival of an item of a specific type and the behavior of the MMAP during that first passage.

3.1. The types of arrivals

We consider the label of an arrival at its arriving epoch, the label of the last arrival before time t, or the label of the first arrival after time t. These are three ways to observe the arrivals. Let $C^0 = \{\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{n_0}\}$, where n_0 is a finite positive integer. Let $L(t)$ be the label of the last arrival before time t. $(L(t), J(t))$ is a Markov process with the infinitesimal generator

$$Q = \begin{pmatrix} D_0 + D_{\boldsymbol{h}_1} & \cdots & D_{\boldsymbol{h}_{n_0}} \\ \vdots & \ddots & \vdots \\ D_{\boldsymbol{h}_1} & \cdots & D_0 + D_{\boldsymbol{h}_{n_0}} \end{pmatrix}.$$

Theorem 3.1. *For the stationary version of the MMAP, we have the following results.*

(a) *Backward looking: The probability that the last arrival before an arbitrary time t is marked by $\boldsymbol{h}$ is given by* $-\theta D_{\boldsymbol{h}} D_0^{-1}\boldsymbol{e}$, $\boldsymbol{h} \in C^0$.

(b) *Forward looking: The probability that the first arrival after an arbitrary time t is marked by $\boldsymbol{h}$ is given by* $-\theta D_0^{-1} D_{\boldsymbol{h}}\boldsymbol{e}$, $\boldsymbol{h} \in C^0$.

(c) *At the arrival: The probability that an arbitrary arrival is of type $\boldsymbol{h}$ is given by* $\lambda_{\boldsymbol{h}}^*/\lambda^*$, *where* $\lambda_{\boldsymbol{h}}^* = \theta D_{\boldsymbol{h}}\boldsymbol{e}$, $\boldsymbol{h} \in C^0$.

Proof. Part (a) is from the stationary probability distribution of Q, which is given by $(-\theta D_{\boldsymbol{h}_1} D_0^{-1},\ -\theta D_{\boldsymbol{h}_2} D_0^{-1}, \ldots, -\theta D_{\boldsymbol{h}_{n_0}} D_0^{-1})$.

Part (b) is obtained from the fact that the probability that the next arrival is of type $\boldsymbol{h}$ is

$$\int_0^\infty \exp\{D_0 t\} D_{\boldsymbol{h}}\, \mathrm{d}t = -D_0^{-1} D_{\boldsymbol{h}}, \ \ \boldsymbol{h} \in C^0.$$

Also, from the above equation, the transition matrix of the embedded Markov chain (L_n, J_n) at arrival epochs is

$$\boldsymbol{P}_{\mathrm{e}} = \begin{pmatrix} -D_0^{-1} D_{\boldsymbol{h}_1} & \cdots & -D_0^{-1} D_{\boldsymbol{h}_{n_0}} \\ \vdots & \ddots & \vdots \\ -D_0^{-1} D_{\boldsymbol{h}_1} & \cdots & -D_0^{-1} D_{\boldsymbol{h}_{n_0}} \end{pmatrix}.$$

The left invariant vector of $\boldsymbol{P}_{\mathrm{e}}$ leads to part (c). □

Next, we consider more complicated types of arrivals. Given a string $(k_1, \ldots, k_i)$, $1 \leqslant k_1, \ldots, k_i \leqslant n_0$, a run of type $(k_1, \ldots, k_i)$ is defined as i consecutive arrivals for which the jth arrival is marked by $\boldsymbol{h}_{k_j}$, $1 \leqslant j \leqslant i$. Similarly to Theorem 3.1, the probabilities of a run of type $(k_1, \ldots, k_i)$ are

(a) Backward looking: $\theta(-D_{\boldsymbol{h}_{k_1}} D_0^{-1}) \cdots (-D_{\boldsymbol{h}_{k_i}} D_0^{-1})\boldsymbol{e}$.

(b) Forward looking: $\theta(-D_0^{-1} D_{\boldsymbol{h}_{k_1}}) \cdots (-D_0^{-1} D_{\boldsymbol{h}_{k_i}})\boldsymbol{e}$.

