Classification of Markov Processes of Matrix $M/G/1$ type with a Tree Structure and its Applications to the $MMAP[K]/G[K]/1$ Queue

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ABSTRACT

The purpose of this paper is to study the classification problem of discrete time and continuous time Markov processes of matrix $M/G/1$ type with a tree structure. We begin this paper by developing a computational method to find whether a Markov process of matrix $M/G/1$ type with a tree structure is positive recurrent, null recurrent, or transient. The method is then used to study the impact of the last-come-first-served general preemptive resume (LCFS-GPR) service discipline on the stability of the $MAP/PH/1$ queue. The later portion of the paper identifies some sufficient conditions for positive recurrence and transience of Markov processes of matrix $M/G/1$ type with a tree structure. The results are used to show that the discrete time or continuous time $MMAP[K]/G[K]/1$ queue or the continuous time $MMAP[K]/PH[K]/S$ queue with a work conserving service discipline is stable if and only if its traffic intensity is less than one, unstable if its traffic intensity is larger than one.

Key words: Markov process, tree structure, Lyapunov function, mean drift method, Matrix analytic methods, discrete or continuous time queueing system.

1. Introduction

Markov processes of matrix $M/G/1$ type with a tree structure was introduced in Takine, Sengupta, and Yeung [17] as a generalization of Markov processes of matrix $M/G/1$ type (Neuts [13, 15]). Since the queueing processes of a number of queueing systems can be formulated into such Markov processes, their study attracted considerable attention from researchers in recent years (Gajrat, et al. [3], HE [5, 6], HE and Alfa [8], and Takine, Sengupta, and Yeung [17], Yeung and Alfa [18], Yeung and Sengupta [19], etc.). However, little has been done on the classification of such Markov processes, except for some special cases (Gajrat, et al. [3], HE [7], Neuts [15], and Takine, et al. [17]). Thus, there is a need for more study of the classification problem, especially for queueing applications.

Previous studies on the classification problem of Markov processes with a tree structure (or random string) can be found in Gajrat, et al. [3], HE [7], and Malyshev [10, 11]. The differences between this work and that of Gajrat and Malyshev are 1) we consider an auxiliary random variable (or an underlying process) while they did not; 2) we exploit the special *M/G/*1 type tree structure while they considered a more general structure. Since the auxiliary random variable is considered, it is difficult to generalize their results directly. Because of the special *M/G/*1 type tree structure, it is possible to obtain results that do not hold for the models considered in Gajrat, et al. [3] and Malyshev [10, 11]. HE [7] gives a complete solution to the classification problem of Markov processes of *M/G/*1 type with a tree structure and a number of queueing systems with multiple types of customers. The objective of this paper is to generalize the main results obtained in HE [7] to Markov processes with an auxiliary random variable. As shall be shown, such a generalization is not straightforward because the auxiliary random variable is considered.

In the first part of this paper (Sections 3, 4, and 5), we develop a computational method that can be used to classify Markov processes of matrix *M/G/*1 type with a tree structure. This method is based on a set of stochastic matrices $\mathbf{G} = \{G(1), \cdots, G(K)\}$ that can be computed using an iterative method. Once the set $\mathbf{G}$ is obtained, the classification of the Markov process of interest is determined completely by the Perron-Frobenius eigenvalue of a non-negative matrix. The computational method is used to analyze the impact of the last-come-first-served general preemptive resume (LCFS-GPR) service discipline on the stability of a discrete time *MAP/PH/*1 queue (Takine, et al. [17] and Yeung and Alfa [18]).

The second part of this paper (Sections 6, 7, 8, and 9) identifies some sufficient conditions for positive recurrence and transience of Markov processes of interest. The usefulness of these results is illustrated by using them to prove sufficient conditions for positive recurrence and transience of the discrete time and continuous time *MMAP*[*K*]/*G*[*K*]/1 queue and the continuous time *MMAP*[*K*]/*PH*[*K*]/*S* queue with a work conserving service discipline. Note that work conserving service disciplines include first-come-first-served (FCFS), LCFS non-preemption, LCFS preemptive resume, priority without preemption, priority with preemptive resume, processor sharing, etc. For these queueing systems of interest, let λ_k be the arrival rate of type k customers and μ_k the service rate of type k customers. It will be shown that the queueing systems are positive recurrent if and only if the traffic intensity $\rho = \lambda_1/\mu_1+\cdots+\lambda_K/\mu_K<1$, transient if $\rho>1$. The queueing systems considered in this paper were studied in HE [5, 6], HE and Alfa [8], Yeung and Alfa [18] by assuming that the steady state can be reached. This paper shows that the steady state of these queueing systems can be reached if $\rho<1$, a condition that was used without rigorous proof.

Matrix analytic methods and the mean drift method are the main mathematical tools used in this paper (Neuts [14, 15], Fayolle, et al. [2], and Meyn and Tweedy [12]). These methods have been used in the past to study classification problems associated with some Markov processes and queueing models. However, the way they are being used in this paper is different from others. For instance, the Lyapunov functions used in this paper are constructed by taking advantage of the *M/G/*1 type tree structure (Sections 3 and 6).

This paper focuses primarily on discrete time Markov processes of matrix *M/G/*1 type with a tree structure. Later in this paper, we shall use “Markov chain” for “discrete time Markov process”. But all the results can be extended to continuous time Markov processes of matrix *M/G/*1 type with a tree structure. While most of the Markov processes associated with queueing systems studied in this paper are discrete time Markov processes, the one associated with the continuous time *MMAP*[*K*]/*PH*[*K*]/*S* queue is a continuous time Markov process. We use this one to illustrate how to solve the classification problem of continuous time Markov processes of interest.

The rest of the paper is organized as follows. In Section 2, a discrete time Markov process of matrix *M/G/*1 type with a tree structure is defined. In Section 3, a criterion for a classification of the Markov process of interest is identified. Section 4 develops a computational method for classification purpose. Section 5 uses the method developed in Section 4 to study the *MAP/PH/*1 queue with an LCFS-GPR service discipline. The impact of LCFS-GPR on the stability of the queueing system is analyzed numerically. In Section 6, simple sufficient conditions for a classification of Markov processes of interest are found. In Sections 7, 8, and 9, the theory developed in Section 6 is utilized to study the classification problem of the discrete time *MMAP*[*K*]/*G*[*K*]/1 queue, the continuous time *MMAP*[*K*]/*G*[*K*]/1 queue, and the continuous time *MMAP*[*K*]/*PH*[*K*]/*S* queue, respectively. Finally, Section 10 summarizes this paper.

2. Markov Chain of Matrix *M/G/*1 Type with a Tree Structure

The following discrete time Markov process of matrix *M/G/*1 type with a tree structure was first introduced in Takine, et al. [17]. Consider a discrete time two-dimensional Markov chain $\{(X_n, \eta_n), n\geq 0\}$ in which the values of X_n are represented by the nodes of a *K*-ary tree, and η_n takes integer values between 1 and *m*, where *m* is a positive integer. X_n is referred to as the node and η_n is referred to as the auxiliary variable of the Markov chain at time *n*. Next, a full description of the transitions of the Markov chain is given.

The *K*-ary tree of interest is a tree for which each node has a parent and *K* children, except the root node of the tree. The root node is denoted as 0. Strings of integers between 1 and *K* are used to represent nodes of the tree. For instance, the *k*th child of the root node has a representation of *k*, the *l*th child of the node *k* has a representation of *kl*, and so on.

Let $\aleph = \{J: J=k_1k_2\cdots k_n, 1\leq k_i \leq K, 1\leq i \leq K, n>0\}\cup\{0\}$. Any string $J\in\aleph$ is a node in the *K*-ary tree. The length of a string *J* is defined as the number of integers in the string and is denoted by $|J|$, the only exception is that $|J| = 0$ for $J = 0$. Let $N(J, k)$ be the number of times that the integer *k* appears in the string *J*. The following two operations related to strings in $\aleph$ are used in this paper.

Addition operation: for $J = k_1\cdots k_n \in\aleph$ and $H = h_1\cdots h_i \in\aleph$, then $J+H = k_1\cdots k_nh_1\cdots h_i \in\aleph$.

Subtraction operation: for $J = k_1\cdots k_n \in\aleph$, $H = k_i\cdots k_n \in\aleph$, $i>0$, then $J-H = k_1\cdots k_{i-1}$

$\in \aleph$.

The Markov chain (X_n, η_n) takes values in $\aleph \times \{1, 2, \cdots, m\}$. To be called a homogenous Markov chain of matrix *M/G/*1 type with a tree structure, (X_n, η_n) transits at each step only to its parent node or a descendent of its parent node. All possible transitions and their corresponding probabilities are given as follows. If $(X_n, \eta_n) = (J+k, i)$ for $1 \le k \le K$ and $1 \le i, i' \le m$, then

1) $(X_{n+1}, \eta_{n+1}) = (J+H, i')$ with probability $a_{(i,i')}(k, H)$ for $H \in \aleph$.

Note that transition probabilities depend only on the last integer in the string representing the current node $J+k$. If $(X_n, \eta_n) = (0, i)$ for $1 \le i, i' \le m$,, then

2) $(X_{n+1}, \eta_{n+1}) = (H, i')$ with probability $b_{(i,i')}(H)$ for $H \in \aleph$.

In matrix form, transition probabilities are represented as:

1)' $A(k, H)$ is an $m \times m$ matrix with elements $a_{(i,i')}(k, H)$, $1 \le k \le K$, for $H \in \aleph$;
2)' $B(H)$ is an $m \times m$ matrix with elements $b_{(i,i')}(H)$ for $H \in \aleph$.

For convenience, we shall occasionally use $A(J, H)$ to denote the transition probability matrix from the node J to the node H for any $J, H \in \aleph$. Throughout this paper, we assume that the Markov chain $\{(X_n, \eta_n), n \ge 0\}$ is irreducible. We now introduce some notation and functions.

$$
\begin{aligned}
&A(k) = \sum_{J \in \aleph} A(k, J), \quad 1 \le k \le K; \\
&A^{*(1)}(k, j) = \sum_{J \in \aleph} A(k, J) N(J, j), \quad B^{*(1)}(j) = \sum_{J \in \aleph} B(J) N(J, j), \quad 1 \le k, j \le K.
\end{aligned} \tag{2.1}
$$

Then we have $A(k)\mathbf{e} = \mathbf{e}$, $1 \le k \le K$, and $\sum_{J \in \aleph} B(J)\mathbf{e} = \mathbf{e}$, where $\mathbf{e}$ is the vector with all components one. We assume that $A^{*(1)}(k, j)$ and $B^{*(1)}(k)$, $1 \le k, j \le K$, are finite. Let $\theta(k)$ be the left eigenvector of the matrix $A(k)$ corresponding to the eigenvalue 1, i.e., $\theta(k)A(k) = \theta(k)$ with $\theta(k)\mathbf{e} = 1$, $1 \le k \le K$. Define

$$
p(k, j) = \theta(k) A^{*(1)}(k, j)\mathbf{e}, \quad 1 \le k, j \le K. \tag{2.2}
$$

Let P be a $K \times K$ matrix with elements $p(k, j)$, $1 \le k, j \le K$. Intuitively, $p(k,1) + p(k,2) + \ldots + p(k, K) - 1$ represents the average distance the Markov chain moved away from the root node in one transition, provided that the conditional distribution in each state is given by $\theta(k)$. Denote by

$sp(P)$ the Perron-Frobenius eigenvalue (i.e., the eigenvalue with the largest modulus) of the matrix P. If m=1 (i.e., η_n = 1 for all $n \geq 0$), the following theorem has been proved in HE [7] (Theorem 3.2, HE [7]).

Theorem 2.1 Assume that m=1 and the matrix P is irreducible. Then the Markov chain of interest is positive recurrent if $sp(P) < 1$; null recurrent if $sp(P) = 1$; and transient if $sp(P) > 1$.

Theorem 2.1 shows that, when P is irreducible, the classification of the Markov chain $\{(X_n, \eta_n), n \geq 0\}$ is determined completely by the Perron-Frobenius eigenvalue $sp(P)$ of the matrix P. Note that the matrix P defined here is different from the matrix P in HE [7]. In fact, $P = \Lambda\text{-M}+I$ in this paper and $P=\text{M}^{-1}\Lambda$ in HE [7], where Λ and M are defined in HE [7] for the case with m=1 and I is the identity matrix. Both of them provide the same information for a complete classification of the Markov chain of interest.

In light of Theorem 2.1, Dr. B. Sengupta and Qi-Ming HE conjectured that Theorem 2.1 holds for m>1 and K>1. Unfortunately, this conjecture is untrue and a counterexample is presented in Section 4. The reason is that the transitions between nodes change the nature of the auxiliary random variable (or the underlying Markov chain). Thus, stationary distributions $\{\theta(k), 1 \leq k \leq K\}$ do not measure correctly the proportion of time the auxiliary random variable stays in each state. Thus, the estimate of the average distance moved away from the root node in each transition is inaccurate.

Since Theorem 2.1 is untrue for m>1 and K>1, generalizations of the results obtained in HE [7] are not straightforward. Thus, we need to take two different approaches in Section 3 and Section 6 to study the problem.

3. A General Criterion for Classification

In this section, we shall extend Theorem 2.1 to the case with $m \geq 1$ and $K \geq 1$. For that purpose, a set of stochastic matrices $\mathbf{G} = \{G(1), \cdots, G(K)\}$ are introduced to construct a matrix $P(\mathbf{G})$ to replace the matrix P in Theorem 2.1. The necessity of using these matrices in $\mathbf{G}$ is partially justified by the counterexample (Example 4.4) to Theorem 2.1 when $m \geq 1$ and $K \geq 1$.

