MANUFACTURING TO ORDER WITH RANDOM YIELD AND COSTLY INSPECTION

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This study considers a situation where a contractor receives an order that it commits to satisfy in full. The fulfillment of the contract requires manufacturing and inspection. Because the number of defective units within a produced lot is not known in advance, it is possible that after examining the lot, it is learned that the number of conforming units is short of the demand. If so, further manufacturing and inspection are required. Once enough conforming units are found, the inspection terminates, and the remaining uninspected units, as well as all defectives, are scrapped.

Whereas previous "multiple production runs" studies implicitly assumed that inspection costs are negligible, we include these costs as a key part of the problem. It turns out that the optimal production lot size depends on the inspection cost. Our model is very general: We provide a framework to calculate the optimal batch and the expected number of inspections for any yield pattern, as well as for any inspection procedure. We also provide results and numerical examples concerning specific yield patterns that are common in practice.

INTRODUCTION

As systems become increasingly complex, consisting of an enormous number of components, the rule of "zero defectives" becomes an absolute necessity because otherwise large and expensive systems may become idle as a result of the failure of relatively simple components. This reality leads to thorough and expensive inspection procedures, which at times may actually be more costly than production itself. The necessity of elaborate inspection is accentuated in environments with orders for a small number of custom-made products.

The need for inspection is a result of the presence of randomness in the production yields. Such models may be dichotomized, roughly, into two classes: (1) models allowing shortages, with a penalty for goods undelivered on time, and (2) models requiring the manufacturer to satisfy the demand in full (see Yano and Lee 1995 for a taxonomy). When demand is "rigid," as in (2), several production runs may need to be attempted until the number of usable units is sufficient. This leads to a problem referred to as "multiple lotsizing production to order" (MLPO; see Grosfeld-Nir and Gerchak 1996 and references therein).

Whereas previous MLPO models did not consider inspection issues/costs, we focus on the implications of such costs for optimal lot sizing. In our model, inspection may be terminated before all units produced are inspected if the fixed-size order has been satisfied. That differs from standard inspection settings (e.g., Grant and Leavenworth 1988, Montgomery 1985). Also, we allow the manufacturer to control, effectively, the size of lot that enters inspection, not as a parameter but as a decision variable. Thus lot sizing and inspection are optimized simultaneously in a general yield setting.

The MLPO with inspection problem is thus as follows. A contractor commits to satisfy an order in full. Its problem is to minimize the total of manufacturing and inspection cost. Its decisions include two interrelated parts: the determination of number of units (the lot) to be manufactured, and the determination of the inspection procedure. Because the number of good units in the initial batch may be smaller than the outstanding demand, several production runs may be necessary. The inspection of a batch terminates once the demand is satisfied, and the remaining noninspected units as well as all defective ones are worthless and must be scrapped.

Concerning the yield pattern, it is worthwhile to start with the examination of two extremes. One is binomial yield, which assumes that the quality of units within a batch is independent of each other and of the unit's place in the lot. That is the most common random yield model (Yano and Lee 1995, Grosfeld-Nir and Gerchak 1996). The second pattern is interrupted geometric (IG) yield, which assumes that once some unit is defective, all subsequent units are also defective, and the probability that a unit following a good one will be defective is fixed throughout the lot. IG yield

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is realistic when production is in runs, and if the process becomes "out of control," all units produced after that are defective (Porteus 1986, 1990). In addition to binomial and IG yields we examine the discrete uniform (DU) yield. The DU illustrates situations of "common cause" yield, often found in operations where thermal or chemical treatment is given to an entire "batch." When a batch is ready for inspection, the units are examined, one at a time, until the demand is satisfied or all units are exhausted. It turns out that different inspection procedures are optimal for different yield patterns. For binomial yield the order of inspection is immaterial. This is also true for any common cause yield, such as DU. On the other hand, with IG yield, once a defective unit is detected, it is clear that only units produced earlier are candidates for examination.