(c) At the arrival:

$$\frac{\theta(-D_{\boldsymbol{h}_{k_1}} D_0^{-1}) \cdots (-D_{\boldsymbol{h}_{k_{i-1}}} D_0^{-1}) D_{\boldsymbol{h}_{k_i}} \boldsymbol{e}}{\lambda^*}.$$

3.2. Palm functions and peakedness functions

The Palm measure and peakedness function are important descriptors of point processes (see Daley and Vere-Jones, 1988; Eckberg, 1983). Neuts and Latouche (1986) discussed the Palm and peakedness functions of the superposition of independent BMAPs. In this subsection, we study the Palm and peakedness functions for each type of arrivals and their relations with the BMAP consisting of arrivals of all types.

The probability distribution vector of the phase immediately after an arbitrary arrival marked by $\boldsymbol{h}$ is given by $(\theta D_{\boldsymbol{h}}\boldsymbol{e})^{-1}\theta D_{\boldsymbol{h}}$. So, the probability distribution vector of the phase immediately after an arrival of type k is given by

$$\theta_{\mathrm{arr}}(k) = \frac{\theta\overline{D}(k)}{\lambda_k^*}, \quad 1\leqslant k\leqslant K. \tag{19}$$

$\theta_{\mathrm{arr}} = (\lambda^*)^{-1}\theta\overline{D}$ is the probability distribution vector of the phase immediately after an arbitrary arrival (of any type).

With these probability vectors, we can define Palm functions for stochastic processes of items of different types at different arrival epochs. Let $H_{k_1}(k_2,t) = E[N_{k_2}(t)|$ an item of type k_1 arrives at time 0], the expected number of items of type k_2 arriving in $(0,t)$, given that there is an item of type k_1 arriving at time 0, $1\leqslant k_1,\ k_2\leqslant K$. Then

$$H_{k_1}(k_2,t) = \lambda_{k_2}^* t + \theta_{\mathrm{arr}}(k_1)[I-\exp\{Dt\}]\boldsymbol{d}(k_2), \quad 1\leqslant k_1,\ k_2\leqslant K. \tag{20}$$

The Palm function of $N_{\mathrm{all}}(t)$, i.e., the total number of arrivals in an interval $(0,t)$, given that there is an arrival at time 0, is given by

$$H(t) = \lambda^* t + \theta_{\mathrm{arr}}[I-\exp\{Dt\}]\boldsymbol{d}. \tag{21}$$

Theorem 3.2. *We denote by* $p_k = \lambda_k^*/\lambda^*$, *the probability that an arbitrary item is of type* k, $1\leqslant k\leqslant K$. *Then we have*

$$H(t) = \sum_{k_1=1}^{K} p_{k_1}\left[\sum_{k_2=1}^{K} H_{k_1}(k_2,t)\right]. \tag{22}$$

Proof. Eq. (22) is obvious from Eqs. (20) and (21). □

Note. For the superposition of K independent MAPs, Eq. (22) simplifies to

$$H(t) = \lambda^* t + \sum_{k=1}^{K} p_k[H_k(k,t) - \lambda_k^* t], \tag{23}$$

since $H_{k_1}(k_2,t) = \lambda_{k_2}^* t$ for $1\leqslant k_1,k_2\leqslant K$ and $k_1 \neq k_2$.

The peakedness function of a point process is the ratio of the variance and the mean of the number of busy servers in a stationary infinite server queue to which the arrival stream is hypothetically offered (see Eckberg, 1983). Apparently, with different types of service times, the peakedness functions are different. For simplicity, we consider the case where the servers have the same exponential service time. Let $h^*_{k_1}(k_2,s)$ be the LS transform of $H_{k_1}(k_2,t)$. For $\mathrm{Re}(s)>0$, we define the functions,

$$\begin{aligned} z_{\exp,k_1}(k_2,s) &= \delta(k_1,k_2)+h^*_{k_1}(k_2,s)-\frac{\lambda^*_{k_2}}{s} \\ &= \delta(k_1,k_2)-\theta_{\mathrm{arr}}(k_1)D(sI-D)^{-1}(\boldsymbol{e}\theta-D)^{-1}\overline{D}(k_2)\boldsymbol{e} \end{aligned} \tag{24}$$

for $1\leqslant k_1,\ k_2\leqslant K$, where $\delta(i,j)$ is 1 if $i=j$ and 0 otherwise. According to Eckberg (1983), $z_{\exp,k_1}(k_2,\mu)$ shows the arrival rate of items of type k_2 after the arrival of a type k_1 item. Thus, $z_{\exp,k}(k,\mu)$ is defined as the peakedness function of the point process consisting of items of type k, $1\leqslant k\leqslant K$. The relationship between the peakedness functions of the process $N_{\mathrm{all}}(t)$ and $\{N_k(t),\ 1\leqslant k\leqslant K\ \}$ is given in the following theorem.