Let $\mathbf{G} = \{G(1), \cdots, G(K)\}$ where $G(1), G(2), \cdots, G(K)$ are $m \times m$ sub-stochastic matrices, i.e., $G(k) \geq 0$ and $G(k)\mathbf{e} \leq \mathbf{e}$, $1 \leq k \leq K$. Let $\mathfrak{R}$ be a set of elements $\mathbf{G}$ for which $\{G(1), \cdots, G(K)\}$ are stochastic matrices and satisfy the following equations, $1 \leq k \leq K$, ($J = k_1 \cdots k_{|J|}$)

$$G(k) = A(k,0) + \sum_{J \in \aleph,\, J \neq 0} A(k,J) G(k_{|J|}) G(k_{|J|-1}) \cdots G(1). \qquad (3.1)$$

Let $\mathbf{G}^* = \{G^*(1), \cdots, G^*(K)\}$ be the minimal nonnegative solution to equation (3.1). According to Takine, et al. [17], the (i, i')th element of the matrix $G^*(k)$ is the probability that the Markov chain will eventually reach the node J in the state (J, i'), given that the Markov chain is in the state $(J+k, i)$ initially. It has been proved in Takine, et al. [17] that the set $\mathbf{G}^*$ is unique and all matrices in the set $\mathbf{G}^*$ are stochastic if the Markov chain is recurrent. Define a sequence $\{G(J)[n], J\in\aleph\}$ as follows. Let $G(0)[n] = I$, for $n\geq 0$, $G(J)[0] = 0$ for $J\in\aleph$ and $J\neq 0$, and

$$G(J)[n+1] = \sum_{H\in\aleph} A(J,H)G(H)[n]. \tag{3.2}$$

(Note that I is the identity matrix.) It is easy to show that $\{G(J)[n], n\geq 0\}$ is a uniformly bounded and nondecreasing sequence for $J\in\aleph$. If we consider the root node as an absorption node, $G(J)[n]$ can be interpreted as the probability that the Markov chain reaches the root node within n transitions, given that the Markov chain is in the node J initially. Denote by $\{G^*(J), J\in\aleph\}$ the limits of $\{G(J)[n], J\in\aleph\}$. The limit $G^*(J)$ is the conditional probability that the Markov chain will eventually reach the root node 0, given that the Markov chain is initially in the node J, for $J\in\aleph$. Then we have $G^*(J) = G^*(k_{|J|})\cdots G^*(k_1)$ because of the special $M/G/1$ type tree structure.

For any set $\mathbf{G} = \{G(1), \cdots, G(K)\}\in\Re$, define $G(J) = G(k_{|J|})\cdots G(k_1)$ for all $J=k_1\cdots k_{|J|}\in\aleph$. It is clear $G(J)\geq G(J)[0] = 0$, i.e., the (i, j)th element of $G(J)$ is larger than or equal to the (i, j)th element of $G(J)[0]$ for $1\leq i, j\leq m$. Using equation (3.2), it can be proved inductively that $G(J)\geq G(J)[n]$, which leads to $G(J)\geq G^*(J)$. This leads to the following relationship between the set $\mathbf{G}^*$ and the set $\Re$.

Lemma 3.1 The set $\Re$ is always nonempty. If the Markov chain $\{(X_n, \eta_n), n\geq 0\}$ is recurrent, $\Re = \{\mathbf{G}^*\}$. If the Markov chain is transient, $\Re$ has at least one element, but the set $\mathbf{G}^*$ is not in $\Re$. For any set $\mathbf{G}$ in $\Re$, $G(k)\geq G^*(k)$.

Proof. First, we prove that the set $\Re$ is nonempty. Let R_+ be the set of nonnegative real numbers. Denote by $R_+^{Kmm} = R_+\times R_+\times\cdots\times R_+$, i.e., the cross product of kmm number of the set R_+. Let

$$E = \left\{ (x_{1,1},\cdots,x_{1,m},x_{2,1},\cdots,x_{2,m},\cdots,x_{m,1},\cdots,x_{m,m}) : \begin{array}{l} 0\leq x_{i,j}, 1\leq i\leq m, 1\leq j\leq m, \\ \sum_{j=1}^{m} x_{i,j} = 1, 1\leq i\leq m. \end{array} \right\}.$$

Let $E^K = E\times E\times\cdots\times E$, i.e., the cross product of K number of the set E. It is easy to see that each element in E^K corresponds to a unique set $\mathbf{G}$ with K stochastic matrices. The right hand side of equation (3.1) defines a mapping from E^K to E^K. It can be proved that this mapping is continuous. Since E^K is a bounded closed convex set in R_+^{Kmm}, the mapping has at least one fixed point on E^K by the well known Brouwer's fixed point theorem. Thus, there is at least one set of stochastic matrices that satisfy equation (3.1). Therefore, the set $\Re$ is nonempty.

If the Markov chain $\{(X_n, \eta_n), n\geq 0\}$ is recurrent, all matrices in the set $\mathbf{G}^*$ are stochastic. Therefore, the set $\mathbf{G}^*$ is in $\Re$. For each $G(k) \in \mathbf{G} \in \Re$, since $G(k)\geq G^*(k)$ and $G(k)$ and $G^*(k)$ are stochastic, we must have $G(k)=G^*(k)$. Thus, the set $\mathbf{G}^*$ is the only element in $\Re$. If the Markov chain $\{(X_n, \eta_n), n\geq 0\}$ is transient, at least one matrix in the set $\mathbf{G}^*$ is not stochastic. Thus, the set $\mathbf{G}^*$ is not in $\Re$. This completes the proof of Lemma 3.1.

For any set $\mathbf{G} = \{G(1), \cdots, G(K)\} \in \Re$, define the following $m\times m$ matrices:

$$\begin{aligned} &N(0, j, \mathbf{G}) = 0, \quad 1\leq j \leq K;\\ &N(k, j, \mathbf{G}) = \delta(k, j)I, \quad 1\leq k, j \leq K; \end{aligned} \tag{3.3}$$

$$N(J, j, \mathbf{G}) = \delta(k_{|J|}, j)I + \sum_{n=1}^{|J|-1} G(k_{|J|})\cdots G(k_{n+1})\delta(k_n, j), \quad J = k_1\cdots k_{|J|}, |J|\geq 2, 1\leq j\leq K;$$

$$p(k, j, \mathbf{G}) = \sum_{J\in\aleph} A(k, J)\, N(J, j, \mathbf{G}), \quad 1\leq k,\ j\leq K,$$

where $\delta(k, j) = 1$, if $k=j$; 0, otherwise. Define an $mK\times mK$ matrix $P(\mathbf{G})$ by

$$P(\mathbf{G}) = \begin{pmatrix} p(1,1,\mathbf{G}) & \cdots & p(1,K,\mathbf{G}) \\ \vdots & \vdots & \vdots \\ p(K,1,\mathbf{G}) & \cdots & p(K,K,\mathbf{G}) \end{pmatrix}. \tag{3.4}$$

Intuitively, the matrix $P(\mathbf{G})$ represents the average distance (less one) the Markov chain moved away from the root node in one transition, given that the set $\mathbf{G}$ is used to represent the change of state of the auxiliary variable. For any set $\mathbf{G}$ in $\Re$, we now prove that the Perron-Frobenius eigenvalue of the matrix $P(\mathbf{G})$, $sp(P(\mathbf{G}))$, provides information for a complete classification of the Markov chain of interest. For the vector $\mathbf{x} = (x_1, \cdots, x_{mK})$, note that $(\mathbf{x})_i$ represents the ith element of the vector $\mathbf{x}$.

Theorem 3.2 For any set $\mathbf{G}\in\Re$, if the matrix $P(\mathbf{G})$ is irreducible, then the Markov chain of matrix *M/G/*1 type with a tree structure - $\{(X_n, \eta_n), n\geq 0\}$ - defined in Section 2 is

1) positive recurrent if and only if $sp(P(\mathbf{G})) < 1$;
2) null recurrent if and only if $sp(P(\mathbf{G})) = 1$;

3) transient if and only if $sp(P(\mathbf{G})) > 1$.

Proof. We first prove that $sp(P(\mathbf{G}))<1$ is a necessary and sufficient condition for positive recurrence of the Markov chain of interest, i.e., part 1). Then we prove part 3). As a result of Part 1) and Part 3), Part 2) is obtained. This proof is parallel to the proof of Theorem 3.2 in HE [7], but some technical details are different.

To prove that $sp(P(\mathbf{G}))<1$ is sufficient for ergodicity of the Markov chain, the mean-drift method is applied. The idea of mean-drift method is to find a Lyapunov function (or a test function) $\mathbf{f}(J)$, a positive column vector of the size m, defined on $\aleph$ such that every element of $\mathbf{f}(J)$ is positive, $\mathbf{f}(J) \to \infty$ when $|J| \to \infty$, and

$$\mathbf{E}[\mathbf{f}(X_{n+1}) - \mathbf{f}(X_n) \mid X_n = J] = \sum_{H\in\aleph} A(J,H)\mathbf{f}(H) - \mathbf{f}(J) < -\varepsilon\mathbf{e} \tag{3.5}$$

holds for all but a finite number of J in $\aleph$ for some positive ε. If so, the Markov process is positive recurrent Foster's criterion (see Theorem 2.2.3 in Fayolle, et al. [2]). Suppose that $sp(P(\mathbf{G}))<1$. Denote by $\mathbf{b}$ the right eigenvector corresponding to $sp(P(\mathbf{G}))$. Since $P(\mathbf{G})$ is irreducible, the vector $\mathbf{b} = (b_1, \cdots, b_{mK})^{\mathrm{T}}$ (where "T" represents the transpose of matrix) is a positive vector, i.e., every element of $\mathbf{b}$ is positive. Let $\mathbf{b}_k = (b_{m(k-1)+1}, \cdots, b_{mk})^{\mathrm{T}}$. Based on the tree structure of the Markov chain of interest, the following Lyapunov function is introduced. For any $J\in\aleph$ and $J \neq 0$, define

$$\mathbf{f}(J) = \sum_{k=1}^{K} N(J,k,\mathbf{G})\mathbf{b}_k. \tag{3.6}$$

Since we can use $\mathbf{f}(J)+\mathbf{f}(0)$ as the Lyapunov function used in equation (3.5), without loss of generality, we choose $\mathbf{f}(0) = 0$. Since $\mathbf{b}_k$ is positive and $G(k)\mathbf{e}=\mathbf{e}$ for all k, $\mathbf{f}(J) \to \infty$ when $|J| \to \infty$, $1\le k\le K$. It is clear that the function $\mathbf{f}(J)$ is quasi-additive, i.e., $\mathbf{f}(J+H) = \mathbf{f}(H) + G(h_{|H|})\cdots G(h_1)\mathbf{f}(J)$ for $J\in\aleph$ and $H = h_1\cdots h_{|H|} \in\aleph$. Applying the above Lyapunov function, the left hand side of inequality (3.5) becomes, for $X_n = J+k$ with $J=k_1\cdots k_n$ and $1\le k\le K$,

$$\begin{aligned}
&\sum_{H\in\aleph} A(k,H)\mathbf{f}(J+H) - \mathbf{f}(J+k) \\
&= \sum_{H\in\aleph} A(k,H)[\mathbf{f}(H) + G(h_{|H|})\cdots G(h_1)\mathbf{f}(J)] - \mathbf{f}(k) - G(k)\mathbf{f}(J) \\
&= \sum_{H\in\aleph} A(k,H)\mathbf{f}(H) - \mathbf{f}(k) + \left[\sum_{H\in\aleph} A(k,H)G(h_{|H|})\cdots G(h_1) - G(k)\right]\mathbf{f}(J) \\
&= \sum_{H\in\aleph} A(k,H)\mathbf{f}(H) - \mathbf{f}(k) = \sum_{j=1}^{K} p(k,j,\mathbf{G})\mathbf{b}_j - \mathbf{b}_k = (sp(P(\mathbf{G}))-1)\mathbf{b}_k < -\varepsilon\mathbf{e},
\end{aligned} \tag{3.7}$$

where $\varepsilon = 0.5(1-sp(P(\mathbf{G}))\min_i\{(\mathbf{b})_i\}>0$. In the above evaluation, equalities in equations (3.1) and (3.3) are used. Thus, inequality (3.5) holds for all $J\in\aleph$ but J=0. Since $B^{*(1)}(k)$, $1\le k\le K$, are finite, the left hand side of inequality (3.5) is finite for J=0. Therefore, the Markov chain is positive recurrent.

To prove the necessity of $sp(P(\mathbf{G}))<1$ for ergodicity, denote by $v_i(J)$ the first passage time from the node J to the root node 0, given that the initial state of the auxiliary random variable is i. Let $\mathbf{v}(J) = (v_1(J), \ldots, v_m(J))^T$. Note that $\mathbf{v}(0) = 0$. According to Foster's criterion, when the Markov chain is ergodic, $\{\mathbf{v}(J), J\in\aleph\}$ are finite and satisfy

$$\mathbf{v}(J) = \mathbf{e} + \sum_{H\in\aleph} A(J,H)\mathbf{v}(H). \tag{3.8}$$

Because of the tree structure and the transition pattern of the Markov chain, the function $\mathbf{v}(J)$ is quasi-additive, i.e., $\mathbf{v}(J+H)=\mathbf{v}(H)+G^*(H)\mathbf{v}(J)$ which implies that $\mathbf{v}(J) = \sum_{k=1}^{K} N(J,k,\mathbf{G}^*)\mathbf{v}(k)$, where $\mathbf{G}^* = \{G^*(1), \cdots, G^*(K)\}$ are the minimal nonnegative solution of equation (3.1). If the Markov chain is positive recurrent, these matrices are stochastic. Equation (3.8) leads to, for $J = k>0$,

$$-\mathbf{e} = \sum_{H\in\aleph} A(k,H)\mathbf{v}(H) - \mathbf{v}(k) = \sum_{j=1}^{K} p(k,j,\mathbf{G}^*)\mathbf{v}(j) - \mathbf{v}(k). \tag{3.9}$$

Let $\mathbf{v} = ((\mathbf{v}(1))^T, \ldots, (\mathbf{v}(K))^T)^T$. Equation (3.9) leads to $P(\mathbf{G}^*)\mathbf{v} = \mathbf{v}-\mathbf{e} < \mathbf{v}$. Since $\mathbf{v}(k)\ge\mathbf{e}$, $1\le k\le K$, $\mathbf{v}$ is positive. That implies that $sp(P(\mathbf{G}^*))<1$. By Lemma 3.1, $\mathbf{G}^*$ is the only element in $\Re$ if the Markov chain is positive recurrent. Then $sp(P(\mathbf{G}))<1$ for any set $\mathbf{G}\in\Re$. This completes the proof of Part 1).

We now prove Part 3), i.e., the Markov chain is transient if and only if $sp(P(\mathbf{G}))>1$. The idea is to look at the probabilities that the Markov chain will ever reach the root node from any other node. If the Markov process is transient, these probabilities should be less than one.