THE MODEL

A contractor commits to satisfy an order for D units in its entirety. The cost associated with the manufacturing of a lot of size N is $\alpha + \beta N$; the parameters α and β are referred to as "setup cost" (fixed ordering cost) and "variable cost" (per-unit processing cost), respectively. Inspection can commence only when the lot is complete. *After a batch is completed, units are inspected one at a time until the demand is satisfied or until the status of all units is ascertained (physically or inferentially).* Inspection cost is γ per unit. If inspection reveals that the number of usable units within a batch is short of the outstanding demand, the problem repeats itself with the demand remaining. Defective units, or noninspected units remaining after the demand has been satisfied, have no value and are scrapped. The objective is to determine, for each D, the optimal lot N_D that minimizes the expected sum of setup, variable, and inspection costs.

We denote by $Y_N, Y_N \leq N$, the (random) yield associated with the processing of a lot N, and by $p(y,N) \equiv \mathbf{P}\{Y_N = y\}$, the corresponding PMF. Thus, $p(y,N)$ is the probability that there are y usable units among the N units that have been processed. Although our formulation will be general (allowing any yield pattern), we shall also consider in detail the following specific yield patterns:

Binomial: $p(y,N) = \binom{N}{y} \theta^y (1-\theta)^{N-y}, \quad 0 \leq y \leq N;$

Discrete Uniform (DU): $p(y,N) = \dfrac{1}{N+1}, \quad 0 \leq y \leq N;$

Interrupted Geometric (IG):

$$p(y,N) = \begin{cases} \theta^y(1-\theta), & \text{if } 0 \leq y < N; \\ \theta^N, & \text{if } y = N. \end{cases}$$

We refer to a sequence $N_1, N_2, \ldots$, as a "production policy." A production policy specifies, for each D, the lot N_D to be processed. We define an "inspection policy" as a rule that specifies, for each D, the order in which the N units at hand should be inspected. For example, if $D = 3$ and $N = 10$, the inspection rule may be: "Inspect first the 6th unit; if it is good, inspect the 8th: otherwise, inspect the 3rd unit, etc.; stop when three good units were found." Another example of an inspection policy is "choose units at random." Note that for binomial and DU yields "random inspection" is intuitively optimal, whereas that is not the case for IG yield. We wish to emphasize that the formulation below fits any production and inspection policies, and thus in particular it fits the optimal rules.

We denote by $I_D(y,N), 0 \leq y \leq N$, the expected number of inspections if demand is D and there are y good units among the N waiting for inspection. We define $I_D(N)$ to be the expected number of inspections corresponding to a lot N when the demand is D. That is,

$$I_D(N) = \sum_{y=0}^{N} p(y,N) I_D(y,N). \tag{1}$$

EXAMPLE 1. Suppose that $D = 1$ and $N = 2$, and that yield is either binomial or DU. Then, units to be inspected may be selected at random. Thus

$I_1(0,2) = 2$; $I_1(1,2) = 0.5 \times 1 + 0.5 \times 2 = 1.5$;

$I_1(2,2) = 1$.

Now, if yield is binomial (θ), then

$$I_1(2) = 2p(0,2) + 1.5p(1,2) + p(2,2)$$
$$= 2 \times (1-\theta)^2 + 1.5 \times 2\theta(1-\theta) + \theta^2 = 2 - \theta.$$

If, on the other hand, yield is DU, then

$I_1(2) = 2 \times (1/3) + 1.5 \times (1/3) + 1 \times (1/3) = 1.5.$

We denote by T_D the expected total number of inspections, until the demand D is satisfied, if some prespecified production and inspection policies are being followed throughout. We write $T_D(N)$ instead of T_D to mark explicitly that the initial lot size equals N.

EXAMPLE 2. Continuation of Example 1. First note that

$$T_1(2) = p(0,2)[I_1(0,2) + T_1(2)] + p(1,2)I_1(0,2)$$
$$+ p(2,2)I_1(2,2)$$
$$= I_1(2)/[1 - p(0,2)].$$

Therefore, for binomial yield: $T_1(2) = (2-\theta)/[1-(1-\theta)^2] = 1/\theta$; and, for DU: $T_1(2) = 1.5/(2/3) = 2.25$.