Theorem 3.3. *We denote by $z_{\exp}(\mu)$ the peakedness function for the BMAP which consists of items of all types of the MMAP. Then we have*

$$z_{\exp}(\mu) = 1-\theta_{\mathrm{arr}}D(\mu I-D)^{-1}(\boldsymbol{e}\theta-D)^{-1}D^*\boldsymbol{e} \tag{25}$$

$$= \sum_{k=1}^{K} p_k z_{\exp,k}(k,\mu)+\sum_{k_1=1}^{K} p_{k_1}\left[\sum_{k_2=1,k_1\neq k_2}^{K} z_{\exp,k_1}(k_2,\mu)\right]. \tag{26}$$

Proof. This is obvious from Eqs. (22) and (24). □

Note. For the superposition of K independent BMAPs, Eq. (26) simplifies to $z_{\exp}(\mu)=\sum_{k=1}^{K} p_k z_{\exp,k}(k,\mu)$.

Eq. (26) shows that the peakedness function of the BMAP consisting of all the arrivals is the average of peakedness functions of each type of items. This reveals a limitation in the value of the peakedness function when different types of items are not distinguished. The reason is that the average does not reflect the behavior of the arrival processes of individual types of items. Therefore, Eq. (26) indicates the needs to look at the peakedness functions of individual types of items in order to get insights into the point process.

3.3. Terminating processes

We consider a stopping time τ of $\{J(t),L(t),\boldsymbol{N}(t),\ t\geqslant 0\}$ of the MMAP. A terminating process is defined as a stochastic process which is identical with the MMAP before time τ and stops in $J(\tau)$ after time τ. For the terminating process, let $\tilde{J}(t)$ be its phase at time t and the vector $\tilde{\boldsymbol{N}}(t)$ be the numbers of arrivals of different types at time t. We have

$$\tilde{J}(t)=\begin{cases} J(t) \text{ if } t<\tau, \\ J(\tau) \text{ if } t\geqslant\tau \end{cases} \quad\text{and}\quad \tilde{\boldsymbol{N}}(t)=\begin{cases} \boldsymbol{N}(t) \text{ if } t<\tau, \\ \boldsymbol{N}(\tau) \text{ if } t\geqslant\tau. \end{cases} \tag{27}$$

We denote by $F^*(z,s)$ the joint transform of $(\tilde{N}_1(\tau),\ldots,\tilde{N}_K(\tau))$ and τ, i.e.,

$$F^*(z,s)=\int_0^\infty \mathrm{e}^{-st}\boldsymbol{E}z^{(\tilde{N}_1(t),\ldots,\tilde{N}_K(t))}\,\mathrm{d}\boldsymbol{P}\{\tau<t\}. \tag{28}$$

First, let

$$\tau=\inf\{t:\ N_1(t)\geqslant 1\},$$

i.e., the first passage time to the arrival of an item of type 1.

Theorem 3.4. *For the first passage time until the arrival of an item of type* 1, *we have*

$$F^*(z,s)=\left(sI-D_0-\sum_{\boldsymbol{h}\in C^0,h_1=0} z^{\boldsymbol{h}}D_{\boldsymbol{h}}\right)^{-1}\left[\sum_{\boldsymbol{h}\in C^0,h_1>0} z^{\boldsymbol{h}}D_{\boldsymbol{h}}\right]\boldsymbol{e}. \tag{29}$$

Proof. Let $F(z,t)$ be the joint generating function of $(\tilde{N}_1(\tau),\ldots,\tilde{N}_K(\tau))$, given that $\tau=t$. By conditioning on the first arrival of the MMAP, we have

$$F(z,t)=\int_0^t \exp\{uD_0\}\left(\sum_{\boldsymbol{h}\in C^0,h_1=0} z^{\boldsymbol{h}}D_{\boldsymbol{h}}\right)F(z,t-u)\,\mathrm{d}u$$

$$+\exp\{tD_0\}\left[\sum_{\boldsymbol{h}\in C^0,h_1>0} z^{\boldsymbol{h}}D_{\boldsymbol{h}}\right]\boldsymbol{e}.$$

