

PARTIAL ORDERS AND THE MATRIX R IN MATRIX ANALYTIC METHODS*

QI-MING HE[†]

Abstract. This paper studies the matrix R , which is the minimal nonnegative solution to a nonlinear matrix equation, raised in matrix analytic methods. Based on some partial orders defined on the transition matrix of Markov chains of GI/M/1 type, the monotonicity of the corresponding matrix R and its Perron–Frobenius eigenvalue is investigated. The results are useful in estimating tail probabilities of stationary distributions of Markov chains of GI/M/1 type and constructing upper bounds for the matrix R . Applications to the GI/MAP/1 queue are discussed as well.

Key words. matrix analytic methods, queueing theory, partial order, GI/MAP/1 queue, non-negative matrix

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1. Introduction. Let $\{A_n, n \geq 0\}$ be a sequence of $m \times m$ nonnegative matrices whose summation matrix A is substochastic or stochastic. $\{A_n, n \geq 0\}$ is called a substochastic or stochastic sequence accordingly. Let R , an $m \times m$ matrix, be the minimal nonnegative solution to the equation

$$(1.1) \quad X = \sum_{n=0}^{\infty} X^n A_n.$$

Based on some partial orders defined on the set of all substochastic and stochastic sequences, this paper presents some characterizations of the matrix R . Applications to Markov chain theory, matrix analytic methods, and the GI/MAP/1 queue are explored.

The interest in the minimal nonnegative solution to (1.1) comes from the study of Markov chains of GI/M/1 type (see (2.1)), which often arise in the modeling of stochastic systems. A classical example is the GI/M/1 queue in which the embedded Markov chain for the queue length at arrival epochs has the GI/M/1-type structure. More complicated examples are Markov chains of GI/M/1 type with matrix transition blocks ($\{A_n, n \geq 0\}$) raised in the study of the GI/PH/1 and GI/MAP/1 queues (Neuts [16]). The matrix R is important since the stationary distribution of a Markov chain of GI/M/1 type, when it exists, has a matrix–geometric solution which can be expressed in terms of R and another constant vector. Early papers (see Purdue [20]) studied (1.1) and its minimal nonnegative solution. It has been proved that, under some conditions, the *Perron–Frobenius eigenvalue* (the eigenvalue with the largest real part) of R is less than one so that the stationary distribution of the Markov chain exists and is unique. More recently, Gail, Hantler, and Taylor [6] found all power bounded solutions of (1.1). Their work greatly extended our understanding of the solutions to (1.1). Although previous studies gained some insights into the minimal nonnegative solution of (1.1), there are still some important issues that need to be explored (see Neuts [19]). This paper addresses two of those issues.

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[†]Department of Industrial Engineering, DalTech, Dalhousie University, Halifax, NS, Canada, B3J 2X4 (Qi-Ming.He@dal.ca).

The first issue is related to the comparison of the stationary distributions of Markov chains of GI/M/1 type. Similar to stochastic comparison of Markov chains (see Daley [5] and Keilson and Kester [9]), some partial orders shall be defined on the transition blocks $(\{A_n, n \geq 0\})$ of Markov chains of GI/M/1 type. When the transition blocks of two Markov chains of GI/M/1 type are partially ordered, the relationship between their corresponding matrix R shall be investigated. Since the stationary distributions of the two Markov chains of GI/M/1 type can be expressed in terms of their matrix R , the relationship between the two stationary distributions can be investigated accordingly. For example, by introducing the stochastically larger order, a general inequality for the matrix R under this partial order shall be proved. Implications of the results, especially to the stationary distribution, are then discussed (see section 4).

The second issue is related to the computation of the matrix R . Some algorithms have been developed for computing R (see Neuts [16], Latouche and Ramaswami [10], Ramaswami [21], and Ramaswami and Taylor [22]). Essentially, these algorithms generate a sequence of matrices which, usually nondecreasing, converges to R . That is, the resulting sequence converges to R from “below.” An interesting problem is to find an algorithm which generates a sequence converging to R from “above.” In this paper, some upper bounds of R and some schemes for computing R are presented. The proposed schemes generate sequences of matrices that have a Perron–Frobenius eigenvalue larger than that of R , and the sequence converges to R under certain conditions. The resulting sequence may not be decreasing, but its matrices are “above” the limiting matrix R in the sense of the Perron–Frobenius eigenvalue.

In this paper, the focus is on the matrix R . It is worthwhile to mention that the approach used in this paper can be used to study a dual problem of (1.1) raised from the study of Markov chains of M/G/1 type, i.e., the minimal nonnegative solution G to the equation

$$(1.2) \quad X = \sum_{n=0}^{\infty} A_n X^n.$$

The reason is that there is a one-on-one mapping between the minimal nonnegative solutions of (1.1) and (1.2) according to Asmussen and Ramaswami [2]. More studies of the matrix G can be found in Akar and Sohraby [1], Bini and Meini [4], Latouche and Stewart [11], and Lucantoni [13], where different algorithms for computing the matrix G are proposed and analyzed.

The rest of this paper is organized as follows. In section 2, some definitions and some classical results are presented. In section 3, a partial order defined on the set of substochastic and stochastic sequences is introduced. Section 4 focuses on the stochastically larger order and some applications to Markov chains of GI/M/1 type. Section 5 presents some inequalities of R and suggests a scheme for computing R . A few numerical examples are presented in section 5 as well. Section 6 discusses the moment generating order, functional monotonicity, and functional dominance. Section 7 is devoted to the GI/MAP/1 queue.

2. Preliminaries. In this section, a discrete Markov chain of GI/M/1 type, a simple algorithm for computing R , and some classical results on nonnegative matrices are introduced.

Markov chains of GI/M/1 type. A Markov chain P is of GI/M/1 type if it has a transition matrix

$$(2.1) \quad P = \begin{pmatrix} \bar{A}_0 & A_0 & & & \\ \bar{A}_1 & A_1 & A_0 & & \\ \bar{A}_2 & A_2 & A_1 & A_0 & \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where all the blocks in P are $m \times m$ matrices. The state space of P is $\{(n, j), n \geq 0, 1 \leq j \leq m\}$. The state set $\{(n, j), 1 \leq j \leq m\}$ is called the level n . One of the major features of the Markov chain P is that, for each transition, the first index n can increase at most by one. This feature determines a special structure of the stationary distribution of P . Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \dots)$ be the stationary distribution of P , where $\boldsymbol{\pi}_n$ is an m -dimensional vector for $n \geq 0$. When P is irreducible and positive recurrent, it has been found (see Neuts [16, Chapter 1]) that $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R^n$, $n \geq 0$, where the vector $\boldsymbol{\pi}_0$ is the unique nonnegative solution to the equations

$$(2.2) \quad \mathbf{x} = \mathbf{x} \sum_{n=0}^{\infty} R^n \bar{A}_n \quad \text{and} \quad \mathbf{x}(\mathbf{I} - R)^{-1} \mathbf{e} = 1,$$

where \mathbf{e} is the column vector with all components one, and \mathbf{I} is the unit matrix. The solution $\boldsymbol{\pi}$ is called the *matrix-geometric solution*. Such a solution exists for (irreducible and positive recurrent) Markov chains of GI/M/1 type with more complicated boundary statuses. In the context of the Markov chain P , R_{ij} ($1 \leq i, j \leq m$) is interpreted as the mean total number of visits to state $(n + 1, j)$ before returning to level n , given that the Markov chain started in state (n, i) . It is clear from the special structure of P that R_{ij} is the same for all positive n .