We now develop a useful recursion for computing $T_D(N)$. For any inspection policy

$$T_D(N) = p(0,N)[I_D(0,N) + T_D(N)]$$
$$+ \sum_{y=1}^{D-1} p(y,N)[I_D(y,N) + T_{D-y}]$$
$$+ \sum_{y=D}^{N} p(y,N) I_D(y,N).$$

Combined with Equation (1), that leads to

$$T_D(N)=\frac{I_D(N)+\sum_{y=1}^{D-1}p(y,N)T_{D-y}}{1-p(0,N)}. \quad (2)$$

Comment. In Equation (2) and other following equations, the summation is stated to be "from 1 to $D-1$." It should be noted that if $N<D$, the summation is actually "from 1 to N." Mathematically, what we write is precise, because $p(y,N)=0$ for $y>N$. Some authors write such equations in two parts, one for $N<D$ and the other for $N\geq D$. For simplicity, we prefer one expression, but caution is advised.

It is easy to see that $T_D(N)\geq D$ and $T_D\geq D$ for any yield structure. But unless some assumptions on yield type are made (see below), $T_D(N)$, as a function of N, may not be well behaved.

Next we formulate the MLPO with inspection problem. We define U_D to be the expected costs incurred in satisfying a demand D, when prespecified production and inspection policies are followed throughout. We write $U_D(N)$ instead of U_D to mark explicitly that $N_D=N$ (N represents the current lot). The following holds for any production and inspection policies:

$$U_D(N)=\alpha+\beta N+p(0,N)[\gamma I_D(0,N)+U_D(N)]$$
$$+\sum_{y=1}^{D-1}p(y,N)[\gamma I_D(y,N)+U_{D-y}]$$
$$+\sum_{y=D}^{N}\gamma I_D(y,N).$$

This leads to

$$U_D(N)=\frac{\alpha+\beta N+\gamma I_D(N)+\sum_{y=1}^{D-1}p(y,N)U_{D-y}}{1-p(0,N)}. \quad (3)$$

Note that, using the last equation, the optimal policy can be calculated recursively in D. However, it is possible to rewrite (3) in a way that makes it simpler and more meaningful.

We denote by V_D and $V_D(N)$ the values corresponding to U_D and $U_D(N)$, respectively, if $\gamma=0$. Then (3) becomes

$$V_D(N)=\frac{\alpha+\beta N+\sum_{y=1}^{D-1}p(y,N)V_{D-y}}{1-p(0,N)}. \quad (4)$$

Note that V_D represents the production costs alone, thus corresponding to the basic MLPO problem (see Grosfeld-Nir and Gerchak 1996). We now show that $U_D(N)$, the expected costs with inspection, can be decomposed into two parts: $V_D(N)$, the expected production costs; and $\gamma T_D(N)$, the expected inspection costs. This nice but nonobvious result splits the total costs between production and inspection, which can be also useful for cost accounting.

THEOREM 1. *For any production and inspection policies,*

$$U_D(N)=V_D(N)+\gamma T_D(N). \quad (5)$$

PROOF. We shall prove the result assuming $N\geq D$. The proof for $N<D$, $D\geq 2$, is similar. We shall use induction in D. For $D=1$, using (3),

$$U_1(N)=\frac{\alpha+\beta N+\gamma I_1(N)}{1-p(0,N)}.$$

Note that from (4), $V_1(N)=(\alpha+\beta N)/[1-p(0,N)]$, and that from (2), $T_1(N)=I_1(N)/[1-p(0,N)]$, so

$$U_1(N)=V_1(N)+\gamma T_1(N).$$

For $D+1$ we must prove that $U_{D+1}(N)=V_{D+1}(N)+\gamma T_{D+1}(N)$. From (3) we have

$$U_{D+1}(N)=\frac{\alpha+\beta N+\gamma I_{D+1}(N)+\sum_{y=1}^{D}p(y,N)U_{D+1-y}}{1-p(0,N)}.$$