Taking LS transforms on both sides of the above equation with respect to t, we obtain Eq. (29). □

For the particular case where τ is a nonnegative random variable which is *independent* of the MMAP, it is clear that

$$F^*(z,s)=\int_0^\infty \exp\left\{-x\left(sI-D_0-\sum_{\boldsymbol{h}\in C^0} z^{\boldsymbol{h}}D_{\boldsymbol{h}}\right)\right\}\mathrm{d}\boldsymbol{P}\{\tau<x\}, \tag{30}$$

Specifically, let τ have a PH-distribution with representation (α,T). We define an MMAP* as the superposition of the MMAP and the PH-renewal process with interarrival time τ. The MMAP* has $K+1$ types of items and coefficient matrices

$$D_0\oplus T,\qquad D_{\boldsymbol{h}}\otimes I,\quad \boldsymbol{h}\in C^0,\ -I\otimes T\boldsymbol{e}\alpha, \tag{31}$$

where $\otimes$ and $\oplus$ are *Kronecker product* and the *Kronecker sum* of matrices respectively. Items of type k, $1\leqslant k\leqslant K$, are from the MMAP and the items of type $K+1$ correspond to the renewals in the PH-renewal process.

Theorem 3.4 implies that the terminating process of the MMAP with a stopping time τ which is independent of the MMAP and has a PH-distribution function is equivalent to the terminating process of the MMAP* with the stopping time as the first passage time to the arrival of an item of type $K+1$.

4. Closure properties

The superposition of two independent MMAPs is also an MMAP. We further show that the MMAP is closed under a random time transformation, a random type change and a specific thinning operation. We also introduce the compound MMAP as a generalization of the compound Poisson process.

4.1. A random time transformation

Let $I(t)$ be an m_1-state irreducible Markov process with infinitesimal generator Q. We define a piece-wise constant function $\mu(\cdot)$ which takes the value μ_j whenever the process is in the state j. The stochastic process

$$\tau(t) = \int_0^t \mu(x)\,\mathrm{d}x$$

is a random time change.

Denoted by M the matrix $\mathrm{diag}(\mu_1,\ldots,\mu_{m_1})$. In Neuts (1993), the transformation $\tau(t)$ is called a (Q,M)-time change. For $t\geqslant 0$, $x\geqslant 0$ and $1\leqslant i,\ i'\leqslant m_1$, let $\phi_{ii'}(x,t)$ be the probability $P\{\tau(t)\leqslant x,\ I(\tau(t)) = i' \mid I(0) = i\}$. The matrix with these elements is denoted by $\Phi(x,t)$.

Lemma 4.1. *For $x\geqslant 0$, the Laplace transform $\Phi^*(s,t) = \int_0^\infty e^{-sx}\Phi(dx,t)$ is given by*

$$\Phi^*(s,t) = \exp\{(Q - sM)t\}. \tag{32}$$

Proof. By conditioning on the transitions in $(t, t+\delta t)$,

$$\phi_{ii'}(x,t+\delta t) = (1 - Q_{ii'}\delta t)\phi_{ii'}(x - \delta t\mu_{i'}, t) + \sum_{k=1,k\neq i'}^{m} \phi_{ik}(x - \mathrm{O}(\delta t), t)Q_{ki'}\delta t + \mathrm{o}(\delta t),$$

where $\mathrm{o}(\delta t)\to 0$ as $\delta t\to 0$, $\min\{\mu_i\}\delta t\leqslant \mathrm{O}(\delta t)\leqslant \max\{\mu_i\}\delta t$. Routinely, it follows that

$$\frac{\partial\Phi(x,t)}{\partial t} + \frac{\partial\Phi(x,t)}{\partial x}M = \Phi(x,t)Q \tag{33}$$

with $\Phi(0,t) = 0$. Taking LS transforms on both sides of Eq. (33) with respect to x, we obtain

$$\frac{\partial\Phi^*(s,t)}{\partial t} = \Phi^*(s,t)(Q - sM).$$

Upon integration, the stated result follows immediately. □

We now define a new stochastic process $(J_n, L_n, \tilde{\tau}_n, n\geqslant 0)$ by $\tilde{\tau}_n = \tau(\tau_n)$.