Continuous-time Markov chains of GI/M/1 type are defined similarly. The infinitesimal generator of a continuous-time Markov chain of GI/M/1 type has a similar structure to that of the matrix P in (2.1). It is worth mentioning that the results obtained in this paper also hold for Markov processes of GI/M/1 type with minor changes.

An algorithm for computing R . A simple algorithm for computing the matrix R is as follows. Let $R[0] = 0$ and

$$(2.3) \quad R[k + 1] = \sum_{n=0}^{\infty} (R[k])^n A_n, \quad k \geq 0.$$

It is easy to see that $\{R[k], k \geq 0\}$ is nondecreasing and converges to R from below, when R is unique. Although there are other algorithms proposed for computing R for some special cases (see Latouche and Ramaswami [10] and Ramaswami and Taylor [22]), this algorithm is easy to use when $\{A_n, n \geq 0\}$ are known. In this paper, (2.3) will be used repeatedly in proving properties about the matrix R .

Nonnegative matrices. In this paper, some results of nonnegative matrices are used repeatedly. For convenience, those results are summarized here (see Gantmacher [7]). Assume that X is an irreducible nonnegative matrix. Let $sp(X)$ denote the Perron–Frobenius eigenvalue of X . Then $sp(X)$ is positive and its corresponding eigenvector has positive elements as well. $sp(X)$ is strictly increasing with respect to each element of X . For any two constants c_1 and c_2 , if $\mathbf{u}c_1 \leq \mathbf{u}X \leq \mathbf{u}c_2$, where \mathbf{u} is a

nonzero nonnegative vector, then $c_1 \leq sp(X) \leq c_2$. For a stochastic or substochastic sequence $\{A_n, n \geq 0\}$, define

$$(2.4) \quad A^*(z) = \sum_{n=0}^{\infty} z^n A_n, \quad z \in [0, 1).$$

If A , the summation matrix of $\{A_n, n \geq 0\}$, is irreducible, $A^*(z)$ is irreducible for all $0 < z < 1$. Denote by $\theta(z)$ the eigenvector corresponding to $sp(A^*(z))$ with $\theta(z)\mathbf{e} = 1$. $\theta(z)$ is unique and positive. Also, $\log(sp(A^*(e^{-s})))$ is convex with respect to $s (> 0)$ (see Neuts [16, Chapter 1]). This last property implies that $sp(A^*(z))$, as a function of z , has at most one intersection with the linear function z in $[0, 1)$. This further implies that the eigenvector of $sp(R)$, denoted by $\theta(sp(R))$ with $\theta(sp(R))\mathbf{e} = 1$, is unique and positive.

3. A partial order and the matrix R . Partial ordering is an important tool in characterizing stochastic systems in applied probability (see Marshall and Olkin [14] as well as Shaked and Shanthikumar [24]). Various partial orders have been introduced and studied. For example, the majorization of vectors and matrices is discussed in Marshall and Olkin [14]. In Ridder [23], functional monotonicity and functional dominance are introduced with applications to matrix analytic methods.

Let M_m be the set of all sequences $\{A_n, n \geq 0\}$ of dimension m whose summation is an irreducible substochastic or stochastic matrix. In this section, a partial order is defined on M_m in order to study the monotonicity of the matrix R . The partial order is defined in such a way that the Perron–Frobenius eigenvalue of the matrix R can play an important role. Relationships between the matrix R and its corresponding eigenvalues of partially ordered sequences shall be derived.

Define a function ϕ (from $[0, 1)$ to $(0, \infty)$) as

$$(3.1) \quad \phi(x) = \sum_{n=0}^{\infty} \phi_n x^n,$$

where $\{\phi_n, n \geq 0\}$ are nonnegative and finite, $\phi_0 = 1$, and the summation converges for all x in $[0, 1)$.

DEFINITION 3.1. For $\{A_n, n \geq 0\}$ and $\{B_n, n \geq 0\}$ in M_m , if

$$(3.2) \quad \sum_{i=0}^n \phi_{n-i} A_i \geq \sum_{i=0}^n \phi_{n-i} B_i \quad \text{for all } n \geq 0,$$

then $\{A_n, n \geq 0\}$ is called smaller than $\{B_n, n \geq 0\}$ with respect to ϕ , denoted as $\{A_n, n \geq 0\} \leq_{\phi} \{B_n, n \geq 0\}$. It can be verified that (3.2) indeed defines a partial order on M_m with transitivity and reflective properties. The following result shows why the partial order \leq_{ϕ} is useful in the study of the matrix R .

LEMMA 3.1. Consider $\{A_n, n \geq 0\}$ and $\{B_n, n \geq 0\}$ in M_m satisfying $\{A_n, n \geq 0\} \leq_{\phi} \{B_n, n \geq 0\}$. Let R_a and R_b be the minimal nonnegative solutions to (1.1) corresponding to $\{A_n, n \geq 0\}$ and $\{B_n, n \geq 0\}$, respectively. (Indexes “ a ” and “ b ” shall be used to distinguish variables corresponding to the two sequences.) Assuming that $sp(R_a) \leq 1$, $\phi(R_a)$, and $\phi(R_b)$ are well defined, then

$$(3.3) \quad \phi(R_a)R_a^n \geq \phi(R_a)R_b^n \quad \text{and} \quad \phi(R_b)R_a^n \geq \phi(R_b)R_b^n, \quad \text{for } n \geq 1,$$

and $sp(R_a) \geq sp(R_b)$. Furthermore, denote by $\theta_a = \theta(sp(R_a))$ and $\theta_b = \theta(sp(R_b))$, for $n \geq 1$,

$$(3.4) \quad \theta_a R_a^n = (sp(R_a))^n \theta_a \geq \theta_a R_b^n \quad \text{and} \quad \theta_b R_a^n \geq \theta_b R_b^n = (sp(R_b))^n \theta_b.$$

Proof. To prove the first inequality in (3.3), the fact that the nondecreasing sequence generated by (2.3) converges to R_b of $\{B_n, n \geq 0\}$ shall be used. First, it has

$$(3.5) \quad \phi(R_a)R_a = \phi(R_a) \sum_{n=0}^{\infty} R_a^n A_n \geq \phi(R_a)A_0 \geq \phi(R_a)B_0 = \phi(R_a)R_b[1],$$

where $A_0 \geq B_0 = R_b[1]$ by definition (see (2.3) and (3.2)). The commutativity of $\phi(R_a)$ and R_a yields, for $n \geq 1$,

$$(3.6) \quad \phi(R_a)R_a^n = R_a^{n-1}\phi(R_a)R_a \geq R_a^{n-1}\phi(R_a)R_b[1] \geq \dots \geq \phi(R_a)(R_b[1])^n.$$