If the Markov chain is transient, the matrix $G^*(k)$ is not stochastic for at least one k. Denote by $\mathbf{g}(k) = \mathbf{e}-G^*(k)\mathbf{e}$ (≥ 0), $1\le k\le K$. Because of the special tree structure, $G^*(J) = G^*(k_{|J|})\cdots G^*(k_1)$ leads to

$$G^*(J)\mathbf{e} = G^*(-k_1 + J)G^*(k_1)\mathbf{e} = G^*(-k_1 + J)\mathbf{e} - G^*(-k_1 + J)\mathbf{g}(k_1)$$
$$= \cdots = \mathbf{e} - \sum_{t=1}^{|J|}\prod_{n=|J|}^{t+1} G^*(k_t)\mathbf{g}(k_t) \geq \mathbf{e} - \sum_{t=1}^{|J|}\prod_{n=|J|}^{t+1} G(k_n)\mathbf{g}(k_t), \tag{3.10}$$

where the fact that $G[k] \geq G^*[k]$, $1 \leq k \leq K$, is used. Note that in equations (3.10), (3.11), and (3.13), when $t+1$ is larger than $|J|$, the product of matrices is defined as I. Use the above inequality to obtain

$$\mathbf{e} - \mathbf{g}(k) = G^*(k)\mathbf{e} = \sum_{J \in \aleph} A(k, J)G^*(J)\mathbf{e} \geq \sum_{J \in \aleph} A(k, J)\left[\mathbf{e} - \sum_{t=1}^{|J|}\prod_{n=|J|}^{t+1} G(k_n)\mathbf{g}(k_t)\right]$$
$$\Rightarrow \ \mathbf{g}(k) \leq \sum_{j=1}^{K} p(k, j, \mathbf{G})\mathbf{g}(j) = (P(\mathbf{G})\mathbf{g})_k, \tag{3.11}$$

where $\mathbf{g} = ((\mathbf{g}(1))^T, (\mathbf{g}(2))^T, \ldots, (\mathbf{g}(K))^T)^T$ and $(P(\mathbf{G})\mathbf{g})_k$ is kth block of the vector $P(\mathbf{G})\mathbf{g}$. If $(P(\mathbf{G})\mathbf{g})_k = \mathbf{g}(k)$ for all k, then $P(\mathbf{G})\mathbf{g} = \mathbf{g}$. Since $P(\mathbf{G})$ is irreducible and $\mathbf{g}$ is nonzero and nonnegative, $sp(P(\mathbf{G})) = 1$ and the vector $\mathbf{g}$ is positive (uniqueness). Then it can be proved that the Markov chain of interest is null recurrent by Theorem 2.2.1 in Fayolle, et al. [2] (similar to the proof of Part 1)). This is a contradiction. Therefore, $P(\mathbf{G})\mathbf{g} \geq \mathbf{g}$ and $P(\mathbf{G})\mathbf{g} \neq \mathbf{g}$, which implies $sp(P(\mathbf{G}))>1$.

Now suppose that $sp(P(\mathbf{G}))>1$. We shall show that at least one sequence $\{G(J)[n], n\geq 0\}$, for any $J \in \aleph$, does not converge to a stochastic matrix. Suppose that the Markov chain is recurrent or the sequence $\{G(J)[n], n\geq 0\}$, for $J \in \aleph$, does converge to a stochastic matrix. Then sequence $\{G(k)[n], n\geq 0\}$ converges to $G^*(k)$ and, $G^*(k)$ is stochastic, $1 \leq k \leq K$. Then the set $\mathbf{G} = \mathbf{G}^*$. First, let $p(k)[n]$ be the (matrix) probability that the Markov chain reaches the node J for the first time at the nth transition, given that the Markov chain is in the node $J+k$ initially. Then

$$G(J)[n] = \sum_{t=1}^{n-1} p(k_{|J|})[t]G(J - k_{|J|})[n-t] \leq \left(\sum_{t=1}^{n-1} p(k_{|J|})[t]\right)G(J - k_{|J|})[n-1]$$
$$\leq G(k_{|J|})[n]G(J - k_{|J|})[n] \leq \cdots \leq \prod_{t=|J|}^{1} G(k_t)[n]. \tag{3.12}$$

In the above equation, the fact that $\{G(J)[n]\}$ is nondecreasing with respect to n is used. Let $\mathbf{g}(k)[n] = \mathbf{e} - G(k)[n]\mathbf{e}$, $1 \leq k \leq K$. Since $sp(P(\mathbf{G}))>1$, there is $A(k, J) \neq 0$ for at least one $J>0$. Since $G(k)[0] = 0$, it can be proved inductively that $\mathbf{g}(k)[n] \geq 0$, $\mathbf{g}(k)[n] \neq 0$ for all $n>0$ and at least one k, $1 \leq k \leq K$. Then we have

$$
\begin{aligned}
G(k)[n+1]\mathbf{e} &= \sum_{J\in\aleph} A(k,J)G(J)[n]\mathbf{e} \le \sum_{J\in\aleph} A(k,J)\prod_{t=|J|}^{1} G(k_t)[n]\mathbf{e} \\
&\le \sum_{J\in\aleph} A(k,J)\left[\mathbf{e} - \sum_{t=1}^{|J|}\prod_{i=|J|}^{t+1} G(k_i)[n]\mathbf{g}(k_t)[n]\right] \\
&= \mathbf{e} - \sum_{J\in\aleph} A(k,J)\sum_{t=1}^{|J|}\prod_{i=|J|}^{t+1} G^*(k_i)\mathbf{g}(k_t)[n] \\
&\quad + \sum_{J\in\aleph} A(k,J)\left[\sum_{t=1}^{|J|}\prod_{i=|J|}^{t+1} G^*(k_i)\mathbf{g}(k_t)[n] - \sum_{t=1}^{|J|}\prod_{i=|J|}^{t+1} G(k_i)[n]\mathbf{g}(k_t)[n]\right] \\
&= \mathbf{e} - \mathbf{g}(k)[n] - \left[\sum_{j=1}^{K} p(k,j,\mathbf{G}^*)\mathbf{g}(j)[n] - \mathbf{g}(k)[n]\right] + O([\max_{1\le i\le Km}\{(\mathbf{g}[n])_i\}]^2)
\end{aligned}
\tag{3.13}
$$

where $\mathbf{g}[n] = (\mathbf{g}(1)[n]^T, \cdots, \mathbf{g}(K)[n]^T)^T$. Since $P(\mathbf{G})$ is irreducible, $P(\mathbf{G}^*)\mathbf{x} \le c\mathbf{x}$ does not hold for any $0 < c < sp(P(\mathbf{G}))$ and any nonnegative vector $\mathbf{x}$. Take $c = 1+0.5(sp(P(\mathbf{G}^*))-1)$. Then for the nonzero and nonnegative vector $\mathbf{g}[n]$, there exists (k, i) such that $(P(\mathbf{G}^*)\mathbf{g}[n])_{(k-1)m+i} > c(\mathbf{g}(k)[n])_i$. Then

$$
(G[k](n+1)\mathbf{e})_i < (G(k)[n]\mathbf{e})_i - 0.5[sp(P(\mathbf{G}^*)-1](\mathbf{g}(k)[n])_i + O([\max_{1\le i\le Km}\{(\mathbf{g}[n])_i\}]^2) \tag{3.14}
$$

holds for at least one pair of (k, i), $1\le k\le K$ and $1\le i\le m$, when n goes to infinity. When n is large enough or equivalently $\{\mathbf{g}(k)[n], 1\le k\le K\}$ are small enough, inequality (3.14) implies that, for at least one pair of (k, i), $(G(k)[n+1]\mathbf{e})_i<(G(k)[n]\mathbf{e})_i$, which contradicts the fact that the sequence $\{G(k)[n], n\ge 0\}$ is nondecreasing. Therefore, at least one $G^*(k)$ is not stochastic, i.e., the Markov chain is transient. This completes the proof of part c).

Part b) holds since part a) and part c) are true. This completes the proof of the theorem.

Note: For Part 1) of Theorem 3.2, the irreducibility condition on the matrix $P(\mathbf{G})$ can be changed to that $P(\mathbf{G})$ has a positive right eigenvector corresponding to its Perron-Frobenius eigenvalue, which is a weaker condition. However, in the proof of Part 3), we do need the irreducibility of $P(\mathbf{G})$. In Theorem 3.4, the irreducibility condition on $P(\mathbf{G})$ is replaced by the weaker condition when additional conditions are imposed on the structure of the Markov chain of interest.

Theorem 3.2 shows that if we can find a set $\mathbf{G}\in\Re$, then the classification problem becomes easy. But such a set $\mathbf{G}$ cannot be found explicitly except for m=1. Nonetheless, Theorem 3.2 is useful for developing a computation method to solve the classification problem. The following corollary shows a useful relationship between the set $\mathbf{G}^*$ and the classification of the Markov chain of interest.

Corollary 3.3 Suppose that $\mathbf{G}^* = \{G^*(1), \ldots, G^*(K)\}$ are the minimal nonnegative solution to equation (3.1). Define $P(\mathbf{G})$ for any set $\mathbf{G}$ of K substochastic or stochastic matrices satisfying equation (3.1). If the matrix $P(\mathbf{G}^*)$ is irreducible, then the Markov chain of interest is 1) positive recurrent if and only if $sp(P(\mathbf{G}^*)) < 1$; 2) null recurrent or transient if and only if $sp(P(\mathbf{G}^*)) = 1$.

Proof. If the Markov chain of interest is positive recurrent and null recurrent, the results are obtained by Lemma 3.1 and Theorem 3.2. If the Markov chain is transient, by equation (3.10) and (3.11), it can be proved that $P(\mathbf{G}^*)\mathbf{g} = \mathbf{g}$ with $\mathbf{g}\geq 0$ and $\mathbf{g}\neq 0$ (the vector $\mathbf{g}$ is defined after equation (11)), which implies $sp(P(\mathbf{G}^*)) = 1$. This completes the proof.

Finally in this section, we consider Markov chains which can only travel a bounded distance in one transition, i.e., $A(k, J)=0$ when $|J|>d$, where d (≥ 2) is a fixed positive number. This case is interesting since it includes the quasi-birth-and-death (QBD) Markov chain of matrix *M/G/*1 type with a tree structure as a special case. In addition, the condition on $P(\mathbf{G})$ is weaker, i.e., instead of irreducibility, $P(\mathbf{G})$ is only required to have a positive right eigenvector corresponding to its Perron-Frobenius eigenvalue.

Theorem 3.4 For the Markov chain of matrix *M/G/*1 type with a tree structure - $\{(X_n, \eta_n), n\geq 0\}$ - defined in Section 2, assume that $A(k, J)=0$ when $|J|>d$, where d (≥ 2) is a fixed positive number. For any set $\mathbf{G}\in\Re$, if the matrix $P(\mathbf{G})$ has a positive right eigenvector corresponding to its Perron-Frobenius eigenvalue, then the Markov chain is

1)' positive recurrent if and only if $sp(P(\mathbf{G})) < 1$;
2)' null recurrent if and only if $sp(P(\mathbf{G})) = 1$;
3)' transient if and only if $sp(P(\mathbf{G})) > 1$.

Proof. The proof of Part 1)' is the same as that of Part 1) of Theorem 3.2. If $sp(P(\mathbf{G})) > 1$, there exists a positive vector $\mathbf{b}$ such that $P(\mathbf{G})\mathbf{b} > \mathbf{b}$, i.e., every element of $P(\mathbf{G})\mathbf{b}$ is larger that the corresponding element of $\mathbf{b}$. Using the Lyapunov function defined in equation (3.6), it can be proved that $\mathbf{E}[\mathbf{f}(X_{n+1}) - \mathbf{f}(X_n) \mid X_n = J] = \sum_{H\in\aleph} A(J,H)\mathbf{f}(H) - \mathbf{f}(J) > \varepsilon\mathbf{e}$ for all $J\neq 0$ for some positive ε. Since $A(k, J)=0$ when $|J|>d$, the Markov chain is transient by Theorem 2.2.7 in Fayolle, et al. [2]. If $sp(P(\mathbf{G})) = 1$, there exists a positive vector $\mathbf{b}$ such that only $P(\mathbf{G})\mathbf{b} = \mathbf{b}$. Using the Lyapunov function defined in equation (3.6), it can be proved that $\mathbf{E}[\mathbf{f}(X_{n+1})-\mathbf{f}(X_n) \mid X_n = J] = \sum_{H\in\aleph} A(J,H)\mathbf{f}(H) - \mathbf{f}(J) \leq 0$ for all $J\neq 0$. According to Theorem 2.2.1 in Fayolle, et al. [2], the Markov chain is recurrent. Combining all these results, we obtain Theorem 3.4. This completes the proof.

Note: When $m=1$, the set $\mathbf{G} = \{1, \cdots, 1\}$ is a set of solution to equation (3.1). As a result, a simple and explicit solution to the classification problem is obtained (Theorem 2.1 or Theorem 3.2 in HE [7]).

Note: When K=1 and m≥1, $p(1,1)$ defined by equation (2.2) has been used widely for a classification of the Markov chain of interest. Condition $p(1,1)<1$ for positive recurrence is called Neuts condition in matrix analytic methods. Now, we prove that $sp(P(\mathbf{G}))$ and $p(1,1)$ are equivalent when K=1. For this case, $P(\mathbf{G}) = p(1,1,\mathbf{G})$, where $\mathbf{G} = \{G(1)\}$. It can be proved that

$$p(1,1,\mathbf{G}) = (A(1)-G(1))(I-G(1)+\mathbf{eg})^{-1} + [\sum_{J\in\aleph,\, J\neq 0} |J|\, A(1,J)]\mathbf{eg}\,, \tag{3.15}$$

where vector **g** is the left invariant vector of the matrix $G(1)$. Note that J (>0) is a string of ones when K=1. Further, it can be proved that $\theta(1)P(\mathbf{G}) = \theta(1) + [p(1,1)-1]\mathbf{g}$. When $P(\mathbf{G})$ is irreducible, $sp(P(\mathbf{G}))<1$ if and only if $p(1,1)<1$, $sp(P(\mathbf{G}))=1$ if and only if $p(1,1)=1$, and $sp(P(\mathbf{G}))>1$ if and only if $p(1,1)>1$. Thus, $sp(p(1,1,\mathbf{G}))$ and $p(1,1)$ are equivalent.

Note: The proofs of Theorems 3.2 and 3.4 suggest that the main results obtained in this section can be extended to some cases for which the matrix $P(\mathbf{G})$ is reducible. But the problem can be quite complicated and we leave it for future research.

4. A Computational Approach

Theorem 3.2 shows that the classification of the Markov chain is completely determined by $sp(P(\mathbf{G}))$ for any $\mathbf{G}\in\Re$, under some conditions on the matrix $P(\mathbf{G})$. Therefore, to find whether the Markov chain is positive recurrent, null recurrent, or transient, we only need to find $sp(P(\mathbf{G}))$ for a set $\mathbf{G}\in\Re$. However, there are no explicit expressions for matrices in the set $\mathbf{G}\in\Re$. Thus, we develop a computational approach to solve the classification problem. The idea is to use the minimal nonnegative solution $\mathbf{G}^*$ and another set $\mathbf{G}\in\Re$ which will be specified later.