Now, incorporating the induction hypothesis (5), we have

$$U_{D+1}(N)$$
$$=\frac{\alpha+\beta N+\gamma I_{D+1}(N)+\sum_{y=1}^{D}p(y,N)[V_{D+1-y}+\gamma T_{D+1-y}]}{1-p(0,N)}.$$

Note that

$$V_{D+1}(N)=\frac{\alpha+\beta N+\sum_{y=1}^{D}p(y,N)V_{D+1-y}}{1-p(0,N)},$$

so, using (3),

$$U_{D+1}(N)=V_{D+1}(N)+\gamma\frac{I_{D+1}(N)+\sum_{y=1}^{D}p(y,N)T_{D+1-y}}{1-p(0,N)}$$
$$=V_{D+1}(N)+\gamma T_{D+1}(N). \quad \square$$

Because for arbitrary yields the function $V_D(N)$ may itself not be quasi-convex (Grosfeld-Nir and Gerchak 1996), the optimization of $U_D(N)$ cannot in general be expected to be "elegant," even if the addition of $T_D(N)$ would not make things more complex. But when $\beta>0$ (the only interesting case), $U_D(\infty)=\infty$, and therefore the optimal lot size N_D will always be finite. So all we need is to search over a finite range of integers. In fact, it can be shown that

$$N_D\leq\min\left\{N:\alpha+\beta N+\gamma>\max_{i\leq i<N}U_D(i)\right\}.$$

This is the case because the luckiest situation occurs when the initial run contains enough good units, resulting in total costs of at least $\alpha+\beta N+\gamma$ (at least one unit needs to be inspected; in the IG model, inspecting one unit may be enough—if we are lucky), and thus the optimal costs will be at least that high. If that is not satisfied, N is too high. So because the $U_D(i)$ are successively computed, the above can be checked and search terminated accordingly.

Most important, for specific yields of interest much more can be said about the objective function and solution, as we see below. In these special cases, the addition of inspection costs (to the production costs) does not turn out to complicate the search for optimal lot size by much.

INSPECTION IN RANDOM ORDER

When yield is binomial or DU, selecting the next unit to be inspected at random is optimal. Actually, random inspection is also optimal for any common cause yield pattern, as all units are equally likely to be defective. Note that the situation is completely different when yield is IG. Then, if a certain unit is defective, all subsequent units are defective.

Next, we develop useful formulas for the calculation of $I_D(N)$. First note that if $y<D$, there will be exactly N inspections. However, if $y \geqslant D$, between D and $N-y+D$ inspection will be needed. Also, if $y \geqslant D$, the event "exactly x inspections are performed" is identical to the event "$D-1$ good units were found in the first $x-1$ inspections, and, the next (last) inspection detected a good unit." The probability, in $x-1$ trials, to detect $D-1$ good units (and $x-D$ bad ones), out of y good units (and $N-y$ bad ones), is hypergeometric. It should be multiplied by the probability of a success in the last trial, given there were $D-1$ successes:

$$I_D(y,N) = \begin{cases} N, & \text{if } 0 \leqslant y < D, \\ \sum_{x=D}^{D+N-y} x \dfrac{\binom{y}{D-1}\binom{N-y}{x-D}}{\binom{N}{x-1}} \dfrac{y-D+1}{N-x+1}, & \text{if } D \leqslant y \leqslant N. \end{cases} \tag{6}$$

Next, we show that there is a surprisingly simple way to compute $I_D(y,N)$, $D \leqslant y \leqslant N$. Note that $I_D(N,N)=D$, and for $D \leqslant y < N$, one can write the following recursion:

$$I_D(y,N) = 1 + \frac{y}{N} I_{D-1}(y-1, N-1) + \left(1 - \frac{y}{N}\right) I_D(y, N-1). \tag{7}$$

In particular,

$$I_1(y,N) = 1 + \left(1 - \frac{y}{N}\right) I_1(y, N-1), \quad 1 \leqslant y < N. \tag{7'}$$

This enables us to prove the following theorem.