Suppose that (3.6) is true for k . For $k + 1$, by induction, it has

$$(3.7) \quad \begin{aligned} \phi(R_a)R_a &= \phi(R_a) \sum_{n=0}^{\infty} R_a^n A_n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} \phi_i R_a^i \right) R_a^n A_n = \sum_{n=0}^{\infty} R_a^n \left(\sum_{i=0}^n \phi_i A_{n-i} \right) \\ &\geq \sum_{n=0}^{\infty} R_a^n \left(\sum_{i=0}^n \phi_i B_{n-i} \right) = \sum_{n=0}^{\infty} \phi(R_a)R_a^n B_n \geq \sum_{n=0}^{\infty} \phi(R_a)(R_b[k])^n B_n \\ &= \phi(R_a) \sum_{n=0}^{\infty} (R_b[k])^n B_n = \phi(R_a)R_b[k+1]. \end{aligned}$$

The exchange of the summations in (3.7) is valid since the coefficients $\{\phi_n, n \geq 0\}$ and matrices involved are nonnegative. Similar to (3.6), it can be proved that $\phi(R_a)R_a^n \geq \phi(R_a)(R_b[k+1])^n$ for $n \geq 1$. Thus, $\phi(R_a)R_a^n \geq \phi(R_a)(R_b[k])^n$ holds for $n \geq 1$ and $k \geq 1$. Since $R_b[k]$ converges to R_b monotonically, the first inequality in (3.3) is obtained. Multiplying θ_a on both sides of the first inequality in (3.3) yields

$$(3.8) \quad \theta_a \phi(R_a)R_a^n = \phi(sp(R_a))\theta_a (sp(R_a))^n \geq \phi(sp(R_a))\theta_a R_b^n.$$

Since $\phi(x)$ is positive, $\theta_a sp(R_a) \geq \theta_a R_b$, which implies that $sp(R_a) \geq sp(R_b)$ and the first inequality of (3.4) holds.

To prove the second inequalities in (3.3) and (3.4), define the following sequence:

$$(3.9) \quad \underline{R}_a[1] = \sum_{n=0}^{\infty} R_b^n A_n \quad \text{and} \quad \underline{R}_a[k+1] = \sum_{n=0}^{\infty} (\underline{R}_a[k])^n A_n, \quad k \geq 1.$$

It can be proved that, for $n \geq 1$,

$$(3.10) \quad \phi(R_b)R_b^n \leq \phi(R_b)(\underline{R}_a[1])^n \quad \text{and} \quad \phi(R_b)R_b^n \leq \phi(R_b)(\underline{R}_a[k])^n, \quad k \geq 1.$$

Then it is only needed to prove that the sequence $\{\underline{R}_a[k], k \geq 1\}$ converges to R_a . By definition and induction,

$$(3.11) \quad \begin{aligned} \theta_a \underline{R}_a[1] &= \sum_{n=0}^{\infty} \theta_a R_b^n A_n \leq \theta_a \sum_{n=0}^{\infty} (sp(R_a))^n A_n = sp(R_a)\theta_a, \\ \theta_a (R_a[1])^n &\leq (sp(R_a))^n \theta_a, \\ \theta_a \underline{R}_a[k+1] &= \sum_{n=0}^{\infty} \theta_a (\underline{R}_a[k])^n A_n \leq \theta_a \sum_{n=0}^{\infty} (sp(R_a))^n A_n = sp(R_a)\theta_a, \\ \theta_a (\underline{R}_a[k+1])^n &\leq (sp(R_a))^n \theta_a, \quad n \geq 1. \end{aligned}$$

On the other hand, by (2.3) and (3.9), $\underline{R}_a[1] \geq A_0 = R_a[1]$. By induction, it can be proved that $\underline{R}_a[k] \geq R_a[k]$ for $k \geq 1$.

Since the sum of $\{A_n, n \geq 0\}$ is an irreducible matrix, θ_a is positive. By (3.11), the sequence $\{\underline{R}_a[k], k \geq 1\}$ is uniformly bounded. Denote by \underline{R}_a the limit matrix of a converging subsequence of $\{\underline{R}_a[k], k \geq 1\}$. Then $R_a \leq \underline{R}_a$ and $sp(R_a)\theta_a \leq \theta_a \underline{R}_a \leq sp(\underline{R}_a)\theta_a$. Thus, $sp(R_a)\theta_a = \theta_a \underline{R}_a$, which implies that $sp(\underline{R}_a) = sp(R_a)$. If $R_a < \underline{R}_a$, $\underline{R}_a - R_a$ is nonzero and nonnegative. Then $\theta_a(\underline{R}_a - R_a)$ is nonzero and nonnegative since θ_a is positive. This is a contradiction. Thus, $R_a = \underline{R}_a$. Since this is true for any converging subsequence of $\{\underline{R}_a[k], k \geq 1\}$, it concludes that the sequence $\{\underline{R}_a[k], k \geq 1\}$ converges to R_a . \square

There is a variety of selections of the function $\phi(x)$. For applications to Markov chains of GI/M/1 type and queueing theory, functions of the form $(1 - x)^{-k}$, $k \geq 0$, are of interest, especially when $k = 0$ and 1. When $k = 0$, $\phi(x) = 1$. $\{A_n, n \geq 0\} \leq_\phi \{B_n, n \geq 0\}$ implies that $A_n \geq B_n$ for all n . Lemma 3.1 implies that $R_a \geq R_b$. This result is useful, but the condition is too strong to be held by any two different stochastic sequences. Therefore, the focus of the next two sections will be on the case $\phi(x) = (1 - x)^{-1}$. The partial order \leq_ϕ with $\phi(x) = (1 - x)^{-1}$, in some sense, bears some analogy to the classical stochastically larger order (see Shaked and Shanthikumar [24]) and it is relatively easy to check whether or not two sequences in M_m are partially ordered.

4. Stochastically larger order and the matrix R . In this section, the focus is on the “stochastically smaller (larger)” order defined by $\phi(x) = (1 - x)^{-1}$, and denoted by \leq_{st} (\geq_{st}). This partial order is important not only because it has a scale case counterpart (see Shaked and Shanthikumar [24]) but also because it induces some useful results for the stationary distribution of Markov chains of GI/M/1 type. Furthermore, it suggests some schemes for computing R (see section 5).

For the stochastically smaller order, $\phi_n = 1$ for all n . If $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$, (3.2) becomes $A_0 + A_1 + \dots + A_n \geq B_0 + B_1 + \dots + B_n$ for all $n \geq 0$. When n goes to infinity, it leads to $A \geq B$, where A and B are the sums of $\{A_n, n \geq 0\}$ and $\{B_n, n \geq 0\}$, respectively. Lemma 3.1 leads to the following results of the matrix R .