Define a sequence $\{\mathbf{G}^*[n], n\geq 0\}$ as follows. Let $G^*(k)[0] = 0$, $1\leq k\leq K$, and

$$G^*(k)[n+1] = A(k,0) + \sum_{J\in\aleph,\, J\neq 0} A(k,J)\prod_{i=|J|}^{1} G^*(k_i)[n], \quad 1\leq k\leq K. \tag{4.1}$$

We first prove the following result.

Lemma 4.1 For the sequence $\mathbf{G}^*[n] = \{G^*(1)[n], \cdots, G^*(K)[n]\}$ generated by using equation (4.1), $\{P(\mathbf{G}^*[n]), n\geq 0\}$ is a nondecreasing sequence and so is $\{sp(P(\mathbf{G}^*[n])), n\geq 0\}$. The sequence $\{sp(P(\mathbf{G}^*[n])), n\geq 0\}$ converges to $sp(P(\mathbf{G}^*))$ monotonically.

Proof. The results are obtained by the fact that the sequence $\{G^*(k)[n], n\geq 0\}$ converges to $G^*(k)$ monotonically for $1\leq k\leq K$. This completes the proof.

Now we generate a sequence $\{\hat{\mathbf{G}}[n], n \geq 0\}$ by using equation (4.1) with $\hat{\mathbf{G}}[0]$ as the initial set of matrices, where $\hat{\mathbf{G}}[0]$ is a given set of K stochastic matrices. It is easy to see that $\hat{\mathbf{G}}[n]$ is a set of stochastic matrices for all n.

Lemma 4.2 For sequences $\{\mathbf{G}^*[n], n\geq 0\}$ and $\{\hat{\mathbf{G}}[n], n \geq 0\}$, we have $G^*(k)[n]\leq \hat{G}(k)[n]$, $1\leq k\leq K$, $P(\mathbf{G}^*[n])\leq P(\hat{\mathbf{G}}[n])$, and $sp(P(\mathbf{G}^*[n]))\leq sp(P(\hat{\mathbf{G}}[n]))$, for $n\geq 0$. Suppose that $\{\hat{\mathbf{G}}[n], n \geq 0\}$ converges to $\hat{\mathbf{G}}$. Then $\hat{\mathbf{G}} \in \Re$. If the Markov chain of interest is positive recurrent or null recurrent, $\mathbf{G}^* = \hat{\mathbf{G}}$ and $sp(P(\mathbf{G}^*)) = sp(P(\hat{\mathbf{G}}))$; otherwise, $1 = sp(P(\mathbf{G}^*)) < sp(P(\hat{\mathbf{G}}))$.

Proof. Note that $\hat{G}(k)[0]\geq G^*(k)[0] = 0$, $1\leq k\leq K$. It can be proved inductively that $\hat{G}(k)[n]\geq G^*(k)[n]$, $1\leq k\leq K$, for all $n\geq 0$. Then the results are obtained by using Corollary 3.3 and Lemma 4.1. This completes the proof.

Based on Lemmas 4.1 and 4.2, the following algorithm is developed for classifying the Markov chain defined in Section 2. Let ε and ε' be small positive numbers (e.g., $\varepsilon=\varepsilon'=10^{-10}$).

Algorithm 4.3

1) Start with $\mathbf{G}^*[0] = \{0, \cdots, 0\}$ and $\hat{\mathbf{G}}[0] = \{I, \cdots, I\}$ (Note that these stochastic matrices in $\hat{\mathbf{G}}[0]$ can be chosen differently).
2) Compute $\mathbf{G}^*[n]$ and $\hat{\mathbf{G}}[n]$ using equation (4.1) respectively.
3) If $\max_{k,i,j}\{(G^*(k)[n+1]-G^*(k)[n])_{i,j}\}<\varepsilon$, stop; Otherwise, go to Step 2.
4) Find Perron-Frobenius eigenvalues $sp(P(\mathbf{G}^*[n]))$ and $sp(P(\hat{\mathbf{G}}[n]))$.

If the matrix $P(\hat{\mathbf{G}}[n])$ is irreducible, we have the following conclusions. If $sp(P(\mathbf{G}^*[n]))\leq sp(P(\hat{\mathbf{G}}[n]))<1-\varepsilon'$, the Markov chain of interest is positive recurrent; if $1-\varepsilon' < sp(P(\mathbf{G}^*[n])) \leq 1 < 1+\varepsilon' < sp(P(\hat{\mathbf{G}}[n]))$, the Markov chain is transient; if $1-\varepsilon' < sp(P(\mathbf{G}^*[n])) \leq 1 \leq sp(P(\hat{\mathbf{G}}[n]))<1+\varepsilon'$, the Markov chain can be considered to be null recurrent. If $A(k, J)=0$ when $|J|>d$, where d (≥ 2) is a fixed positive number, and the matrix $P(\hat{\mathbf{G}}[n])$ has a positive right eigenvector associated with its Perron-Frobenius eigenvalue, we have similar conclusions.

Theoretically, $sp(P(\hat{\mathbf{G}}[n]))$ provides information for a complete classification of the Markov chain of interest, when n is large enough. We bring the set $\mathbf{G}^*[n]$ into the algorithm since its computation is stable. The eigenvalue $sp(P(\mathbf{G}^*[n]))$ can also be used for accuracy check. In using Algorithm 4.3, we have to be cautious about cases such that the sequence $\{\hat{\mathbf{G}}[n], n \geq 0\}$ does not converge. In that case, the algorithm fails to provide reliable information for a classification. We should also be cautious about the null recurrent case.

We are now able to show that Theorem 2.1 does not hold when m>1 and K>1. There are counterexamples for which $sp(P(\mathbf{G}))$<1 but $sp(P)$>1 ($sp(P)$ is defined in Section 2) and counterexamples for which $sp(P(\mathbf{G}))$>1 but $sp(P)$<1. One of them is presented as follows.

Example 4.4 Consider a Markov chain of matrix *M/G/*1 type with a tree structure with m=3, K=2, and transition matrices (only those nonzero matrices):

$$A(1,0)=\begin{pmatrix}0.05 & 0.05 & 0.2\\ 0 & 0 & 0\\ 0 & 0.3 & 0\end{pmatrix},\ A(1,1)=\begin{pmatrix}0.05 & 0.05 & 0.1\\ 0.5 & 0 & 0.1\\ 0 & 0 & 0.2\end{pmatrix},\ A(1,11)=\begin{pmatrix}0.1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0.1\end{pmatrix},\ A(1,12)=\begin{pmatrix}0.3 & 0.1 & 0\\ 0 & 0.4 & 0\\ 0 & 0 & 0.4\end{pmatrix};$$

$$A(2,0)=\begin{pmatrix}0.4 & 0 & 0.1\\ 0.1 & 0.4 & 0.1\\ 0.1 & 0.2 & 0.2\end{pmatrix},\ A(2,21)=\begin{pmatrix}0.1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0.1\end{pmatrix},\ A(2,22)=\begin{pmatrix}0.3 & 0.1 & 0\\ 0 & 0.4 & 0\\ 0 & 0 & 0.4\end{pmatrix}.$$

For this Markov chain, matrices $\{G^*(1), G^*(2)\}$ are stochastic and $sp(P(\mathbf{G})) = 0.991859$. Thus, the Markov chain is positive recurrent, but $sp(P)$=1.011921 > 1. Therefore, $sp(P)$ fails to provide correct information for a classification of the Markov chain.

5. The Discrete Time *MAP/PH/*1 Queue

In this section, we consider a discrete time *MAP/PH/*1 queue with a last-come-first-served general preemptive resume service discipline (LCFS-GPR). This type of queueing system was introduced in Yeung and Alfa [18] (also see Yeung and Sengupta [19]). But the stability conditions of such queueing systems are difficult to find due to the complexity of the LCFS-GPR service discipline. With the computational method developed in Section 4, we are able to analyze the impact of the LCFS-GPR on the stability of the queueing system. Note that the continuous time *MMAP*[*K*]/*PH*[*K*]/1 queue that has multiple types of customers and a LCFS-GPR service discipline can be analyzed in the same way.

The following definition of Markov arrival process (*MAP*) is a discrete time version of the continuous time *MAP* introduced by Neuts ([13]). A discrete time *MAP* is defined by two nonnegative $m \times m$ matrices $\{D_0, D_1\}$, where m is a positive integer. Matrices D_0 and D_1 are nonzero. Let $D = D_0+D_1$. Then the matrix D is called the transition matrix of the underlying Markov chain and it is stochastic. We assume that D is irreducible. Let $\eta(n)$ be the phase of the Markov arrival process at time n, $1 \leq \eta(n) \leq m$.

The service time of a new customer has a discrete time phase-type distribution (PH-distribution) with a matrix representation (α, T), where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_K)$, $\alpha\mathbf{e} = 1$, and T is a $K\times K$ matrix. Let $\mathbf{T}^0 = \mathbf{e} - T\mathbf{e}$. More details about PH-distribution can find in Chapter 2 of Neuts [14].

All customers are served on an LCFS-GPR service discipline. When a customer arrives, it pushes the customer in service (if any) out of the server and starts its own service with a service time (α, T). For the outgoing customer, its current service state is recorded. When the server becomes available to customers in queue, the one who arrived last gets the server. When a customer reenters the server, the distribution of its service time is $((q_{i,1}, \cdots, q_{i,K}), T)$ if it was pushed out of the server in state i, $1\le i\le K$. Let Q be a $K\times K$ matrix with elements $q_{i,j}$. Then the matrix Q is a stochastic matrix and it specifies the service discipline. If $Q = I$, customers are served on a LCFS preemptive resume basis. If $Q = \mathbf{e}\alpha$, customers are served on a LCFS preemptive repeat basis. We shall compare these service disciplines and a few others.

Let $X(n)$ be the string consisting of the states of customers in queue at the beginning of period n (string of integers between 1 and K). For instance, when $X(n) = J = k_1\cdots k_{|J|}$, the service state of the current customer is $k_{|J|}$, the service state of the last customer in queue will be $k_{|J|-1}$, i.e., when this customer reenters the server, its service starts in state $k_{|J|-1}$, and so on. Then it is easy to see that $(X(n), \eta(n))$ is a Markov chain of matrix $M/G/1$ type with a tree structure. The transition blocks of that Markov chain are given by

$$\begin{aligned} &A(k,0) = (\mathbf{T}^0)_k D_0, \quad A(k,j) = (\mathbf{T}^0)_k \alpha_j D_1 + T_{k,j} D_0, \\ &A(k,jl) = \left(\sum_{i=1}^{K} T_{k,i} q_{i,j} \alpha_l \right) D_1, \quad 1 \le k, j, l \le K. \end{aligned} \tag{5.1}$$

Then we can use the algorithm developed in Section 4 to study the impact of the matrix Q on the stability of the queueing system. A numerical example is presented next.

Example 5.1 Consider a $MAP/PH/1$ queue with a LCFS-GPR service discipline with the following arrival process and service time distribution:

$$m = K = 2,\ D_0 = \begin{pmatrix} 0.7 & 0 \\ 0 & 1-\lambda \end{pmatrix},\ D_1 = \begin{pmatrix} 0.07 & 0.23 \\ 0.5\lambda & 0.5\lambda \end{pmatrix},\ \alpha = (0.2, 0.8),\ T = \begin{pmatrix} 0.6 & 0.3 \\ 0.2 & 0.1 \end{pmatrix}.$$

We consider the following three LCFS-GPR service disciplines:

$$Q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q(2) = \begin{pmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{pmatrix}, \quad Q(3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \alpha.$$

The Perron-Frobenius eigenvalues $sp(P(\mathbf{G}))$ corresponding to the above matrix Qs, as a function of the arrival rate λ, are shown in Table 5.1. Rows 2, 3, and 4 in Table 5.1 show $sp(P(\mathbf{G}))$ for different Q and λ.

Table 5.1 Perron-Frobenius eigenvalues $sp(P(\mathbf{G}))$ for Example 5.1

	λ=0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$Q(1)$	0.792	0.856	0.907	0.949	0.983	1.010	1.031	1.047	1.059	1.065
$Q(2)$	0.768	0.819	0.863	0.904	0.933	0.960	0.982	0.999	1.013	1.020
$Q(3)$	0.758	0.805	0.846	0.882	0.914	0.941	0.963	0.981	0.995	1.002

According to Table 5.1, the difference between $Q(1)$ and $Q(3)$ is clear. For $Q(1)$, $sp(P(\mathbf{G}))$ becomes larger than 1 when λ is between 0.5 and 0.6. For $Q(3)$, $sp(P(\mathbf{G}))$ becomes larger than 1 when λ is between 0.9 and 0.95. Although $sp(P(\mathbf{G}))$ is not the traffic intensity of the corresponding queueing system, its closeness to 1 does show the instability of the queueing system. Thus, Table 5.1 shows that the queueing system goes from stable to unstable when λ goes from 0.5 to 0.6 for $Q(1)$. But the queueing system is still stable for $Q(3)$ until λ is close to 0.95. This implies that the LCFS preemptive repeat service discipline allows a larger utilization than the resume one, in terms of the stability of the queueing system. Partially, the reason is that the service process has a larger probability to complete in state 1 than in that 2 (see the definition of the matrix T), i.e., the decreasing failure rate property (DFR).

The matrix $Q(1)$ represents the LCFS preemptive resume service discipline and $Q(3)$ represents the LCFS preemptive repeat service discipline. The matrix $Q(2)$ represents some service discipline "between" $Q(1)$ and $Q(3)$. Table 5.1 shows that the matrix Q has a huge impact on the stability of the queueing system. The implications of the numerical results need further study.

6. Sufficient Conditions for Positive Recurrence and Transience

The computational method developed in Sections 3 and 4 can be used to classify a Markov chain of matrix *M/G/*1 type with a tree structure computationally. But it does not lead to simple conditions needed for queueing applications. This section shows that Theorem 2.1 holds for $m>1$ and $K>1$ if additional conditions are imposed on matrices $\{A(k), A^{*(1)}(k, j), 1\le k, j\le K\}$ (defined in Section 2). The results can be used to solve the classification problem of some queueing systems with multiple types of customers.