THEOREM 2.

$$I_D(y,N) = \frac{N+1}{y+1} D, \quad D \leqslant y \leqslant N. \tag{8}$$

PROOF. We will use induction over D, and induction over N within it. Consider $D=1$. We must prove that

$$I_1(y,N) = \frac{N+1}{y+1}, \quad 1 \leqslant y \leqslant N. \tag{9}$$

Clearly (9) holds for $N=1$, because then $y=1$ and $I_D(N,N)=D$, $D \leqslant N$. Suppose that (9) holds for N; then, using (7′)

$$I_1(y, N+1) = 1 + \left(1 - \frac{y}{N+1}\right) \frac{N+1}{y+1} = \frac{N+2}{y+1}.$$

Therefore, (9) holds for $D=1$, $1 \leqslant y \leqslant N$.

Suppose that (8) holds for D. For $(D+1)$ we must prove that

$$I_{D+1}(y,N) = \frac{N+1}{y+1}(D+1), \quad D+1 \leqslant y \leqslant N. \tag{10}$$

Clearly, (10) holds for $N=D+1$ because then $y=D+1$ and $I_{D+1}(D+1, D+1) = D+1$. Suppose that (10) holds for N; then, for $N+1$ we have (using (7))

$$\begin{aligned} I_{D+1}(y, N+1) &= 1 + \frac{y}{N+1} I_D(y-1, N) + \left(1 - \frac{y}{N+1}\right) I_{D+1}(y,N) \\ &= 1 + \frac{y}{N+1} \frac{N+1}{y} D + \left(1 - \frac{y}{N+1}\right) \frac{N+1}{y+1}(D+1) \\ &= \frac{N+2}{y+1}(D+1). \end{aligned}$$

COROLLARY 1. *With random inspection $I_D(N)$ becomes*

$$I_D(N) = \sum_{y=0}^{N} p(y,N) I_D(y,N) = N \sum_{y=0}^{D-1} p(y,N) + (N+1) D \sum_{y=D}^{N} \frac{p(y,N)}{y+1}. \tag{11}$$

SPECIAL YIELD STRUCTURES

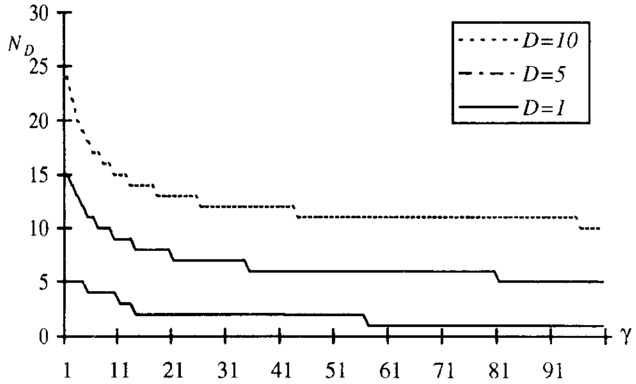
We now demonstrate how the methodology just developed can be used to explore the implications of specific yields of interest. The results are interesting in their own right. We start with DU and binomial yields, for which random inspection is optimal. We then explore the case of IG yield, in which we use an inspection policy proved optimal in previous studies.

Discrete Uniform Yield

Because for DU yield random inspection is optimal, we have, using (11), $I_D(N)=N$, $N<D$; and for $N \geqslant D$,

$$\begin{aligned} I_D(N) &= N \sum_{y=0}^{D-1} p(y,N) + (N+1) D \sum_{y=D}^{N} \frac{p(y,N)}{y+1} \\ &= \frac{ND}{N+1} + D\left(\frac{1}{D+1} + \cdots + \frac{1}{N+1}\right). \end{aligned}$$

Figure 1. N_D as a function of γ for some value of D in Example 3.