THEOREM 4.1. *For $\{A_n, n \geq 0\}$ and $\{B_n, n \geq 0\}$ in M_m , assume that $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$. Then $sp(R_a)\theta_a \geq \theta_a R_b, \theta_b R_a \geq sp(R_b)\theta_b, sp(R_a) \geq sp(R_b)$, and*

(1) *if $sp(R_a) < 1$, then $(\mathbf{I} - R_a)^{-1}R_a \geq (\mathbf{I} - R_a)^{-1}R_b$ and $(\mathbf{I} - R_b)^{-1}R_a \geq (\mathbf{I} - R_b)^{-1}R_b$;*

(2) *if $sp(R_b) < 1$, then $(\mathbf{I} - R_b)^{-1}R_a \geq (\mathbf{I} - R_b)^{-1}R_b$.*

Proof. Consider $\{tA_n, n \geq 0\}$ and $\{tB_n, n \geq 0\}$ for $0 < t < 1$. Their minimal nonnegative solutions to (1.1) are $R_a(t)$ and $R_b(t)$, respectively. It is clear that $R_a(t), R_b(t), sp(R_b(t))$, and $sp(R_a(t))$ are nondecreasing in t and upper bounded by $R_a, R_b, sp(R_b)$, and $sp(R_a)$, respectively. It can be proved that $R_a(t), R_b(t), sp(R_b(t))$, and $sp(R_a(t))$ are continuous with respect to t when $sp(R_a) < 1, sp(R_b) < 1$, and $0 \leq t \leq 1$ (see He [8]). Matrices $(\mathbf{I} - R_a(t))^{-1}$ and $(\mathbf{I} - R_b(t))^{-1}$ are well defined since $sp(R_b(t)) \leq sp(R_a(t)) < 1$. Therefore, all conclusions in Lemma 3.1 hold for $t < 1$. Taking t to 1, the results are obtained. \square

Results given in Theorem 4.1 have many applications. Two examples are shown next. First, recall that $R_a(R_b)$ is the mean number (in matrix form) of visits of the Markov chain $P_a(P_b)$ (see (2.1) for definition) to level $k + 1$ before returning to level k , given that the process started in level k . Intuitively, when $\{A_n, n \geq 0\}$ is stochastically smaller than $\{B_n, n \geq 0\}$, Markov chain P_a is more likely to stay in

level $k + 1$ and higher. Therefore, the mean number of visits to level $k + 1$ and higher should be larger. This intuition leads to the following corollary.

COROLLARY 4.2. *Assume that $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$ and $sp(R_a) < 1$. Then $(\mathbf{I} - R_a)^{-1}R_a \geq (\mathbf{I} - R_b)^{-1}R_b$, where $(\mathbf{I} - R_a)^{-1}R_a$ (or $(\mathbf{I} - R_b)^{-1}R_b$) is the mean total number of visits to level $k + 1$ and higher before returning to level k , given that the Markov chain started in level k .*

Proof. By Theorem 4.1, $(\mathbf{I} - R_a)^{-1}R_a \geq (\mathbf{I} - R_a)^{-1}R_b$. Then, for all $n \geq 1$,

$$(4.1) \quad \begin{aligned} (\mathbf{I} - R_a)^{-1}R_b &= [\mathbf{I} + (\mathbf{I} - R_a)^{-1}R_a]R_b \geq R_b + (\mathbf{I} - R_a)^{-1}R_b^2 \\ &\geq R_b + R_b^2 + \dots + R_b^n + (\mathbf{I} - R_a)^{-1}R_b^{n+1}. \end{aligned}$$

Since $sp(R_b) \leq sp(R_a) < 1$, R_b^n converges to zero. This implies that $(\mathbf{I} - R_a)^{-1}R_b^n$ converges to zero, which leads to the result.

The probabilistic interpretation for $(\mathbf{I} - R_a)^{-1}R_a$ (and $(\mathbf{I} - R_b)^{-1}R_b$) comes from the fact that R^n is the mean number of visits to level $k + n$ before returning to level k , given that the Markov chain started in level k (see Neuts [16, Chapter 1]). \square

Note. When $sp(R_a) = 1$, the Markov chain P_a is transient. The mean number of visits to level $k + 1$ and higher before returning to level k is infinity, given that the Markov chain started in level k .

Corollary 4.2 shows that the mean number of visits to *all higher levels* before the Markov chain returns to its current level is monotone with respect to the stochastically larger order. It is natural to ask if it is also true for *a higher level*; i.e., when $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$, is it true that $R_a \geq R_b$? The answer to this question is negative. A counterexample is given as follows.

Example 4.1. Define $\{A_n, n \geq 0\}$ as follows: $A_n = 0, n \geq 3$,

$$(4.2) \quad A_0 = \begin{pmatrix} 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0.1 \\ 0 & 0 & 0.1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.1 & 0.05 & 0 \\ 0.1 & 0 & 0 \\ 0.1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.2 & 0.2 & 0.25 \\ 0 & 0.5 & 0.2 \\ 0.15 & 0.55 & 0.1 \end{pmatrix}.$$

$\{B_n, n \geq 0\}$ is the same as $\{A_n, n \geq 0\}$ except $(B_0)_{1,2} = 0$ and $(B_1)_{1,2} = 0.15$. Apparently, $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$. However, $R_a \geq R_b$ is not true since

$$(4.3) \quad R_a = \begin{pmatrix} 0.0356 & 0.1269 & 0.1099 \\ 0.1347 & 0.0352 & 0.1097 \\ 0.0156 & 0.0106 & 0.1029 \end{pmatrix} \quad \text{and} \quad R_b = \begin{pmatrix} 0.0155 & 0.0112 & 0.1025 \\ 0.1341 & 0.0399 & 0.1060 \\ 0.0155 & 0.0112 & 0.1025 \end{pmatrix}.$$

Another application of Theorem 4.1 is the comparison of tail probabilities of stationary distributions of Markov chains P_a and P_b of GI/M/1 type. When $sp(R_a) < 1$, for instance, there is an approximation (see Neuts [17])

$$(4.4) \quad R_a^n = (sp(R_a))^n \mathbf{v} \boldsymbol{\theta}_a + o((sp(R_a))^n) \quad \text{as } n \rightarrow \infty,$$

where \mathbf{v} and $\boldsymbol{\theta}_a$ are the right and left eigenvectors of R_a corresponding to $sp(R_a)$ and normalized by $\boldsymbol{\theta}_a \mathbf{e} = \boldsymbol{\theta}_a \mathbf{v} = 1$. Equation (4.4) implies that the stationary distribution $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \dots)$ of Markov chain P_a has the following approximation (see section 2 for the definition):

$$(4.5) \quad \boldsymbol{\pi}_n = (sp(R_a))^n (\boldsymbol{\pi}_0 \mathbf{v}) \boldsymbol{\theta}_a + o((sp(R_a))^n) \quad \text{as } n \rightarrow \infty.$$

Therefore, when $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$, $sp(R_a) \geq sp(R_b)$ and the tail probabilities of P_a are larger than that of P_b . Again, this implies that P_a is more likely to be in higher levels than P_b , although the probabilities of such events are small.