First, we prove a simple sufficient condition for positive recurrence and a simple sufficient condition for transience. Let $\mathbf{z} = (z_1, \cdots, z_K) \in R_+^K$ and define, for $1 \le k \le K$,

$$A^*(k,\mathbf{z}) = \sum_{J\in\aleph} \mathbf{z}^J A(k,J), \quad B^*(\mathbf{z}) = \sum_{J\in\aleph} \mathbf{z}^J B(J), \tag{6.1}$$

where $\mathbf{z}^J = z_{j_1} \cdots z_{j_{|J|}}$ for $J \in \aleph$ and $J \neq 0$, and $\mathbf{z}^J = 1$ if $J=0$.

Lemma 6.1 If there exists an $m\times 1$ positive vector $\mathbf{u}^*$ such that $A^*(k, \mathbf{z})\mathbf{u}^* < z_k\mathbf{u}^*$, i.e., every element of $A^*(k, \mathbf{z})\mathbf{u}^*$ is strictly smaller than its counterpart in $z_k\mathbf{u}^*$, and $B^*(\mathbf{z})\mathbf{u}^* < \infty$ for some $\mathbf{z}$ satisfying $1 < z_k < \infty$, $1 \le k \le K$, then the Markov chain defined in Section 2 is positive recurrent.

Proof. The proof of this lemma is based on the mean drift method. Let

$$\varepsilon = \min_{1\le k\le K} \{ \min_{1\le i\le m} \{(z_k\mathbf{u}^* - A^*(k,\mathbf{z}))_i\}\}. \tag{6.2}$$

By the assumption, ε is positive. Define a Lyapunov function $\mathbf{f}(J) = \mathbf{z}^J\mathbf{u}^*$ for $J \in \aleph$. Since $1 < z_k < \infty$ for $1 \le k \le K$, $\mathbf{f}(J) \to \infty$ when $|J| \to \infty$. Also it has $\mathbf{f}(J+H) = \mathbf{z}^J\mathbf{z}^H\mathbf{u}^*$. Then we have, for all $J \in \aleph$ and $1 \le k \le K$,

$$\begin{aligned} &\mathbf{E}[\mathbf{f}(X_{n+1}) - \mathbf{f}(X_n) \mid X_n = J+k] = \sum_{H\in\aleph} A(J+k,H)\mathbf{f}(J+H) - \mathbf{f}(J+k) \\ &= \sum_{H\in\aleph} A(k,H)\mathbf{f}(J+H) - \mathbf{f}(J+k) = \mathbf{z}^J[A^*(k,\mathbf{z}) - z_k I]\mathbf{u}^* < -\varepsilon\mathbf{e}. \end{aligned} \tag{6.3}$$

For $J=0$, it can be proved that $\mathbf{E}[\mathbf{f}(X_{n+1}) - \mathbf{f}(X_n) | X_n=0] = B^*(\mathbf{z})\mathbf{u}^* - I$. Since $B^*(\mathbf{z})\mathbf{u}^* < \infty$, the Markov chain of interest is positive recurrent by Theorem 2.2.3 in Fayolle, et al. [2] or Foster's criterion. This completes the proof of Lemma 6.1.

Lemma 6.2 If there exists an $m\times 1$ positive vector $\mathbf{u}^*$ such that $A^*(k, \mathbf{z})\mathbf{u}^* \le z_k\mathbf{u}^*$ for some $\mathbf{z}$ satisfying $0 < z_k < 1$ for at least one k, $1 \le k \le K$, then the Markov chain defined in Section 2 is transient.

Proof. The proof of this lemma is based on the mean drift method. The Lyapunov function is defined as $\mathbf{f}(J) = \mathbf{z}^J\mathbf{u}^*$ for $J \in \aleph$. We have, for all $J \in \aleph$ and $1 \le k \le K$,

$$\mathbf{E}[\mathbf{f}(X_{n+1}) - \mathbf{f}(X_n) \mid X_n = J+k] = \mathbf{z}^J[A^*(k,\mathbf{z}) - z_k I]\mathbf{u}^* \le 0. \tag{6.4}$$

Suppose that $0 < z_{k^0} < 1$ for k^0. Then $\mathbf{f}(J) \to 0 < \mathbf{f}(0) = \mathbf{u}^*$, when $|J| \to \infty$ for $J = k^0 \ldots k^0$. Thus, the Markov chain of interest is transient by Theorem 2.2.2 in Fayolle, et al. [2]. This completes the proof of Lemma 6.2.

To use Lemmas 6.1 and 6.2, we need to find the vector $\mathbf{u}^*$. The difficulty in finding such a vector $\mathbf{u}^*$ is that all K inequalities - $A^*(k, \mathbf{z})\mathbf{u}^* < z_k \mathbf{u}^*$ or $A^*(k, \mathbf{z})\mathbf{u}^* \le z_k \mathbf{u}^*$ - must be satisfied simultaneously. In the rest of this section, such a vector $\mathbf{u}^*$ is found under certain conditions on P and $\{A(k), A^{*(1)}(k, j), 1 \le k, j \le K\}$.

To find the vector $\mathbf{u}^*$, we start with the right eigenvector of $sp(P)$. Let $\mathbf{b} = (b_1, \cdots, b_K) \in R_+^K$ be an right eigenvector corresponding to the Perron-Frobenius eigenvalue of the matrix P defined by equation (2.2), i.e., $P\mathbf{b} = sp(P)\mathbf{b}$. Assume that P is irreducible. Then $\mathbf{b}$ is unique and positive. According to HE [7], for m=1, the vector $\mathbf{b}$ shows a direction in which we can compare the movement of the Markov chain towards or away from the root node. For m=1, if the Markov chain moves towards the root node (strictly) faster than moving away from the root node in the direction $\mathbf{b}$, the Markov chain is positive recurrent. If the Markov chain moves away from the root node (strictly) faster than moving towards the root node in the direction $\mathbf{b}$, the Markov chain is transient. It is natural to conjecture that this is true for the case with m>1. But Example 4.4 gives a counterexample. Nonetheless, we still believe that the direction $\mathbf{b}$ is an important direction. In fact, we shall find the vector $\mathbf{u}^*$ based on the vector $\mathbf{b}$.

If m=1, choose $\mathbf{u}^* = 1$. It is immediate to see that the Perron-Frobenius eigenvalue of the matrix P alone determines the classification of the Markov chain. For m>1, the solution of the vector $\mathbf{u}^*$ is not clear. In the rest of this section, we construct such a vector $\mathbf{u}^*$ associated with the vector $\mathbf{b}$ of the matrix P.

We assume that functions defined in equation (6.1) are analytic in an open ball that includes point $\mathbf{e}^T$ in R_+^K, or equivalently these functions are analytic at $\mathbf{z} = \mathbf{e}^T$. Let $z_k(t) = 1 + b_k t$, for $1 \le k \le K$. Then $\mathbf{z}(t) \equiv (z_1(t), \cdots, z_K(t)) = \mathbf{e}^T + t\mathbf{b}^T$. To choose the vector $\mathbf{u}^*$, we shall only consider the vector $\mathbf{z}(t)$ for small t. Let $\xi(k, t)$ be the Perron-Frobenius eigenvalue of the matrix $A^*(k, \mathbf{z}(t))$, $\theta(k, t)$ the corresponding left eigenvector, and $\mathbf{u}(k, t)$ the corresponding right eigenvector. It can be proved that $\xi(k, t)$ is an analytic function with respect to t and $\theta(k, t)$ and $\mathbf{u}(k, t)$ can be chosen as analytic functions with respect to t. Vectors $\theta(k, t)$ and $\mathbf{u}(k, t)$ are normalized by $\theta(k, t)\mathbf{e} = \mathbf{e}$ and $\theta(k, t)\mathbf{u}(k, t) = 1$. Then $\theta(k, 0) = \theta(k)$ and $\mathbf{u}(k, 0) = \mathbf{e}$. Then we have

$$A^{*(1)}(k, \mathbf{b}) \equiv \left. \frac{dA^*(k, \mathbf{z}(t))}{dt} \right|_{t=0} = \sum_{k=1}^{K} b_k A^{*(1)}(k, j) \tag{6.5}$$

Similar to Lemma 1.3.3 in Neuts [14], it can be proved that, for $1\le k\le K$,

$$\xi^{(1)}(k,0)=\left.\frac{d\xi(k,t)}{dt}\right|_{t=0}=\lim_{t\to 0}\theta(k,t)\frac{dA^{*}(k,\mathbf{z}(t))}{dt}\mathbf{u}(k,t)$$
$$=\theta(k)A^{*(1)}(k,\mathbf{b})\mathbf{e}=\sum_{k=1}^{K}p(k,j)b_j=(P\mathbf{b})_k=sp(P)b_k. \tag{6.6}$$

Now, we are ready to state and prove a weaker version of Theorem 2.1 for $m>1$.

Theorem 6.3 Suppose that the Markov chain defined in Section 2 satisfies the following additional conditions:

Condition A: the matrix P is irreducible;
Condition B: $\{A(k), 1\le k\le K\}$ are irreducible;
Condition C: $A^{*(1)}(k,\mathbf{b})\mathbf{e}=(I-A(k))\mathbf{w}+sp(P)b_k\mathbf{e}$, where $\mathbf{w}$ is an $m\times 1$ vector and finite, for $1\le k\le K$.

Then 1) the Markov chain is positive recurrent if $sp(P) < 1$; 2) the Markov chain is transient if $sp(P) > 1$.

Proof. According to Lemmas 6.1 and 6.2, to prove the results, we only have to find the vector $\mathbf{u}^*$. For that purpose, expand functions $\xi(k, t)$, $\mathbf{u}(k, t)$, and $A^*(k, t)$ at t=0 as $\xi(k, t) = 1 + tb_k sp(P) + O(t^2)$, $\mathbf{u}(k, t) = \mathbf{e} + t\,\mathbf{u}^{(1)}(k) + O(t^2)$, and $A^*(k, \mathbf{z}(t)) = A(k) + tA^{*(1)}(k, \mathbf{b}) + O(t^2)$, respectively. The vector $\mathbf{u}^{(1)}(k)$ can be found as follows: $1\le k\le K$,

$$\begin{aligned}
&A^{*}(k,\mathbf{z}(t))\mathbf{u}(k,t)=\xi(k,t)\mathbf{u}(k,t)\\
\Rightarrow\quad&[A(k)+tA^{*(1)}(k,\mathbf{b})][\mathbf{e}+t\mathbf{u}^{(1)}(k)]=[1+t\xi^{(1)}(k,0)][\mathbf{e}+t\mathbf{u}^{(1)}(k)]+O(t^2)\\
\overset{t\to 0}{\Rightarrow}\quad&A^{*(1)}(k,\mathbf{b})\mathbf{e}+A(k)\mathbf{u}^{(1)}(k)=b_k sp(P)\mathbf{e}+\mathbf{u}^{(1)}(k)\\
\Rightarrow\quad&(I-A(k))\mathbf{u}^{(1)}(k)=-b_k sp(P)\mathbf{e}+A^{*(1)}(k,\mathbf{b})\mathbf{e}\\
\Rightarrow\quad&\mathbf{u}^{(1)}(k)=[\theta(k)\mathbf{u}^{(1)}(k)-b_k sp(P)]\mathbf{e}+(I-A(k)+\mathbf{e}\theta(k))^{-1}A^{*(1)}(k,\mathbf{b})\mathbf{e}\\
&\qquad\quad=[\theta(k)\mathbf{u}^{(1)}(k)-\theta(k)\mathbf{w}]\mathbf{e}+\mathbf{w}.
\end{aligned}\tag{6.7}$$

Note that $A(k)\mathbf{e} = \mathbf{e}$ and Condition C is used in the last equality. The invertibility of the matrix $I-A(k)+\mathbf{e}\theta(k)$ can be proved routinely since $A(k)$ is stochastic and irreducible. Now we introduce a vector $\mathbf{u}(t)$ as $\mathbf{u}(t)=\mathbf{e}+t\sum_{k=1}^{K}\omega_k\mathbf{u}^{(1)}(k)$, where $\{\omega_k, 1\le k\le K\}$ are K real numbers. Then

$$
\begin{aligned}
A^*(k,\mathbf{z}(t))\mathbf{u}(t) &= \mathbf{e}+t\,[A^{*(1)}(k,\mathbf{b})\mathbf{e}+A(k)\sum_{j=1}^{K}\omega_j\mathbf{u}^{(1)}(j)]+O(t^2) \\
&= \mathbf{e}+t\left\{A^{*(1)}(k,\mathbf{b})\mathbf{e}+\sum_{j=1}^{K}\omega_j[\theta(j)\mathbf{u}^{(1)}(j)-\theta(j)\mathbf{w}]\mathbf{e}+A(k)(\sum_{j=1}^{K}\omega_j)\mathbf{w}\right\}+O(t^2) \\
&= \mathbf{e}+t\left\{sp(P)b_k\mathbf{e}+\sum_{j=1}^{K}\omega_j[(\theta(j)\mathbf{u}^{(1)}(j)-\theta(j)\mathbf{w})\mathbf{e}+\mathbf{w}]\right\} \\
&\quad +t\left\{(I-A(k))\mathbf{w}+A(k)(\sum_{j=1}^{K}\omega_j)\mathbf{w}-(\sum_{j=1}^{K}\omega_j)\mathbf{w}\right\}+O(t^2) \\
&= \mathbf{e}+t\left\{sp(P)b_k\mathbf{e}+\sum_{j=1}^{K}\omega_j\mathbf{u}^{(1)}(j)\right\}+t(1-\sum_{j=1}^{K}\omega_j)(I-A(k))\mathbf{w}+O(t^2)
\end{aligned}
\tag{6.8}
$$

We choose $\sum_{k=1}^{K}\omega_k=1$. If $sp(P) < 1$, choose t to be positive and small enough so that the absolute value of the sum of all terms associated with t^2 is smaller than $t(1-sp(P))/2$. Then

$$
\begin{aligned}
&A^*(k,\mathbf{z}(t))\mathbf{u}(t) \\
&\le \mathbf{e}+t[\sum_{j=1}^{K}\omega_j(j)\mathbf{u}^{(1)}(j)+b_k\frac{(1+sp(P))}{2}\mathbf{e}]+t^2b_k\frac{(1+sp(P))}{2}\sum_{j=1}^{K}\omega_j\mathbf{u}^{(1)}(j) \\
&= [1+tb_k\frac{(1+sp(P))}{2}][\mathbf{e}+t\sum_{j=1}^{K}\omega_j\mathbf{u}^{(1)}(j)]=[1+tb_k\frac{(1+sp(P))}{2}]\mathbf{u}(t) \\
&= (1+tb_k)\mathbf{u}(t)-tb_k\frac{(1-sp(P))}{2}\mathbf{u}(t)<z_k(t)\mathbf{u}(t).
\end{aligned}
\tag{6.9}
$$

Choose $t>0$ and small enough. Then $z_k(t) = 1 + b_k t > 1$ and $\mathbf{u}(t)$ is positive. Let $\mathbf{u}^* = \mathbf{u}(t)$. Since $B^*(\mathbf{z})$ is analytic at $\mathbf{z} = \mathbf{e}^T$, $B^*(\mathbf{z}(t))\mathbf{u}(t)$ is finite. By Lemma 6.1, the Markov chain is positive recurrent.