Combining this with (2), we obtain for $N < D$

$$T_D(N) = \frac{N + \sum_{y=1}^{N} T_{D-y}/(N+1)}{N/(N+1)} = N + 1 + \frac{1}{N}\sum_{y=1}^{N} T_{D-y},$$

and for $N \geqslant D$,

$$T_D(N) = \frac{I_D(N) + \sum_{y=1}^{D-1} T_{D-y}/(N+1)}{N/(N+1)} = D\left(1 + \left(1 + \frac{1}{N}\right)\sum_{y=D}^{N}\frac{1}{y+1}\right) + \frac{1}{N}\sum_{y=1}^{D-1} T_{D-y}.$$

Function $T_D(N)$ is characterized by the following theorem.

THEOREM 3. *For DU yield, the function $T_D(N)$ is quasi-convex in N; i.e., it is nonincreasing for $N \leqslant N^*$ and increasing for $N \geqslant N^*$ for some $N^* > D$. Asymptotically, $T_D(N)$ is proportional to $D \log N$.*

PROOF. See the appendix.

The following example is used to gain insight into the pattern of N_D and U_D as the inspection cost γ changes.

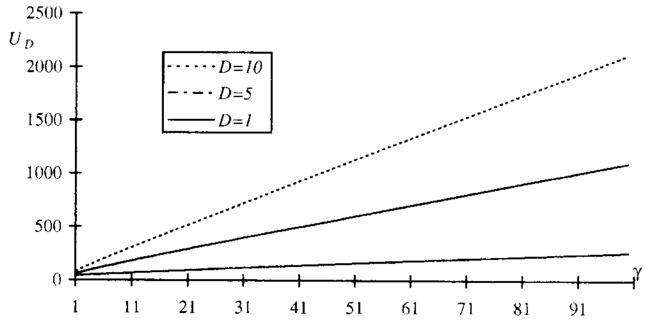
EXAMPLE 3. Let $\alpha = 30$ and $\beta = 1$. The yield is DU. Figure 1 gives N_D as a function of γ for several values of D.

$N_D(\gamma)$ decreases and converges to D as γ increases. Although we have no general proof of that, it can be proved that $N_D(0) \geqslant N_D(\gamma)$ for all γ; $N_D(0)$ has been computed by Anily (1995). Figure 2 shows the expected costs, U_D, for the same values.

Binomial Yield

It is possible to prove formally that for binomial yield, $T_D(N) = D/\theta$, independent of N. (Recall that in Example 2,

Figure 2. U_D as a function of γ in Example 3.



$T_1(2) = 1/\theta$.) However, the proof is tedious, and a simpler, straight-forward argument is available.

With binomial yield and random inspection, the inspector may view her situation as if there is an infinite number of units available for inspection. Further, the quality of each unit is independent of others. Thus the number of inspections required follows a negative binomial distribution (sampling until there are D successes). Thus, the expected number of trials is D/θ. This leads to the following theorem.

THEOREM 4. *When the yield is binomial, the optimal lot for 'MLPO with inspection' (5) is the same as the optimal lot for the basic MLPO problem (4). Thus, also, $U_D = V_D + \gamma D/\theta$.*

PROOF. Simply note that writing $T_D(N) = D/\theta$ in (5) implies that the lot N, which minimizes $V_D(N)$, also minimizes $U_D(N)$. □

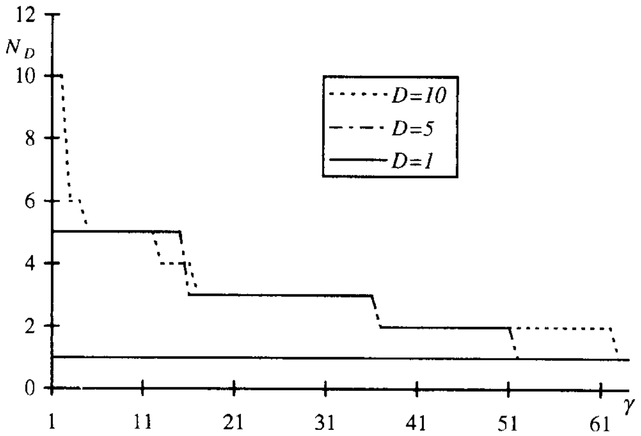
Interrupted Geometric Yield

The key observation for this yield structure is that the optimal lot does not exceed the demand, i.e., $N_D \leqslant D$. Thus, in the search for optimal policy, via (5), only lot sizes not exceeding D are considered. It remains to specify the inspection policy, which determines $T_D(N)$. Clearly, the inspection policy aims at minimizing $T_D(N)$ for each N.