Note. For any partial order \leq_ϕ defined by (3.1) and (3.2), $sp(R_a) \geq sp(R_b)$ if $\{A_n, n \geq 0\} \leq_\phi \{B_n, n \geq 0\}$. This implies that the same conclusion for tail probabilities holds for the partial order ϕ . Cases of interest include $\phi(x) = (1 - x)^{-k}$ for $k \geq 0$. It is clear that the condition in (3.2) is weaker for large k than for small k , but they all imply the monotonicity of the Perron–Frobenius eigenvalue of the minimal nonnegative solution to (1.1).

5. Upper bounds of the matrix R and related issues. As was mentioned in section 1, upper bounds for the matrix R are useful, and it is even more interesting to develop algorithms generating sequences which converge to R from above. To address these issues, the stochastically larger order is explored further in this section. Although algorithms generating sequences which converge to R from above are not developed in this paper, some upper bounds are found and a scheme is developed. The scheme generates a sequence, whose matrices have Perron–Frobenius eigenvalues larger than $sp(R)$, converging to R under some conditions.

Consider $\{A_n, n \geq 0\}$ and $\{B_n, n \geq 0\}$ in \mathbf{M}_m . Define the following sequence for R_b :

$$(5.1) \quad \hat{R}_b[1] = \sum_{n=0}^{\infty} R_a^n B_n, \quad \hat{R}_b[k + 1] = \sum_{n=0}^{\infty} (\hat{R}_b[k])^n B_n \quad \text{for } k \geq 1.$$

When $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$, the sequence generated by using (5.1) is expected to converge to R_b monotonically. The sequence indeed converges to R_b under some conditions but, unfortunately, it is not always a monotone sequence.

PROPERTY 5.1. *Assume that $\{A_n, n \geq 0\} \leq_{st} \{B_n, n \geq 0\}$ and $sp(R_a) < 1$. Then*

$$(5.2) \quad \hat{R}_b[k] \geq R_b[k] \quad \text{and} \quad sp(R_b) \leq sp(\hat{R}_b[k]) \leq sp(R_a), \quad k \geq 1.$$

If the sequence generated by using (5.1) converges, it converges to R_b .

Proof. The first inequality in (5.2) is proved as follows:

$$(5.3) \quad \hat{R}_b[1] \geq B_0 = R_b[1], \text{ by induction, } \hat{R}_b[k] \geq R_b[k].$$

To prove the second part, the following inequalities are proved first, for $n, k \geq 1$,

$$(5.4) \quad (\mathbf{I} - R_b)^{-1} R_b^n \leq (\mathbf{I} - R_b)^{-1} (\hat{R}_b[k])^n \quad \text{and} \quad (\mathbf{I} - R_a)^{-1} (\hat{R}_b[k])^n \leq (\mathbf{I} - R_a)^{-1} R_a^n.$$

By Theorem 4.1, it has

$$(5.5) \quad (\mathbf{I} - R_b)^{-1} R_b = \sum_{n=0}^{\infty} (\mathbf{I} - R_b)^{-1} R_b^n B_n \leq \sum_{n=0}^{\infty} (\mathbf{I} - R_b)^{-1} R_a^n B_n \leq (\mathbf{I} - R_b)^{-1} \hat{R}_b[1],$$

$$\begin{aligned} (\mathbf{I} - R_b)^{-1} R_b^n &= R_b^{n-1} (\mathbf{I} - R_b)^{-1} R_b \\ &\leq R_b^{n-1} (\mathbf{I} - R_b)^{-1} \hat{R}_b[1] \leq \dots \leq (\mathbf{I} - R_b)^{-1} (\hat{R}_b[1])^n. \end{aligned}$$

By induction, the first part of (5.4) is proved. To prove the second part, first notice

$$\begin{aligned}
 (\mathbf{I} - R_a)^{-1}R_a &= \sum_{n=0}^{\infty} R_a^n \left(\sum_{i=0}^n A_i \right) \geq \sum_{n=0}^{\infty} R_a^n \left(\sum_{i=0}^n B_i \right) \\
 (5.6) \qquad &= (\mathbf{I} - R_a)^{-1} \sum_{n=0}^{\infty} R_a^n B_n = (\mathbf{I} - R_a)^{-1} \hat{R}_b[1], \\
 (\mathbf{I} - R_a)^{-1}R_a^n &\geq (\mathbf{I} - R_a)^{-1}(\hat{R}_b[1])^n, \quad n \geq 0.
 \end{aligned}$$

Using (5.1), the second part of (5.4) can be proved by induction. By (5.4), it is easy to see that the Perron–Frobenius eigenvalues of the matrices in the sequence generated by (5.1) are between $sp(R_a)$ and $sp(R_b)$. When the sequence generated by (5.1) converges, it converges to a nonnegative solution of (1.1) associated with the sequence $\{B_n, n \geq 0\}$. Since the minimal nonnegative solution of (1.1) with the largest eigenvalue less than one is unique (see Neuts [16, Theorem 1.3.3]), the sequence generated by (5.1) converges to the matrix R_b . \square

The above result shows that the Perron–Frobenius eigenvalue plays an important role in generating new sequences which converge to R from “above,” since the most important feature of the initial matrix for (5.1) is that its Perron–Frobenius eigenvalue is larger than that of the matrix R_b . Following this idea, another sequence can be constructed as follows, for $0 < s < 1$:

$$(5.7) \quad \hat{R}_b[1] = \sum_{n=0}^{\infty} s^n B_n = B^*(s), \quad \hat{R}_b[k+1] = \sum_{n=0}^{\infty} (\hat{R}_b[k])^n B_n, \quad k \geq 1.$$

PROPERTY 5.2. Assume that $1 > s > sp(R_b)$. For the sequence generated by (5.7),

$$(5.8) \quad \hat{R}_b[k] \geq R_b[k], \quad sp(R_b) \leq sp(\hat{R}_b[k]) \leq sp(B^*(s)) \leq s.$$

If the sequence converges, it converges to R_b .