If $sp(P) > 1$, choose t to be ***negative*** and small enough so that the absolute value of all terms associated with t^2 is smaller than $-t(sp(P)-1)/2$. Similar to equation (6.9), $A^*(k,\mathbf{z}(t)) < z_k\mathbf{u}(t)$, for $1\le k\le K$. Since $t<0$ and small enough in absolute value, $0<z_k(t)=1 + b_k t<1$, $1\le k\le K$. Let $\mathbf{u}^* = \mathbf{u}(t)$. By Lemma 6.2, the Markov chain is transient. Note that the selection of $\{\omega_j\}$ implies that the vector $\mathbf{u}^*$ can be chosen differently. This completes the proof of Theorem 6.3.

The additional conditions in Theorem 6.3, especially Condition C, are restrictive. But Theorem 6.3 is still useful for some queueing applications in Section 7 and 8. The Taylor expansion idea used in the proof of Theorem 6.3 shall be used again in Section 9.

7. The Discrete Time *MMAP*[*K*]/*G*[*K*]/1 Queues

The queueing model considered in this section is the discrete time version of the continuous time *MMAP*[*K*]/*G*[*K*]/1 queue introduced in HE [5] whose classification problem will be studied in Section 8. First, we give a detailed description of the discrete time Markov arrival process with marked transitions (*MMAP*[*K*]).

The following definition of *MMAP*[*K*] is a discrete time version of the continuous time *MMAP*[*K*] introduced by Marcel Neuts (see HE and Neuts [9]). The *MMAP*[*K*] was also introduced in Asmussen and Koole [1]. A discrete time Markov arrival process with marked transitions is defined by a set of nonnegative $m \times m$ matrices $\{D_J, J \in \aleph\}$, where m is a positive integer. The matrix D_J is the (matrix) arrival rate of a batch J, $= k_1 k_2 \ldots k_{|J|}$, i.e., the arrival rate of a batch with a type k_1 customer, a type k_2 customer, …, and a type $k_{|J|}$ customer. The matrix D_0 is not stochastic. Let

$$D = D_0 + \sum_{J \in \aleph,\, J \neq 0} D_J, \quad D^*(\mathbf{z}) = \sum_{J \in \aleph} \mathbf{z}^J D_J. \tag{7.1}$$

Then the matrix D is called the transition matrix of the underlying Markov chain and it is stochastic. We assume that D is irreducible. We also assume that matrix $D^*(\mathbf{z})$ is analytic at $\mathbf{z} = \mathbf{e}^T$. Let $\eta(n)$ be the phase of the underlying Markov process at time n, $1 \le \eta(n) \le m$. Let θ be the left invariant vector of the matrix D, i.e., $\theta D = \theta$ and $\theta \mathbf{e} = 1$. The average number of type k customers who arrived in a period is given by

$$\lambda_k = \theta \sum_{n=1}^{\infty} n \Big(\sum_{J:\, N(J,k)=n} D_J \Big) \mathbf{e} = \theta \sum_{J \in \aleph} D_J N(J,k) \mathbf{e}, \ \ 1 \le k \le K. \tag{7.2}$$

When a batch of customers arrive in the queueing system, they join the queue according to their order in the batch. The nth customer in J is a type k_n customer, $1 \le n \le |J|$. Thus, the queue of such a queueing system is represented by a string of integers that are the types of the customers in these positions. For example (for K=2), let $q(t)$ be the queue string at time t. Then $q(t) = 122$ implies that there are 3 customers waiting in the system at time t: the customer who arrived first is of type 1; the customer who arrived second is of type 2; and the customer who arrived last is of type 2.

The service time of a type k customer has a discrete density function $f_k(n)$ with z-transform $f^*_k(z)$ and mean $1/\mu_k = \sum_n n f_k(n)$, $1 \le k \le K$. We assume that $f^*_k(z)$ is analytic at z=1.

We consider the discrete time *MMAP*[*K*]/*G*[*K*]/1 queue with a work conserving service discipline (including FCFS, LCFS non-preemptive, LCFS preemptive resume, priority with non-preemption, priority with preemptive resume, processor sharing, etc.). Since the classification problem of these queueing systems is the same, we shall focus on the queue with a LCFS non-preemptive service discipline. That is: once a customer enters the server, its service will not be interrupted. Upon the completion of its service, a customer leaves the queueing system immediately. The customer in queue who arrived last enters the server. Denote by $\rho = \lambda_1/\mu_1 + \cdots + \lambda_K/\mu_K$, the traffic intensity of the queueing system.

We observe the queueing system at departure epochs. Let $q(n)$ be the queue string right after the nth departure; $\eta(n)$ the state of the underlying chain of the arrival process right after the nth departure. It is easy to see that $(q(n), \eta(n))$ is a Markov chain of matrix *M/G/*1 type with a tree structure.

Theorem 7.1 The Markov chain $\{(q(n), \eta(n)), n\geq 0\}$ of a discrete time $MMAP[K]/G[K]/1$ queue with a work conserving service discipline is positive recurrent if and only if $\rho<1$; transient if $\rho>1$.

Proof. Let $N(k)$ be the number of batches that arrived during the service period of a type k customer. Let $J(k)$ be the string consisting of all customers who arrived during the service period of a type k customer. Then $J(k) = J_1+\cdots+J_{N(k)}$, where J_i is the string of customers who arrived in the ith period of the service time. The (matrix) distribution of $J(k)$ is given as, for $H\in\aleph$,

$$A(k,J) = (\mathbf{P}\{J(k)=H\}) = \sum_{n=1}^{\infty} f_k(n)\left(\sum_{\{J_1,\cdots,J_n\}:\, J_1+\cdots+J_n=H} D_{J_1}\cdots D_{J_n}\right). \tag{7.3}$$

The transition of $\{q(n), n\geq 0\}$ is given by $q(n) = (J+k) - k + J(k) = J + J(k)$, given that $q(n) = J+k$, and $q(n+1)$ is an arbitrary batch if $q(n) = 0$. Transition matrices $\{A(k, J), 1\leq k\leq K, J\in\aleph\}$ of the Markov chain $\{(q(n), \eta(n)), n\geq 0\}$ are given by equation (7.3). The transition matrices $\{B(J), J\in\aleph\}$ can be written but omitted since they are not used directly. By equation (7.3), we obtain

$$A^*(k,\mathbf{z}) = \sum_{J\in\aleph} \mathbf{z}^J A(k,J) = f_k^*(D^*(\mathbf{z})), \quad A(k) = f_k^*(D). \tag{7.4}$$

By equation (7.4), it is easy to see that θ is the left invariant vector of $A(k)$, $1\leq k\leq K$. To find the matrix P, we have, for $1\leq k, j\leq K$,

$$\begin{aligned} A^{*(1)}(k,j) &= \sum_{H\in\aleph} N(H,j)A(k,H) = \sum_{n=1}^{\infty} f_k(n)\left(\sum_{H\in\aleph} N(H,j) \sum_{\{J_1,\cdots,J_n\}:\, J_1+\cdots+J_n=H} D_{J_1}\cdots D_{J_n}\right) \\ &= \sum_{n=1}^{\infty} f_k(n)\left(\sum_{i=0}^{n-1} D^i \Big[\sum_{J\in\aleph} N(J,j)D_J\Big] D^{n-1-i}\right); \end{aligned} \tag{7.5}$$

$$p(k,j) = \theta A^{*(1)}(k,j)\mathbf{e} = \sum_{n=1}^{\infty} f_k(n)\left(n\theta\sum_{J\in\aleph} N(J,j)D_J\mathbf{e}\right) = \frac{\lambda_j}{\mu_k}, \quad 1\leq k, j\leq K.$$

Thus, we obtain the matrix P explicitly in terms of the arrival rates and service rates. Intuitive, equation (7.5) makes sense since a work conserving service discipline is applied. It is easy to obtain that $sp(P) = \rho$ and the corresponding right eigenvector is $\mathbf{b} = (1/\mu_1, \cdots, 1/\mu_K)^T$. Let $\mathbf{z}(t) = \mathbf{e}^T + t\mathbf{b}^T$. Then we have the following equalities:

$$\theta \frac{dD^*(\mathbf{z}(t))}{dt}\bigg|_{t=0} \mathbf{e} = \theta \sum_{k=1}^{K} b_k \frac{\partial D^*(\mathbf{z})}{\partial z_k}\bigg|_{\mathbf{z}=(1,\cdots,1)} \mathbf{e} = \sum_{k=1}^{K} b_k \lambda_k = \rho = sp(P). \tag{7.6}$$

$$\begin{aligned} A^{*(1)}(k,\mathbf{b})\mathbf{e} &= \frac{dA^*(k,\mathbf{z}(t))}{dt}\bigg|_{t=0} \mathbf{e} = \sum_{n=1}^{\infty} f_k(n)[\sum_{i=0}^{n-1} D^i \frac{dD^*(\mathbf{z}(t))}{dt}\bigg|_{t=0} \mathbf{e}] \\ &= \sum_{n=1}^{\infty} f_k(n)[(I-D^n)(I-D+\mathbf{e}\theta)^{-1} + \mathbf{e}\theta n] \frac{dD^*(\mathbf{z}(t))}{dt}\bigg|_{t=0} \mathbf{e} \\ &\equiv (I-A(k))\mathbf{w} + b_k sp(P)\mathbf{e}, \end{aligned} \tag{7.7}$$

where $D\mathbf{e}=\mathbf{e}$ is used. The invertibility of matrix $I-D+\mathbf{e}\theta$ can be proved routinely since the matrix D is an irreducible stochastic matrix. Therefore, conditions A, B, and C in Theorem 6.3 are satisfied. Therefore, the Markov chain of interest is positive recurrent if $\rho<1$, transient if $\rho>1$.

We still need to prove that $\rho<1$ is necessary for positive recurrence. The idea is to look at the total work-load in the queueing system at an arbitrary time. The following approach was used by Takine and Hasegawa [16] for continuous time cases. Let $V_i(n, x)$ be probability that the total work-load in the queueing system at the end of the nth period is x and the state of the underlying Markov chain is i, i.e., the distribution of the virtual waiting time. Note that the initial state at n=0 is not considered explicitly since it is not important for our proof. Let $\mathbf{V}(n, x) = (V_1(n, x), \cdots, V_m(n, x))$. Then the following difference equation holds:

$$\begin{aligned} &\mathbf{V}(n+1,0) = \mathbf{V}(n,1)D_0 + \mathbf{V}(n,0)D_0; \\ &\mathbf{V}(n+1,x) = \mathbf{V}(n,x+1)D_0 + \sum_{J\in\aleph,\, J\neq 0} [\sum_{l=1}^{x} \mathbf{V}(n,x+1-l)D_J f(J,l) + \mathbf{V}(n,0)D_J f(J,x)],\ x>0, \end{aligned}$$

(7.8)

where $f(J, x)$ is the joint density function of $f_{k_1}(x), f_{k_2}(x), \cdots,$ and $f_{k_{|J|}}(x)$. Let $\mathbf{V}^*(n, z)$ be the z-transform of the vector $\mathbf{V}(n, x)$ with respect to x. It is easy to obtain

$$\mathbf{V}^*(n+1,z) = [\mathbf{V}^*(n,z)-\mathbf{V}(n,0)]D_0/z+\mathbf{V}(n,0)D_0$$
$$+\sum_{J>0}[\mathbf{V}^*(n,z)-\mathbf{V}(n,0)]D_J f^*(J,z)/z+\mathbf{V}(n,0)\sum_{J>0}D_J f^*(J,z) \quad (7.9)$$
$$= \mathbf{V}(n,0)(1-1/z)[D_0+\sum_{J\in\aleph,\, J\neq 0}D_J f^*(J,z)]+\mathbf{V}^*(n,z)[D_0+\sum_{J\in\aleph,\, J\neq 0}D_J f^*(J,z)]/z,$$

where $f^*(J, z)$ is the z-transform of $f(J, x)$. If the queueing system is positive recurrent, the stationary distribution of the work-load exists. Denote by $\mathbf{V}^*(z)$ the limit of $\{\mathbf{V}^*(n, z), n\geq 0\}$ and $\mathbf{y}(0)$ the limit of $\{\mathbf{V}(n,0), n\geq 0\}$. Then we have

$$\mathbf{V}^*(z) = \mathbf{y}(0)(z-1)[D_0+\sum_{J\in\aleph,\, J\neq 0}D_J f^*(J,z)][zI-D_0-\sum_{J\in\aleph,\, J\neq 0}D_J f^*(J,z)]^{-1}. \quad (7.10)$$

Let $\tau(z)$ be the Perron-Frobenius eigenvalue of matrix $D_0+\sum_{J>0} D_J f^*(J, z)$. Let $\theta(z)$ be the left eigenvalue of $\tau(z)$ and $\mathbf{e}(z)$ the right eigenvalue of $\tau(z)$ with $\theta(z)\mathbf{e} = \theta(z)\mathbf{e}(z) = 1$. Since D is irreducible, it is easy to see that $\mathbf{e}(1) = \mathbf{e}$ and $\theta(1) = \theta$. Similar to HE [5], it can be proved that

$$\tau^{(1)}(1) = \left.\frac{d\tau(z)}{dz}\right|_{z=1} = \theta\left.\frac{d(D_0+\sum_{J\in\aleph,\, J\neq 0}D_J f^*(J,z))}{dz}\right|_{z=1}\mathbf{e} = \rho.$$

$$1 = \mathbf{V}^*(1)\mathbf{e} = \lim_{z\to 1}\mathbf{y}(0)[D_0+\sum_{J\in\aleph,\, J\neq 0}D_J f^*(J,z)]\left[\frac{zI-D_0-\sum_{J\in\aleph,\, J\neq 0}D_J f^*(J,z)}{z-1}\right]^{-1}\mathbf{e} \quad (7.11)$$
$$= \mathbf{y}(0)D\mathbf{e}\frac{\theta\mathbf{e}}{1-\rho}.$$

The Jordan canonical form of matrix is used to derive the last equality in equation (7.11). This leads to $\mathbf{y}(0)\mathbf{e} = 1-\rho > 0$ if the Markov chain is positive recurrent. This implies $\rho<1$.