The optimal inspection procedure in such a case has been found by He et al. (1996). The optimal inspection order could depend on N and θ in a rather complex way, but it has some intuitive properties. In particular, if N is "small" and θ is "not too small," one should start by inspecting the last unit produced. If it is found to be good, we are done; if it is defective, Bayesian revision of the quality of previous units must take place. The following heuristic is shown by He et al. (1996) to work extremely well: Let $n^*(\theta) = \arg\min(|\theta^n - 0.5|) \approx \log_\theta 0.5$. Then, if $N \geqslant n^*(\theta)$, inspect unit $n^*(\theta)$ first; if $N < n^*(\theta)$, and quality of unit N is unknown, inspect unit N first; if the last unit is defective, inspect unit $N/2$ next.

Note that while this yield pattern is related to the setting discussed in Porteus and Angelus (1997), their formulation considers inspection for the purpose of restoring the process to being in control.

Figure 3. N_D as a function of γ in Example 4, with $\theta = 0.8$.



EXAMPLE 4. Assume again that $\alpha = 30$ and $\beta = 1$. Figure 3 shows the dependence of N_D on γ. In Figure 3, $\theta = 0.8$. It is clear that $N_D(\gamma)$ decreases, and eventually converges to 1, as γ increases.

When the success probability θ changes, we see the results in Figure 4. As the success probability increases, the lot size becomes larger, which agrees with intuition.

Figure 5 shows how the costs U_D decrease in the success probability. The benefits of improving the production process when θ is initially small are quite significant.

Together, Figures 4 and 5 show that when production is almost perfect (θ is close to 1), the optimal lot size becomes equal to the demand D, and the total cost is close to $\alpha + \beta D$.

CONCLUDING REMARKS

Previous work on manufacturing to order with random yield did not consider the implications of inspection. If *all* units produced are always inspected, unit inspection cost can indeed be incorporated into the variable cost, and no special treatment is thus required. But when filling orders with particular specifications, there is no point in identifying more good units than were ordered. That fact reduces the inspection costs, but also has implications for the production plan. It is these implications we explored in this work.

Figure 4. N_D as a function of θ when $y = 5$.

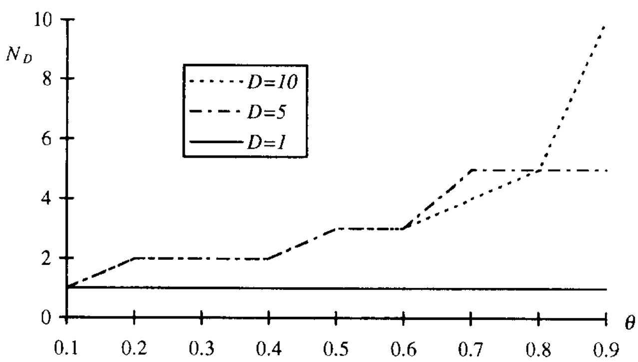
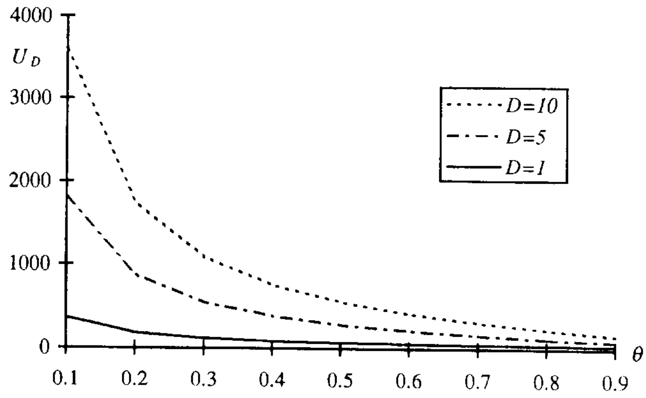


Figure 5. U_D as a function of θ when $\gamma = 5$.