Proof. By definition, it has

$$(5.9) \quad \hat{R}_b[1] = B^*(s) \geq R_b[1] = B_0 \quad \text{and} \quad \hat{R}_b[k] \geq R_b[k], \quad k \geq 1.$$

Recall that $\theta(s)$ is the eigenvector of $B^*(s)$ corresponding to its Perron–Frobenius eigenvalue and $s \geq sp(B^*(s))$. Then

$$\begin{aligned}
 (5.10) \quad \theta(s)\hat{R}[1] &= \theta(s)B^*(s) = \theta(s)sp(B^*(s)) \leq \theta(s)s, \\
 \theta(s)(\hat{R}[1])^n &\leq \theta(s)s^n, \quad n \geq 0.
 \end{aligned}$$

By induction, it can be proved that

$$\begin{aligned}
 (5.11) \quad \theta(s)(\hat{R}[k])^n &\leq \theta(s)(sp(B^*(s)))^n \leq \theta(s)s^n, \quad k, n \geq 1, \\
 sp(\hat{R}[k]) &\leq sp(B^*(s)) \leq s.
 \end{aligned}$$

Replacing $\theta(s)$ by θ_b and s by $sp(R_b)$ and changing the direction of inequalities in (5.10) and (5.11), the other inequality in (5.8) can be proved.

Using the same argument as in Property 5.1, when the sequence generated by (5.7) converges, it converges to R_b . \square

Properties 5.1 and 5.2 show that all the matrices in the sequence generated by (5.1) or (5.7) have their largest eigenvalues in $[sp(R_b), sp(R_a)]$ or $[sp(R_b), s]$ with

$sp(R_a) < 1$ and $s < 1$. This suggests that the generated sequence should converge to R_b so that the convergence condition in Properties 5.1 and 5.2 can be dropped out. This conjecture is supported by numerical examples as well. However, the generated sequence may not be monotone, and a rigorous proof of its convergence is difficult to obtain. In fact, such a proof may involve the theory of multiple dimension contraction mappings and is beyond the scope of this paper. This is why it is assumed in Properties 5.1 and 5.2 that the generated sequence converges. The issue of convergence is left as an open problem.

The following property shows that an extra condition guarantees the monotonicity and convergence of a generated sequence.

PROPERTY 5.3. *For a nonnegative matrix X with $0 < sp(X) < 1$, define $X[0] = X$, $X[k + 1] = B_0 + X[k]B_1 + \dots$ for $k \geq 0$. If $X[0] \geq X[1]$, then $X[k] \geq X[k + 1]$, $k \geq 0$, and $\{X[k], k \geq 0\}$ converges to R_b .*

Proof. If $X[0] \geq X[1]$, then $(X[0])^n \geq (X[1])^n$. Therefore, $X[1] \geq X[2]$ by definition. The rest of the proof is completed by induction and by using Theorem 1.3.3 in Neuts [16]. \square

Usually, to calculate the matrix R associated with a sequence $\{A_n, n \geq 0\}$ in M_m , a sequence $\{B_n, n \geq 0\}$ in M_m is not given to facilitate the computation of R . One has to construct new sequences in M_m from $\{A_n, n \geq 0\}$ for such purposes. Thus, let us focus on a single sequence $\{A_n, n \geq 0\}$ in M_m . The issues of interest are (1) to find some upper bounds of R_a and (2) to find some initial matrices or values for the algorithms defined by (5.1) and (5.7), respectively. The idea is to consider a truncated sequence $\{A_0, A_1, \dots, A_{N-1}, A_N + A_{N+1} + \dots\}$ for a given $N (> 0)$. Denote the minimal nonnegative solution to (1.1) corresponding to the truncated sequence by $R_{(N)}$. It is easy to verify that the truncated sequence is stochastically smaller than $\{A_n, n \geq 0\}$. The computation of $R_{(N)}$ could be much easier than that of R_a , especially when N is small. For instance, when $N = 2$, an efficient algorithm was proposed in Latouche and Ramaswami [10] for computing $R_{(2)}$. Also, $R_{(N)}$ is expected to be an upper bound of R_a . Unfortunately, this is not true in general. A simple counterexample is obtained when $sp(R_a) = 1$ (which implies that $sp(R_{(N)}) = 1$). In fact, numerical results show that no upper bound can be easily found for R_a . However, if $sp(R_{(N)}) < 1$ for a small $N (\geq 2)$, a sequence converging to R_a from “above” in terms of the Perron–Frobenius eigenvalue can usually be generated.

Upper bounds. Consider $N = 2$; a simple upper bound for R_a is suggested. Let $\theta_{(2)}$ be the eigenvector of $sp(R_{(2)})$, and $\theta A = \theta$ and $\theta e = 1$. By Theorem 4.1, $sp(R_{(2)})\theta_{(2)} \geq \theta_{(2)}R_a$. It is easy to prove that $\theta R_a \leq \theta$. Combining the two inequalities yields

$$(5.12) \quad (R_a)_{i,j} \leq \min \left\{ sp(R_{(2)}) \frac{(\theta_{(2)})_j}{(\theta_{(2)})_i}, \frac{(\theta)_j}{(\theta)_i} \right\}.$$

Notice that when $sp(R_{(2)}) = 1$, $\theta_{(2)} = \theta$. The upper bounds provided by (5.12) are useful for some state i where $(\theta)_i$ (or $(\theta_{(2)})_i$) is small compared with other components. Another immediate result is that $sp(R_{(2)}) \geq sp(R_a)$.

A computational scheme. Using the idea of truncation, first find the smallest N for which $sp(R_{(N)}) < 1$. This is equivalent to finding the minimal number N_{min} such that

$$(5.13) \quad \theta \left[\sum_{n=0}^{N-1} nA_n + N \left(\sum_{n=N}^{\infty} A_n \right) \right] e > 1.$$

This condition is given in Neuts [16, Chapter 1].

When N_{min} is small, R_{Nmin} can be computed efficiently using some existing algorithms. With R_{Nmin} , (5.1) can be used to generate a sequence which may converge to R from “above.” Although the generated sequence is not always monotone, numerical results show that it is a decreasing sequence in most of the cases. The sequence generated by using (5.1) can be compared with the nondecreasing sequence generated by using (2.3) to determine when the iteration process for R should be stopped.

It is interesting to know whether or not $R_{(N)}$ is an upper bound of R_a when $sp(R_{(N)}) < 1$ or $sp(R_a) < 1$. The following two examples show that $R_{(N)} \geq R_a$ is generally untrue.

Example 5.1. Define the following stochastic sequence:

$$(5.14) \quad \begin{aligned} A_0 &= \begin{pmatrix} 0 & 0.2 & 0 & 0 \\ 0.1 & 0 & 0.1 & 0 \\ 0 & 0 & 0.5 & 0.1 \\ 0.1 & 0 & 0.1 & 0.5 \end{pmatrix}, & A_1 &= \begin{pmatrix} 0.1 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}, & A_3 &= \begin{pmatrix} 0.2 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The matrix R for $\{A_0, A_1, A_2 + A_3\}$ is denoted as $R_{(2)}$, and $R_{(3)}$ for $\{A_0, A_1, A_2, A_3\}$. It is found that $sp(R_{(2)}) = 1 > sp(R_{(3)}) = 0.8988$, but $R_{(2)} > R_{(3)}$ is not true. This example shows that even when the transition matrix sequence is truncated to $N = 2$, the elements of the matrix R do not necessarily become larger. Therefore, $R_{(2)}$ may not be an upper bound of R_a in general.