Theorem 4.2. *For independent stochastic processes Q and MMAP, the stochastic process $(J_n, I_n, L_n, \tilde{\tau}_n)$ is an MMAP with coefficient matrices*

$$D_0\otimes M^{-1} + I\otimes QM^{-1} \quad \textit{and} \quad D_{\boldsymbol{h}}\otimes M^{-1},\ \boldsymbol{h}\in C^0. \tag{34}$$

Proof. It is obvious that $(J_n, I_n, L_n, \tilde{\tau}_n)$ is a Markov renewal process. The sojourn time in each phase has an exponential distribution since the transform function $\mu(\cdot)$ is a piecewise constant function. The LS transform of the time until the next arrival which is marked by $\boldsymbol{h}$, has a matrix form

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}\left[\int_0^\infty \exp\{D_0 t\}D_{\boldsymbol{h}} \otimes \Phi(x,t)\,\mathrm{d}t\right] \\ = \int_0^\infty (\exp\{D_0 t\}D_{\boldsymbol{h}}) \otimes \Phi^*(s,t)\,\mathrm{d}t \\ = [sI - (D_0 \otimes M^{-1} + I \otimes QM^{-1})]^{-1}(D_{\boldsymbol{h}} \otimes M^{-1}).\end{aligned}$$

This is also the LS transform of the time until the next arrival that is marked by $\boldsymbol{h}$ of the MMAP with the coefficient matrices given in Eq. (34). Similarly, it can be proved that the distributions of the arrival times of a finite number of arrivals are the same for $(J_n, I_n, L_n, \tilde{\tau}_n)$ and the MMAP. Details are omitted. □

4.2. A random-type change

For simplicity, in this Section 4.3 and Section 4.4, we assume that $C^0 = \{1, 2, \dots, K\}$, D_k represents the arrival (matrix) rate of type k items, $1 \leqslant k \leqslant K$, and the batch size is one. Notice that this interpretation of C^0 is different from that was given at the beginning of Section 2. Suppose that upon arrival, an item of type k_1 changes its type to k_2 with probability p_{k_1,k_2}, $1 \leqslant k_1 \leqslant K$ and $1 \leqslant k_2 \leqslant K'$. We call this a *random-type change*.

Theorem 4.3. *The process obtained by the random-type change is itself an MMAP with coefficient matrices*

$$\left(D_0, \sum_{k=1}^{K} p_{k,1} D_k, \dots, \sum_{k=1}^{K} p_{k,K'} D_k\right). \tag{35}$$

Proof. It is obvious that the new process is still an MMAP with the phase process $J(t)$. The distribution of the time until the next arrival of type k is given by

$$\int_0^x \exp\{tD_0\}\left(\sum_{k_1=1}^{K} p_{k_1,k} D_{k_1}\right)\mathrm{d}t, \quad 1 \leqslant k \leqslant K'.$$

This is the same as that of an MMAP with the stated coefficient matrices given by Eq. (35). The same argument used in the proof of Theorem 4.2 leads to the conclusion. □

4.3. *The compound* MMAP

A stochastic process $T(t)$ is a compound MMAP if

$$T(t) = \sum_{k=1}^{K} \sum_{n=1}^{N_k(t)} v_{k,n}, \tag{36}$$

where $(N_1(t),\dots,N_K(t))$ is the counting process of an MMAP with $C^0=\{1,2,\dots,K\}$ (this is for simplicity) and $\{v_{k,n},\ 1\leqslant k\leqslant K,\ n\geqslant 1\}$ are nonnegative random variables. We shall derive the LS transforms of $T(t)$ for the following two cases.

(a) The $\{v_{k,n}, n\geqslant 1\}$ are sequences of i.i.d. random variables, $1\leqslant k\leqslant K$, which are independent of each other and independent of the MMAP. We define $\psi_{j,j'}(x,t) = \boldsymbol{P}\{T(t)<x, J(t)=j' \,|\, J(0)=j\}$ and $\Psi^*(s,t)$ to be the matrix with elements the LS transform with respect to x of $\psi_{j,j'}(x,t)$. If there are n_1 type 1 arrivals, $\dots, n_K$ type K arrivals in $(0,t)$, the LS transform of $T(t)$ is given by

$$\prod_{k=1}^{K}(f_k^*(s))^{n_k},$$

where $f_k^*(s)$ is the LS transform of $v_{k,1}$, $1\leqslant k\leqslant K$. By the law of total probability,

$$\Psi^*(s,t)=\exp\left\{t\left(D_0+\sum_{k=1}^{K}f_k^*(s)D_k\right)\right\}. \tag{37}$$