We provided a method for computing the expected inspection costs for any production and inspection policies. Inspection costs are shown to be particularly simple to compute when units are inspected in random order, which is often optimal. For the important yield structures we considered, the optimal lot size with costly inspection was smaller than when inspection is free. It would be of interest to prove/disprove the generality of this observation.

The examples we provide show what major impact the yield pattern has on the solution of the problem. When yield is binomial, inspection costs have no impact on lot size. For discrete uniform and interrupted geometric yields, the optimal lot size is shown to be decreasing in the unit inspection cost.

Future research may also address inspection order and lot sizing issues in settings with machine restoration, as envisioned by Porteus and Angelus.

APPENDIX

PROOF OF THEOREM 3. This proof consists of three parts. First, we show that function $T_D(N)$ is nonincreasing for $1 \leqslant N < D$. Then we show that $T_D(N)$ is quasi-convex for $N \geqslant D$. Finally, we show that $T_D(D-1) \geqslant T_D(D)$.

First, note that for the DU, $T_D - T_{D-1} \geqslant 2$, i.e., at least two inspections are expected to be needed to find a conforming unit. This is true because $T_1(1) = 2$ and $T_1(N) \geqslant 2$ for $N \geqslant 2$. Inequality $T_D - T_{D-1} \geqslant 2$ implies that $T_D - T_{D-y} \geqslant 2y$. Using the expression of $T_D(N)$, straightforward calculations lead to, for $1 \leqslant N < D-1$,

$$T_D(N+1) - T_D(N)$$

$$= \frac{1}{N(N+1)} \left[N(N+1) - \sum_{y=1}^{N} (T_{D-y} - T_{D-N-1}) \right]$$

$$\leqslant \frac{1}{N(N+1)} \left[N(N+1) - 2\sum_{y=1}^{N} (N+1-y) \right] = 0.$$

Second, for $N \geqslant D$,

$$T_D(N+1) - T_D(N) = \frac{1}{N(N+1)}\left[ND - D\sum_{y=D}^{N}\frac{1}{y+1} - \sum_{y=1}^{D-1} T_{D-y}\right].$$

Suppose that $T_D(N+1) - T_D(N) \geqslant 0$ for some $N \geqslant D$. It can be proved that $T_D(n+1) - T_D(n) > 0$ for all $n > N$. Let $N^* = \min\{N : N \geqslant D \text{ and } T_D(N+1) - T_D(N) \geqslant 0\}$. Then $T_D(N)$ is nonincreasing from D to N^* and increasing after N^*. Therefore $T_D(N)$ is quasi-convex with respect to N for $N \geqslant D$.

Finally, we show that $T_D(D-1) \geqslant T_D(D)$. With straightforward calculations, we obtain

$$T_D(D-1) = D + \frac{1}{D-1}\sum_{y=1}^{D-1} T_{D-y}, \quad \text{and}$$

$$T_D(D) = D + 1 + \frac{1}{D}\sum_{y=1}^{D-1} T_{D-y}.$$

The above two expressions lead to

$$\begin{aligned} T_D(D-1) - T_D(D) &= \frac{1}{D(D-1)}\left[\sum_{y=1}^{D-1} T_{D-y} - D(D-1)\right] \\ &\geqslant \frac{1}{D(D-1)}\left[2\sum_{y=1}^{D-1}(D-y) - D(D-1)\right] = 0, \end{aligned}$$

where the above inequality holds because $T_{D-y} \geqslant 2(D-y)$. Combining the three parts, we see that $T_D(N)$ is nonincreasing from 1 to N^* and increasing after N^* with respect to N for fixed D. □

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