Example 5.2. Define a stochastic sequence:

$$(5.15) \quad \begin{aligned} A_0 &= \begin{pmatrix} 0 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0.1 & 0 & 0.1 & 0 \end{pmatrix}, & A_1 &= \begin{pmatrix} 0.05 & 0 & 0.05 & 0 \\ 0 & 0.2 & 0.05 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 0 & 0.05 & 0.05 & 0 \\ 0 & 0.1 & 0.05 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{pmatrix}, & A_3 &= \begin{pmatrix} 0 & 0.2 & 0.3 & 0 \\ 0 & 0.1 & 0.2 & 0 \\ 0 & 0 & 0 & 0.4 \\ 0.1 & 0 & 0 & 0 \end{pmatrix}, \\ A_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \end{pmatrix}. \end{aligned}$$

The matrix R for $\{A_0, A_1, A_2, A_3 + A_4\}$ is denoted as $R_{(3)}$, and $R_{(4)}$ for $\{A_0, A_1, A_2, A_3, A_4\}$. $sp(R_{(3)}) = 0.258442 > sp(R_{(4)}) = 0.25555$, but $R_{(3)} > R_{(4)}$ is not true.

6. Discussions of other partial orders. In sections 4 and 5, the focus was on the stochastically larger order. It has been shown that the Perron–Frobenius eigenvalue of the minimal nonnegative solution to (1.1) is monotone with the stochastically larger order but the matrix R itself is generally not. It is interesting to find out (1) are there weaker conditions which guarantee the monotonicity of the Perron–Frobenius eigenvalue and (2) what conditions guarantee the monotonicity of R with respect to some partial order? A thorough discussion of these two issues is beyond the scope of this paper. Only two examples are shown below.

Example 6.1. The moment generating order.

DEFINITION 6.1. $\{A_n, n \geq 0\}$ is smaller than $\{B_n, n \geq 0\}$ with respect to the moment generating order if $A^*(z) \geq B^*(z)$ for $z \in [0, 1)$. Denote this order as $\{A_n, n \geq 0\} \leq_m \{B_n, n \geq 0\}$. The moment generating order is weaker than the “ \leq_ϕ ” order introduced in section 3 since $\phi(x)A^*(x) \geq \phi(x)B^*(x)$ implies $A^*(x) \geq B^*(x)$ for $x \in [0, 1)$ as $\phi(x) > 0$. If $\{A_n, n \geq 0\} \leq_m \{B_n, n \geq 0\}$,

$$(6.1) \quad \theta_a sp(R_a) = \theta_a R_a = \theta_a A^*(sp(R_a)) \geq \theta_a B^*(sp(R_a)).$$

This implies that $sp(R_a) \geq sp(B^*(sp(R_a)))$. By the convexity of function $\log(sp(B^*(e^{-s})))$, $sp(R_a) \geq sp(R_b)$. Therefore, the moment generating order implies the monotonicity of the Perron–Frobenius eigenvalue of the matrix R .

Example 6.2. Functional monotone and functional dominance.

Let J be the matrix with all diagonal and underdiagonal elements 1 and all others zero. For vectors \mathbf{u} and \mathbf{v} , if $\mathbf{u}J \leq \mathbf{v}J$, then \mathbf{u} is J -dominated by \mathbf{v} , denoted by $\mathbf{u} \leq_J \mathbf{v}$. A matrix X is J -dominated by Y is $XJ \leq YJ$, denoted by $X \leq_J Y$. For a matrix X with m row vectors $\{\mathbf{x}_i\}$, if $\mathbf{x}_1 \leq_J \mathbf{x}_2 \leq_J \dots \leq_J \mathbf{x}_m$, then X is J -monotone (see Ridder [23] for more details about monotone and dominance orders).

DEFINITION 6.2. A sequence $\{A_n, n \geq 0\}$ is J -dominated by $\{B_n, n \geq 0\}$ if $A_n \leq_J B_n$, for all n , denoted by $\{A_n, n \geq 0\} \leq_J \{B_n, n \geq 0\}$.

PROPERTY 6.1. For $\{A_n, n \geq 0\}$ and $\{B_n, n \geq 0\}$ in M_m , if A_n and B_n and J -monotone for all $n \geq 0$, then R_a and R_b are J -monotone. Furthermore, if $\{A_n, n \geq 0\} \leq_J \{B_n, n \geq 0\}$, then $R_a \leq_J R_b$.

Proof. In Ridder [23], it has been proved that the J -monotone and J -dominance are closed under matrix multiplication and summation. The results are true for the sequences generated by using (2.3). Since R_a and R_b are limits of those sequences, the results follow. \square

Note. The condition for J -domination is very strong since it imposes J -monotone on every pair of matrices $\{A_n, B_n\}$ of the two sequences. Nonetheless, J -domination is a partial order which guarantees the monotonicity of the matrix R . In addition, Property 6.1 finds a nice application in the GI/MAP/1 queue which shall be shown in the next section.

7. Applications to the GI/MAP/1 queue. This section considers a single server queueing system with a Markov arrival process (MAP) as its service process, general independent interarrival times, and a “first-come-first-served” service discipline. The MAP was first introduced in Neuts [15] (also see Lucantoni, Meier-Hellstern, and Neuts [12], and Neuts [18]) as a generalization of the phase-type renewal process. The MAP is defined on a finite irreducible Markov process (called the underlying Markov process) with m states and an irreducible infinitesimal generator D . In the MAP, the sojourn time in state i is exponentially distributed with parameter D_{ii} . At the end of the sojourn time in state i , a transition occurs to state j , $1 \leq j \leq m$, where the transition may or may not represent an arrival. Let D_0 be the (matrix) rate

of transitions without an arrival and D_1 be the rate of transitions with an arrival. D_0 and D_1 are $m \times m$ matrices where D_0 has negative diagonal elements and nonnegative off-diagonal elements, and D_1 is a nonnegative matrix. Then $D = D_0 + D_1$. Using θ to denote the stationary probability vector of the Markov process with the generator D , θ satisfies $\theta D = 0$ and $\theta e = 1$. The stationary service rate of the Markov arrival process is then $\lambda = \theta D_1 e$. The interarrival time between two consecutive customers is random with finite mean and variance. Let $F(t)$ be the distribution function of the interarrival time and denote the Laplace–Stieltjes transform (LST) of $F(t)$ by $f^*(s)$.