Therefore, the Markov chain or the queueing system is positive recurrent if and only if $\rho<1$. This completes the proof of Theorem 7.1.

8. The Continuous Time *MMAP*[*K*]/*G*[*K*]/1 Queues

This section considers the continuous counterpart of the model studied in Section 7. Since some of the derivations and calculations are parallel to that of Section 7, details are omitted.

The Markov arrival process with marked arrivals ($MMAP[K]$) is represented by matrices $\{D_0, D_J, J>0\}$, where D_0 is a matrix with negative diagonal elements and nonnegative off-diagonal elements and $\{D_J, J>0\}$ are nonnegative matrices. Matrix $D = \sum_J D_J$ is an infinitesimal generator of the underlying Markov process. We assume that D is irreducible and matrix $D^*(\mathbf{z})$ is analytic at $\mathbf{z}=\mathbf{e}^{\mathrm{T}}$. Let θ be the stationary distribution of matrix D, i.e., $\theta D = 0$ and $\theta\mathbf{e} = 1$. The service time of a type k customer has a distribution function $F_k(x)$ with Laplace Stieltjes transform $f^*{}_k(s)$ and mean $1/\mu_k$. We assume that $f^*{}_k(s)$ is analytic at s=0. Again, the traffic intensity is defined as $\rho = \lambda_1 / \mu_1 + \cdots + \lambda_K / \mu_K$.

Again, we focus on the queue with a LCFS non-preemptive service discipline. We observe the queueing system at departure epochs. Let $q(n)$ be the queue string right after the nth departure; $\eta(n)$ is the state of the underlying process of the arrival process right after the nth departure. Then $(q(n), \eta(n))$ is a Markov chain of matrix $M/G/1$ type with a tree structure.

Theorem 8.1 The Markov chain $(q(n), \eta(n))$ or the queueing system is positive recurrent if and only if $\rho<1$, transient if $\rho>1$.

Proof. The continuous case is simpler since arrivals and service completions cannot occur at the same time. The z-transform of transition matrices can be obtained as:

$$\begin{aligned} A^*(k,\mathbf{z}) &= \sum_{J\in\aleph} \mathbf{z}^J A(k,J) = \int_0^\infty F_k(dx)\exp\{xD^*(\mathbf{z})\}, \quad 1\le k\le K; \\ A(k) &= \int_0^\infty F_k(dx)\exp\{xD\}, \quad 1\le k\le K. \end{aligned} \tag{8.1}$$

Similar to equation (7.5), the matrix P defined in Section 2 is obtained as

$$p(k,j) = \frac{\lambda_j}{\mu_k}, \quad 1\le k, j\le K. \tag{8.2}$$

Therefore, the matrix P is simple and $sp(P) = \rho$. The right eigenvector corresponding to $sp(P)$ is $\mathbf{b}$ with $b_k=1/\mu_k$. Let $z_k(t) = 1+b_k t$, for $1\le k\le K$. It is easy to prove

$$\begin{aligned} A^{*(1)}(k,\mathbf{b})\mathbf{e} &= \left.\frac{dA^*(k,\mathbf{z}(t))}{dt}\right|_{t=0}\mathbf{e} = \int_0^\infty F_k(dx)\sum_{n=1}^\infty \frac{x^n}{n!}D^{n-1}\left.\frac{dD^*(\mathbf{z}(t))}{dt}\right|_{t=0}\mathbf{e} \\ &= \int_0^\infty F_k(dx)[(\exp\{xD\}-I)(D+\mathbf{e}\theta)^{-1}+\mathbf{e}\theta x]\left.\frac{dD^*(\mathbf{z}(t))}{dt}\right|_{t=0}\mathbf{e} \\ &\equiv (I-A(k))\mathbf{w} + b_k sp(P)\mathbf{e}, \end{aligned} \tag{8.3}$$

where $D\mathbf{e}=0$ is used. By Theorem 6.3, the Markov chain is positive recurrent if $\rho<1$, transient if $\rho>1$. It has been proved in HE [5] that $\rho<1$ is necessary for positive recurrence of the Markov chain. The proof of necessity is similar to the discrete case. Let $\mathbf{V}^*(s)$ be the Laplace Stieltjes transform of the virtual waiting time in steady state, then we have

$$\mathbf{V}^*(s) = -s\mathbf{y}(0)[sI + D_0 + \sum_{J\in\aleph,\, J\neq 0} D_J f^*(J,s)]^{-1}, \tag{8.4}$$

where $f^*(J, s)$ is the Laplace Stieltjes transform of the sum of the service times of all customers in batch J. Using this result, we can prove that $\mathbf{y}(0)\mathbf{e} = 1-\rho > 0$ if the Markov chain is positive recurrent, which implies that $\rho<1$.

Therefore, the Markov chain or the queueing system is positive recurrent if and only if $\rho<1$, transient if $\rho>1$. This completes the proof of Theorem 8.1.

Theorem 8.1 applies to the *MMAP*[*K*]/*G*[*K*]/1 queue with a FCFS service discipline, the *MMAP*[*K*]/*PH*[*K*]/1 queue with a LCFS non-preemptive service discipline, and *MMAP*[*K*]/*G*[*K*]/1 queue with a LCFS preemptive resume service discipline studied in HE [5, 6], and HE and Alfa [8], respectively.

9. The Continuous Time *MMAP*[*K*]/*PH*[*K*]/*S* Queues

This section considers a multiple server queueing system with a Markov arrival process *MMAP*[*K*] and phase-type service times. The service times of different types of customers may have different distribution functions. We consider work conserving service disciplines. To solve the classification problem, we only consider the queue in which all types of customers are served on a LCFS non-preemptive basis. Some notations introduced in previous sections are used.

The arrival process of this queueing system is the same as that of Section 8. There are S identical servers in the queueing system. The service times of type k customers have a common continuous time phase-type distribution (*PH*-distribution) with a matrix representation (α_k, T_k), where α_k is an m_k-dimension vector and T_k is an $m_k\times m_k$ matrix. Let $\mathbf{T}^0{}_k = -T_k\mathbf{e}$. Then $-T_k{}^{-1}\mathbf{T}^0{}_k = \mathbf{e}$. The mean service time is given by $1/\mu_k = -\alpha_k T^{-1}{}_k\mathbf{e}$. Then μ_k is the average service rate of type k customers. Let π_k be the stationary distribution of $T_k+\mathbf{T}^0{}_k\,\alpha_k$. Then $1/\mu_k = \pi_k\mathbf{T}^0{}_k$. More details about *PH*-distribution can be found in Chapter 2 of Neuts [14].

The traffic intensity of the queueing system is defined as $\rho = (\lambda_1/\mu_1 + \cdots + \lambda_K/\mu_K)/S$. We want to show that the queueing system can reach its steady state if $\rho<1$. To avoid heavy notation, we give details to the case with $S=1$ only. An outline for the proof of the multiple server case is given later.

The queueing system of interest is represented by the following four dimensional stochastic process (S=1):

$q(t)$: the string of customers in queue (exclude the one in server, if any), $q(t)\in\aleph\cup\{-1\}$;
$I(t)$: the state of the underlying Markov process D, $1\le I(t)\le m$;
$I_{1,1}(t)$: the type of the customer in service (if any) $1\le I_{1,1}(t)\le K$;
$I_{1,2}(t)$: the phase of the PH-distribution of the current service (if any), $1\le I_{1,2}(t)\le m_{I_{1,1}(t)}$.

If there is no customer in the system at time t, $q(t) = -1$. If there is one customer in the system at time t, $q(t) = 0$, since the customer is in service. If there are customers waiting at time t, $q(t)$ is a string in $\aleph$. It is easy to see that $(q(t), I(t), I_{1,1}(t), I_{1,2}(t))$ is a Markov process with a state space: $\aleph\times\{1, 2, \cdots, m\}\times\cup^{K}_{k=1}\{1, 2, \cdots, m_k\}$. This is a QBD Markov process with a tree structure when $(I(t), I_{1,1}(t), I_{1,2}(t))$ is defined as the auxiliary random variable with $m\overline{m}$ states (where $\overline{m} = m_1+\cdots+m_K$), except that if $q(t) = -1$, the auxiliary variable takes values $\{1, 2, \cdots, m\}$. Furthermore, the infinitesimal generator of the QBD Markov process is defined by the following transition blocks. For $J = k_1\cdots k_n \in\aleph$ and $1\le k\le K$,

$$A(k,k+J) = D_J \otimes I_{\overline{m}\times\overline{m}}, \quad J\in\aleph,\ J\neq 0;$$

$$A(k,k) = D_0 \otimes I_{\overline{m}\times\overline{m}} + I_{m\times m}\otimes\begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_K\end{pmatrix} \equiv D_0\otimes I + I\otimes\hat{T}; \tag{9.1}$$

$$A(k,0) = I_{m\times m}\otimes[\begin{pmatrix}\mathbf{T}_1^0\alpha_k \\ \vdots \\ \mathbf{T}_K^0\alpha_k\end{pmatrix}(0,\cdots,0,I_{m_k\times m_k},0,\cdots,0)] \equiv I\otimes\hat{T}_k^0,$$

where "$\otimes$" represents the Kronecker product of matrices (see Gantmacher [4]) and I_n is an $n\times n$ identity matrix for positive integer n. Transitions associated with the root node 0 and the node J=–1 can be found in HE [6]. They are omitted since they are not used here. The function $A^*(k, \mathbf{z})$ is rewritten in the following way.

$$A^*(k,\mathbf{z}) = z_k[I\otimes(\hat{T}+\frac{\hat{T}_k^0}{z_k})] + z_k[(D_0 + \sum_{J\in\aleph,\, J\neq 0}\mathbf{z}^J D_J)\otimes I]. \tag{9.2}$$

First, it has been proved in HE [6] that $\rho<1$ is necessary for positive recurrence if S=1. The idea is to show that the probability that the queueing system is empty is $1-\rho$, if the queueing system can reach steady state. Therefore, we only need to prove the sufficiency of the condition. For this purpose, we prove the following lemma.

Lemma 9.1 If $\rho<1$, there exists a positive vector $\mathbf{u}^*$ such that

$$A^*(k,\mathbf{z})\mathbf{u}^* \le -\varepsilon\mathbf{e}, \quad 1\le k\le K, \tag{9.3}$$

for some $\mathbf{z}$ with $\{z_k>1, 1\le k\le K\}$ and some positive ε. If $\rho>1$, there exists a positive vector $\mathbf{u}^*$ and $\mathbf{z}$ satisfing equation (9.3) with $\{0<z_k<1, 1\le k\le K\}$ and $\varepsilon\ge 0$.

Proof. Let $b_k=1/\mu_k$ and $z_k(t)=b_kt+1$, for $1\le k\le K$. Let $D^*(t)=\sum_J(\mathbf{z}(t))^J D_J$. Let $\xi(t)$ be the Perron-Frobenius eigenvalue of matrix $D^*(t)$ and $\zeta(k,t)$ the Perron-Frobenius eigenvalue of matrix $\hat{T}^*(k,t)=\hat{T}+\hat{T}_k^0/z_k(t)$, $1\le k\le K$. Then we have $D^*(t)\mathbf{u}_D(t)=\xi(t)\mathbf{u}_D(t)$ and $\hat{T}^*(k,t)\mathbf{u}_T(k,t)=\zeta(k,t)\mathbf{u}_T(k,t)$ for some nonnegative nonzero analytical vector $\mathbf{u}_D(t)$ and $\mathbf{u}_T(k,t)$ with $\mathbf{u}_D(0)=\mathbf{e}$ and $\mathbf{u}_T(k,0)=\mathbf{e}$. Let $\theta(t)$ and $\pi(k,t)$ be the right eigenvector corresponding to $\xi(t)$ and $\zeta(k,t)$ with $\theta(t)\mathbf{e}=1$ and $\pi(k,t)\mathbf{e}=1$, respectively. Then $\theta(0)=\theta$ and $\pi(k,0)=(0,\cdots,0,\pi_k,0,\cdots,0)$. Similar to Lemma 1.3.3 in Neuts [14], it can be proved that

$$\begin{aligned}
\xi^{(1)}(0) &= \lim_{t\to 0}\theta(t)\frac{dD^*(t)}{dt}\mathbf{u}_D(t)=\theta\sum_{k=1}^K b_k\frac{\partial D^*(\mathbf{z})}{\partial z_k}\bigg|_{\mathbf{z}=(1,\cdots,1)}\mathbf{e}=\rho;\\
\zeta^{(1)}(k,0) &= \lim_{t\to 0}\pi(t)\frac{d\hat{T}^*(k,t)}{dt}\mathbf{u}_T(k,t)=-b_k\pi_k\mathbf{T}_k^0=-1,\quad 1\le k\le K.
\end{aligned}\tag{9.4}$$

Then we have the following Taylor expansions at $t=0$:

$$\zeta(k,t)=-t+O(t^2),\quad \mathbf{u}_T(k,t)=\mathbf{e}+t\mathbf{u}_T^{(1)}(k)+O(t^2),\quad 1\le k\le K, \tag{9.5}$$

where $\{\mathbf{u}_T^{(1)}(k), 1\le k\le K\}$ are finite vectors. This leads to, for $1\le k\le K$,

$$\begin{aligned}
&(\hat{T}+\hat{T}_k^0/z_k(t))[\mathbf{e}+t\mathbf{u}_T^{(1)}(k)+O(t^2)]=(-t+O(t^2))[\mathbf{e}+t\mathbf{u}_T^{(1)}(k)+O(t^2)]\\
\Rightarrow\quad & -tb_k\begin{pmatrix}\mathbf{T}_1^0\\ \vdots\\ \mathbf{T}_K^0\end{pmatrix}+t(\hat{T}+\frac{\hat{T}_k^0}{z_k(t)})\mathbf{u}_T^{(1)}(k)=-t\mathbf{e}+O(t^2)\\
\overset{t\to 0}{\Rightarrow}\quad & (\hat{T}+\hat{T}_k^0)\mathbf{u}_T^{(1)}(k)=-\mathbf{e}+b_k\begin{pmatrix}\mathbf{T}_1^0\\ \vdots\\ \mathbf{T}_K^0\end{pmatrix}\equiv-\mathbf{e}+b_k\mathbf{T}^0.
\end{aligned}\tag{9.6}$$