In the stationary version (the initial state is chosen according to the stationary probability vector θ), the expectation of $T(t)$ is given by

$$\boldsymbol{E}_\theta T(t)=\left(\sum_{k=1}^{K}\lambda_k^*\boldsymbol{E}v_{k,1}\right)t.$$

A special case arises when all $\{v_{k,n}\}$ are integer variables, with common density $\{p_{k,j},\ 0\leqslant j<\infty\}$, for $1\leqslant k\leqslant K$. Then $T(t)$ is the counting process of a BMAP with coefficient matrices

$$(D_0,\tilde{D}_{(1)},\dots,\tilde{D}_{(n)},\dots),$$

where $\tilde{D}_{(n)}=\sum_{k=1}^{K}p_{k,n}D_k$, $1\leqslant n<\infty$.

(b) The sequences $\{v_{k,n}, 1\leqslant n<\infty\}$ are the successive interarrival times of a MAP with coefficient matrices (C_{0k},C_{1k}) of order m_k, $1\leqslant k\leqslant K$. We assume that all the stochastic processes are independent with each other. This case is interesting since $\{v_{k,n}, 1\leqslant n<\infty\}$ is a nonrenewal process but explicit results can still be obtained. Let $I_k(t)$ be the phase of the MAP (C_{0k},C_{1k}) at time t, $1\leqslant k\leqslant K$. The stochastic process $(J(t),I_1(t),\dots,I_K(t))$ is a Markov process defined on $[1,m]\times[1,m_1]\times\cdots[1,m_K]$. Let

$$\begin{aligned}\psi_{ji_1\cdots i_K,j'i_1'\cdots i_K'}(x,t)=\boldsymbol{P}\{&T(t)<x,\ J(t)=j',\ I_1(t)=i_1',\dots,I_K(t)=i_K'\\ &|J(0)=j,I_1(0)=i_1,\dots,I_K(0)=i_K\}\end{aligned}$$

and $\Psi(x,t)$ to be the matrix with elements $\psi_{ji_1\dots i_K,j'i_1'\dots i_K'}(x,t)$. By the law of total probability

$$\begin{aligned}\Psi(x,t)=&\int_0^t\int_0^x\sum_{k=1}^{K}[\exp\{D_0u\}D_k\otimes I\otimes\exp\{C_{0k}v\}C_{1k}\otimes I]\Psi(x-v,t-u)\,\mathrm{d}v\,\mathrm{d}u\\&+\exp\{D_0t\}\otimes I.\end{aligned}$$

Let $\Psi^*(s_1,s_2)=\int_0^\infty\int_0^\infty \mathrm{e}^{-s_2 t}\mathrm{e}^{-s_1 x}\Psi(\mathrm{d}x,t)\,\mathrm{d}t$. Taking the joint transforms of both sides of the above equation, we obtain

$$\Phi^*(s_1,s_2)=\left(s_2 I-D_0-\sum_{k=1}^{K}D_k\otimes I\otimes[(s_1 I-C_{0k})^{-1}C_{1k}]\otimes I\right)^{-1}.$$

The inverse transform with respect to s_2 of the above formula is

$$\Psi^*(s,t)=\exp\left\{t\left(D_0\otimes I+\sum_{k=1}^{K}D_k\otimes I\otimes[(sI-C_{0k})^{-1}C_{1k}]\otimes I\right)\right\}. \tag{38}$$

For the stationary version, the expectation of $T(t)$ is given by

$$\boldsymbol{E}_{\tilde{\theta}}T(t)=\left(\sum_{k=1}^{K}\lambda_k\mu_k^*\right)t,$$

where $\tilde{\theta}=\theta\otimes\theta_1\cdots\otimes\theta_K$, θ_k is the stationary probability vector of $C_{0k}+C_{1k}$, and $\mu_k^*=-\theta_k C_{0k}^{-1}\boldsymbol{e}$, $1\leqslant k\leqslant K$.