Consider the embedded Markov chain (q_n, J_n) at the n th arrival epoch, where q_n is the queue length just before the arrival and J_n is the phase of the service process at the arrival epoch. This embedded Markov chain (q_n, J_n) is of GI/M/1 type, since the increase of the queue length is at most one at a time. The one-step transition matrix of (q_n, J_n) is similar to P given in (2.1). Transition blocks $\{A_n, n \geq 0\}$ are defined in terms of D_0 , D_1 , and $F(t)$. The matrix R is then defined as the minimal nonnegative solution to (1.1) (see Neuts [19]). It can be proved that the matrix R is also the minimal nonnegative solution to the following exponential-form equation:

$$(7.1) \quad X = \int_0^\infty \exp\{t(D_0 + XD_1)\}dF(t).$$

Next, the results obtained in previous sections are applied to obtain some interesting results about the matrix R and the GI/MAP/1 queue. For convenience, the following analysis of R begins with (7.1) instead of (1.1). The discussion consists of two parts. The first part includes the results obtained by assuming the stochastically larger order on the interarrival times, and the second part consists of results obtained by imposing a special structure on the matrix representation (D_0, D_1) of the service process.

First, define a new matrix Q from R . Let $\xi > \max\{1, |(D_0)_{i,j}|\}$. By (7.1), it has

$$(7.2) \quad \begin{aligned} R &= \int_0^\infty \exp\{t(-\xi \mathbf{I} + \xi \mathbf{I} + D_0 + RD_1)\}dF(t) \\ &= \sum_{n=0}^\infty \int_0^\infty \frac{e^{-\xi t}(\xi t)^n}{n!}dF(t) \left[\mathbf{I} + \frac{D_0 + RD_1}{\xi} \right]^n \equiv a^*(Q), \end{aligned}$$

where $Q = \mathbf{I} + (D_0 + RD_1)/\xi$ and

$$(7.3) \quad a^*(z) = \sum_{n=0}^\infty z^n a_n, \quad \text{where } a_n = \int_0^\infty \frac{e^{-\xi t}(\xi t)^n}{n!}dF(t) \quad \text{and } \sum_{n=0}^\infty a_n = 1.$$

For the matrix Q , it has

$$(7.4) \quad Q = K_0 + a^*(Q)K_1, \quad \text{where } K_0 = \mathbf{I} + \frac{D_0}{\xi} \quad \text{and } K_1 = \frac{D_1}{\xi}.$$

Thus, the matrix Q is the minimal nonnegative solution to (7.4), or, equivalently, Q is the minimal nonnegative solution to (1.1) with a sequence $\{K_0 + a_0K_1, a_1K_1, a_2K_1, \dots\}$, which has a special structure in the sense that only two matrices K_0 and K_1 are involved. Let $K = K_0 + K_1$. K is an irreducible stochastic matrix. It is sometimes more convenient to deal with (7.4) than (7.1). The following relationship between R and Q ensures the equivalence of the studies of (7.1) and (7.4).

LEMMA 7.1. *The minimal nonnegative solutions to (7.1) and (7.4) are R and Q , respectively. Let $\eta(s)$ be the largest eigenvalue of $K(s) = K_0 + sK_1$. Then $\eta(s)$ is increasing in s and*

$$(7.5) \quad sp(R) = \int_0^\infty \exp\{-tsp(Q)\}dF(t) = f^*(sp(Q)) \quad \text{and} \quad sp(Q) = \eta(sp(R)).$$

Proof. The one-on-one relationship between R and Q is clear from (7.1), (7.2), and (7.3). The results of the Perron–Frobenius eigenvalues are then obtained easily from (7.2) and (7.4). $sp(R) < 1$ if and only if $sp(Q) < 1$ since $f^*(s)$ and $\eta(s)$ are strictly increasing. \square

Now, the focus shifts from R and (7.1) to Q , (7.4), $a^*(s)$, and $K(s)$. The problem becomes simpler since only a function $a^*(s)$ and a linear matrix function $K(s)$ are involved. It is now easy to apply results obtained in sections 4, 5, and 6 to (7.4).

THEOREM 7.2. *Consider two GI/MAP/1 queueing systems, labeled “a” and “b,” respectively, with the same service process. If the interarrival times satisfy $F_a(x) \leq_{st} F_b(x)$ (the usual stochastic order, i.e., $F_a(x) \geq F_b(x)$ for all $x \geq 0$; see Shaked and Shanthikumar [24]), then*

$$(7.6) \quad sp(R_a) \geq sp(R_b) \quad \text{and} \quad -(D_0 + R_a D_1)^{-1} \geq -(D_0 + R_b D_1)^{-1}.$$

Proof. If $F_a(x) \leq_{st} F_b(x)$, then $\{a_n\} \leq_{st} \{b_n\}$. By Definition 3.1, $\{K_0 + a_0 K_1, a_1 K_1, \dots\} \leq_{st} \{K_0 + b_0 K_1, b_1 K_1, \dots\}$. By Theorem 4.1, $sp(Q_a) \geq sp(Q_b)$. Since $\eta(s)$ is increasing, $sp(R_a) \geq sp(R_b)$. By Corollary 4.2, $(\mathbf{I} - Q_a)^{-1} \geq (\mathbf{I} - Q_b)^{-1}$, which implies (7.6). \square

Intuitively, Theorem 7.2 shows that systems with stochastically smaller interarrival times (shorter interarrival times), are more likely to have a long queue since $sp(R_a) \geq sp(R_b)$ (recall the discussion in section 4). The matrix $D_0 + R_a D_1$ is the infinitesimal generator of the waiting time process in the queueing system (see Asmussen and Perry [3]). Theorem 7.2 shows that the mean waiting time is longer for the system with a stochastically smaller interarrival time.

The following theorem shows the monotonicity of the matrix R under the J -monotone order. It further implies the monotonicity of the corresponding stationary distribution of the queue length in the queueing system of interest.

THEOREM 7.3. *Consider two GI/MAP/1 queues, labeled “a” and “b,” respectively, with the same interarrival times. Their service processes have matrix representations (D_0, D_1) and (C_0, C_1) , respectively. If D_0, D_1, C_0 , and C_1 are J -monotone, $D_0 \leq_J C_0$ and $D_1 \leq_J C_1$, then R_a and R_b are J -monotone, and $R_a \leq_J R_b$.*

Proof. The theorem is proved by Property 6.1. \square

Finally, a scheme for computing Q and R is proposed. The idea is to find an upper bound for $sp(Q)$ so that a sequence whose matrices have a Perron–Frobenius eigenvalue larger than $sp(Q)$ can be generated (see (5.7) and (7.4)). Another non-decreasing sequence can be generated by using (2.3). Compare the two sequences to determine when the iteration process for Q should be stopped. An upper bound for $sp(Q)$ can be found as follows: Let $t_0 = 0.5$.

- (i) $s_n = a^*(t_n) = f^*(\xi(1 - t_n))$.
- (ii) If $sp(K(s_n)) \geq t_n$, STOP; if $sp(K(s_n)) < t_n$, go to (iii).
- (iii) $t_{n+1} = (1 + t_n)/2$, go to (i).

Then t_n gives an upper bound of $sp(Q)$. The matrix R can be obtained accordingly.

This scheme is feasible when $f^*(t)$ can be evaluated numerically. The Perron–Frobenius eigenvalue of the matrix $K(s)$ can be found without much difficulty since

it is a linear function of s . This scheme might be useful in improving the accuracy for computing Q and R .

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