4.4. *A thinning of the* MMAP

Let the thinning process be a MAP with coefficient matrices (C_0,C_1). The thinned process is a point process consisting of all arrivals of the MMAP which immediately follow an arrival of the MAP in the superposition process of the two point process. An arrival of the thinned process keeps its mark in the MMAP. It is easily seen that the thinned process is an MMAP with coefficient matrices

$$\tilde{D}_0=\begin{pmatrix} D\oplus C_0 & I\otimes C_1\\ 0 & D_0\oplus(C_0+C_1)\end{pmatrix},$$

$$\tilde{D}_k=\begin{pmatrix} 0 & 0\\ D_k\otimes I & 0\end{pmatrix},\quad 1\leqslant k\leqslant K.$$

Let $\tilde{D}=\sum_{k=0}^{K}\tilde{D}_k$. Then

$$\tilde{D}=\begin{pmatrix} D\oplus C_0 & I\otimes C_1\\ (D-D_0)\otimes I & D_0\oplus(C_0+C_1)\end{pmatrix}.$$

By routine calculations, the stationary probability vector of $\tilde{D}$ is given by (π_1,π_2), where

$$\pi_1=-(\theta\otimes\theta_2)((D-D_0)\otimes I)(D_0\otimes C_0)^{-1},$$

$$\pi_2=-(\theta\otimes\theta_2)(I\otimes C_1)(D_0\otimes C_0)^{-1}$$

and θ_2 is the stationary probability vector of C_0+C_1.

We denote by $\tilde{\lambda}_k^*$, $1\leqslant k\leqslant K$, the arrival rates of the thinned MMAP. The we have

$$\tilde{\lambda}_k^*=-(\theta\otimes\theta_2 C_1)(D_0\otimes C_0)^{-1}(D_k\otimes I)\boldsymbol{e},\quad 1\leqslant k\leqslant K. \tag{39}$$

Thinning of stochastic processes is useful in studying the burstiness of the MMAP (see Neuts, 1993).

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References

Asmussen, S., 1987. Applied Probability and Queues. Wiley, New York.

Asmussen, S., Ramaswami, V., 1990. Probabilistic interpretation of some duality results of the matrix paradigms in queueing theory. Stochastic Models 6, 715–734.

Daley, D.J., Vere-Jones, D., 1988. An Introduction to the Theory of Point Processes. Springer, New York.

Eckberg, A.E., 1983. Generalized peakedness of teletraffic processes. Proc. 10th Int. Teletraffic Conf., Montreal, Canada, Paper No. 4.4B-3.

Keilson, J., Wishart, D.M.G., 1964. A central limit theorem for processes defined on a finite Markov chain. Proc. Cambridge Philos. Soc., vol. 60, pp. 547–567.

Liu, D., Neuts, M.F., 1994. Assessing the effect of bursts of arrivals on the characteristics of a queue. in: Perez Vilaplana, Puri, M.L. (Eds.), Recent Advances in Statistics and Probability, Proc. 4th Int. Meeting of Statistics in the Basque Country – IMSIBAC 4, VSP International Science Publisher, pp. 371–387.

Lucantoni, D.M., 1991. New results on the single server queue with a batch Markovian arrival process. Stochastic Models 7, 1–46.

Lucantoni, D.M., Meier-Hellstern, K.S., Neuts, M.F., 1990. A single server queue with server vacations and a class of non-renewal arrival processes. Adv. Appl. Probab. 22, 676–705.

Narayana, S., Neuts, M.F., 1992. The first two moment matrices of the counts for the Markovian arrival process. Stochastic Models 8, 459–478.

Neuts, M.F., 1979. A versatile Markovian point process. J. Appl. Probab. 16, 764–779.

Neuts, M.F., 1981. Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach. The Johns Hopkins University Press, Baltimore.

Neuts, M.F., 1991. The joint distribution of arrivals and departures in quasi-birth-and-death processes. In: Stewart, N.J. (Ed.), Numerical Solution of Markov Chains, Marcel Dekker, New York, pp. 147–159.

Neuts, M.F., 1993. Burstiness of point processes. Stochastic Models 9, 445–466.

Neuts, M.F., Latouche, G., 1986. The superposition of two PH-renewal processes. In: J. Janssen (Ed.), Semi-Markov Models: Theory and Applications. Plenum Publishers, London, 131–77.

Roberts, J.W. (Ed.), 1991. Performance evaluation and design of multiservice networks. Cost 224 Final Report.