Dividing $\mathbf{u}_T^{(1)}(k)$ into sub-vectors $\{\mathbf{u}_T^{(1)}(k, j), 1 \le j \le K\}$ of sizes $\{m_j, 1 \le j \le K\}$. Then the last equality in equation (9.6) gives

$$
\begin{aligned}
&T_j \mathbf{u}_T^{(1)}(k, j) + \mathbf{T}_j^0 \alpha_k \mathbf{u}_T^{(1)}(k,k) = -\mathbf{e} + b_k \mathbf{T}_j^0, \quad 1 \le j \le K; \\
&\Rightarrow \mathbf{u}_T^{(1)}(k, j) = -T_j^{-1}\mathbf{T}_j^0 \alpha_k \mathbf{u}_T^{(1)}(k,k) - T_j^{-1}\mathbf{e} + b_k T_j^{-1}\mathbf{T}_j^0 = (\alpha_k \mathbf{u}_T^{(1)}(k,k) - b_k)\mathbf{e} - T_j^{-1}\mathbf{e}.
\end{aligned} \tag{9.7}
$$

This leads to $\mathbf{u}_T^{(1)}(k) = (\alpha_k \mathbf{u}_T^{(1)}(k,k) - b_k)\mathbf{e} - \hat{T}^{-1}\mathbf{e}$. Then

$$
\begin{aligned}
\hat{T}_j^0 \mathbf{u}_T^{(1)}(k) &= \hat{T}_j^0[(\alpha_k \mathbf{u}_T^{(1)}(k,k) - b_k)\mathbf{e} - \hat{T}^{-1}\mathbf{e}] = (\alpha_k \mathbf{u}_T^{(1)}(k,k) - b_k)\mathbf{T}^0 - \begin{pmatrix} \mathbf{T}_1^0 \alpha_j T_j^{-1}\mathbf{e} \\ \vdots \\ \mathbf{T}_1^0 \alpha_j T_j^{-1}\mathbf{e} \end{pmatrix} \\
&= (\alpha_k \mathbf{u}_T^{(1)}(k,k) - b_k + b_j)\mathbf{T}^0 \equiv (c_k - b_k + b_j)\mathbf{T}^0.
\end{aligned} \tag{9.8}
$$

Let $\mathbf{u}_T(t) = \mathbf{e} + t\sum_{k=1}^{K} \omega_k \mathbf{u}_T^{(1)}(k)$ for some constants $\{\omega_k, 1 \le k \le K\}$. Then

$$
\begin{aligned}
&(\hat{T} + \frac{\hat{T}_k^0}{z_k(t)})\mathbf{u}_T(t) = -\frac{tb_k}{z_k(t)}\mathbf{T}^0 + t(\hat{T} + \frac{\hat{T}_k^0}{z_k(t)})\left(\sum_{k=1}^{K} \omega_k \mathbf{u}_T^{(1)}(k)\right) \\
&= -\frac{tb_k}{z_k(t)}\mathbf{T}^0 + \left(\frac{t^2\hat{T}_k^0}{z_k(t)} + t(\hat{T} + \hat{T}_k^0)\right)\left(\sum_{k=1}^{K} \omega_k \mathbf{u}_T^{(1)}(k)\right) \\
&= -\frac{tb_k}{z_k(t)}\mathbf{T}^0 + \frac{t^2\hat{T}_k^0}{z_k(t)}\left(\sum_{k=1}^{K} \omega_k \mathbf{u}_T^{(1)}(k)\right) + t\sum_{j=1}^{K} \omega_j (\hat{T} + \hat{T}_j^0 + \hat{T}_k^0 - \hat{T}_j^0)\mathbf{u}_T^{(1)}(j) \\
&= -\frac{tb_k}{z_k(t)}\mathbf{T}^0 + \frac{t^2\hat{T}_k^0}{z_k(t)}\left(\sum_{k=1}^{K} \omega_k \mathbf{u}_T^{(1)}(k)\right) \\
&\qquad + t\left(-\sum_{j=1}^{K} \omega_j \mathbf{e} + \sum_{j=1}^{K} \omega_j b_j \mathbf{T}^0 + \sum_{j=1}^{K} \omega_j (c_j - b_j + b_k - c_j)\mathbf{T}^0\right) \\
&= -t\sum_{j=1}^{K} \omega_j \mathbf{e} - t\left[\frac{1}{z_k(t)} - (\sum_{j=1}^{K} \omega_j)\right] b_k \mathbf{T}^0 + \frac{t^2\hat{T}_k^0}{z_k(t)}\left(\sum_{k=1}^{K} \omega_k \mathbf{u}_T^{(1)}(k)\right).
\end{aligned} \tag{9.9}
$$

If $\rho<1$, choose $\sum_{k=1}^{K} \omega_k = \frac{(1+\rho)}{2}$. Then we choose a small positive t such that

$t < \frac{(1-\rho)}{(1+\rho)} \frac{1}{\max\{b_k\}}$ and $t\left|\left(\frac{\hat{T}_k^0}{z_k(t)} + \frac{1+3\rho}{4}\right)\left(\sum_{k=1}^{K} \omega_k \mathbf{u}_T^{(1)}(k)\right)\right| < \frac{1-\rho}{4}\mathbf{e}$. Then

$$(\hat{T} + \frac{\hat{T}_k^0}{z_k(t)})\mathbf{u}_T(t) < -t\frac{(1+3\rho)}{4}\mathbf{u}_T(t). \tag{9.10}$$

Choose positive t small enough so that $\xi(t) = \rho t + O(t^2) < t(1+7\rho)/8$ and equation (9.10) holds. By equation (9.2), we have, for $1 \le k \le K$,

$$\begin{aligned} & A^*(k, \mathbf{z}(t))(\mathbf{u}_D(t) \otimes \mathbf{u}_T(t)) \\ &= z_k(t)\left\{\xi(t)(\mathbf{u}_D(t) \otimes \mathbf{u}_T(t)) + \mathbf{u}_D(t) \otimes [\hat{T} + \frac{\hat{T}_k^0}{z_k(t)}]\mathbf{u}_T(t)\right\} \\ &< z_k(t)\left\{\frac{(1+7\rho)}{8}(\mathbf{u}_D(t) \otimes \mathbf{u}_T(t)) - t\frac{(1+3\rho)}{4}\mathbf{u}_D(t) \otimes \mathbf{u}_T(t)\right\} \\ &= z_k(t)\left(-t\frac{(1-\rho)}{8}\right)\mathbf{u}_D(t) \otimes \mathbf{u}_T(t) < -t\frac{(1-\rho)}{8}\mathbf{u}_D(t) \otimes \mathbf{u}_T(t) \end{aligned} \tag{9.11}$$

Set $\varepsilon = (1-\rho)/8$ (>0) and $\mathbf{u}^* = \mathbf{u}_D(t) \otimes \mathbf{u}_T(t)$, equation (9.3) holds for $1 \le k \le K$.

If $\rho > 1$, repeat the whole process but with a ***negative*** small t, we obtain $\mathbf{u}^*$ for which the inequality holds with $\varepsilon = (\rho-1)/8$ (>0). This completes the proof Lemma 9.1.

Theorem 9.2 For a continuous time $MMAP[K]/PH[K]/S$ queue with a work conserving service discipline, the Markov process $\{(q(t), I(t), I_{1,1}(t), I_{1,2}(t)), t>0\}$ or the queueing system is positive recurrent if $\rho<1$, transient if $\rho>1$.

Proof. First, Lemmas 6.1 and 6.2 are modified to $A^*(k, \mathbf{z})\mathbf{u}^* < 0$ and $A^*(k, \mathbf{z})\mathbf{u}^* \le 0$, respectively. Then the result for $S=1$ is obtained by Lemma 9.1. The generalization to $S>1$ can be accomplished by using the following expression:

$$A^*(k, \mathbf{z}) = z_k[\sum_{j=1}^{S} I_{m(m_1+\cdots+m_K)^{j-1}} \otimes (\hat{T} + \frac{\hat{T}_k^0}{z_k}) \otimes I_{(m_1+\cdots m_K)^{S-j}}] + z_k[D^*(\mathbf{z}) \otimes I]. \tag{9.13}$$

Then $\mathbf{u}^* = \mathbf{u}_D(t) \otimes \mathbf{u}_T(t) \otimes \cdots \otimes \mathbf{u}_T(t)$ can be used to prove the results in Lemma 9.1 for $S>1$, which leads to this theorem. Note that the boundary nodes are not essential for this problem and are not considered. This completes the proof of Theorem 9.2.

Note 9.1: For $S=1$, it has been proved in HE [6] that $\rho<1$ is necessary for positive recurrence of the corresponding Markov process. The same method can be used to prove that $\rho<1$ is necessary for positive recurrence for the case with $S>1$. In summary, a continuous time *MMAP*[*K*]/*PH*[*K*]/*S* queue with a work conserving service discipline is positive recurrent if and only if $\rho<1$; transient if $\rho>1$.

Note 9.2: Note that the above method does not apply directly in the discrete time *MMAP*[*K*]/*PH*[*K*]/*S* queue with a work conserving service discipline if $S>1$. The main problem with the discrete time case is that arrivals and completions of service can occur at the same time, which makes the corresponding Markov chain more complicated. Nonetheless, we believe, the same conclusion can be proved for the discrete time *MMAP*[*K*]/*PH*[*K*]/*S* queue with a work conserving service discipline. But details need further study.

10. Conclusion

In this paper, we showed that the classification of Markov chains of matrix *M/G/*1 type with a tree structure is determined by the Perron-Frobenius eigenvalue of a nonnegative matrix. A computational method was developed for computing the Perron-Frobenius eigenvalue of interest. Also in this paper, we proved two sufficient conditions for positive recurrence and transience of Markov chains of matrix *M/G/*1 type with a tree structure. Using these results, we have proved in this paper that discrete time and continuous time *MMAP*[*K*]/*G*[*K*]/1 queues with a work conserving service discipline is positive recurrent if and only if the traffic intensity is less than one, transient if the traffic intensity is larger than one. We also proved that the *MMAP*[*K*]/*PH*[*K*]/*S* ($S>1$) queue is positive recurrent if $\rho<1$ and transient if $\rho>1$.

Future research includes finding refined conditions for Theorem 2.1 if $m>1$ and $K>1$, especially for the null recurrent case. Results of that sort can be used to study the classification problem of *MMAP*[*K*]/*G*[*K*]/1 queues if $\rho = 1$, a case which needs more attention.

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Extra Examples

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Example 5.1 Consider a *MAP/PH/1* queue with a LCFS-GPR service discipline with the following arrival process and service time distribution:

$$m = K = 2,\ D_0 = \begin{pmatrix} 0.7 & 0 \\ 0 & 1-\lambda \end{pmatrix},\ D_1 = \begin{pmatrix} 0.07 & 0..23 \\ 0.5\lambda & 0.5\lambda \end{pmatrix},\ \alpha = (0.4,0.6),\ T = \begin{pmatrix} 0.3 & 0.6 \\ 0.2 & 0.1 \end{pmatrix}.$$

We consider the following LCFS-GPR service disciplines:

$$Q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad Q(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\alpha,\quad Q(3) = \begin{pmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{pmatrix},\quad Q(4) = \begin{pmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{pmatrix}.$$

The Perron-Frobenius eigenvalues $sp(P(\mathbf{G}))$ as a function of the arrival rate λ corresponding to the above matrix Qs are shown in Figure 5.1. In Figure 5.1, the horizontal axis is the λ and the vertical axis is $sp(P(\mathbf{G}))$.

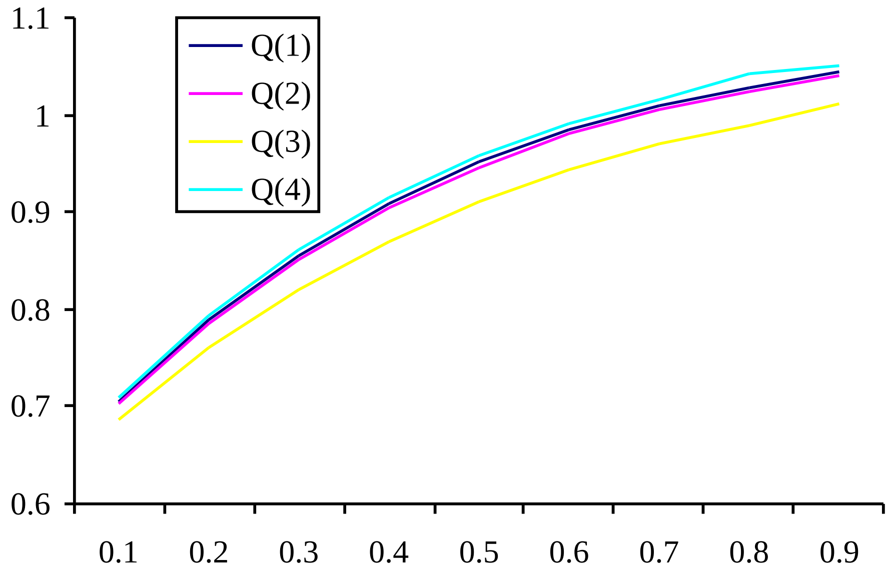

Figure 5.1 Perron-Frobenius eigenvalues $sp(P(\mathbf{G}))$

In Figure 5.1, the difference between $Q(3)$ and the rest is more clear. For $Q(1)$, $Q(2)$, and $Q(4)$, $sp(P(\mathbf{G}))$ crosses 1 from below for λ between 0.6 and 0.7. For $Q(3)$, $sp(P(\mathbf{G}))$ crosses 1 from below for λ between 0.8 and 0.9. Although the absolute value of $sp(P(\mathbf{G}))$ may not be an indicator of traffic intensity of the corresponding queueing system, its position to 1 does show the stability of the queueing system. Thus, Figure 5.1 show that the queueing systems go from stable to unstable when λ goes from 0.6 to 0.7 for *Q*(1), *Q*(2), and *Q*(4). But the queueing system is still stable for *Q*(3) until λ is close to 0.9. The reason is that the service process has a larger probability to complete in state 1 than in that 2 (see the definition of matrix T). The implication of the numerical results need further study.

Example 4.2 Consider a Markov chain of *M/G/1* type with a tree structure with *m*=2, *K*=2, and transition matrices (only those nonzero matrices):

$$A(1,0)=\begin{pmatrix}0.4 & 0.1\\ 0.2 & 0.1\end{pmatrix},\ A(1,1)=\begin{pmatrix}0.1 & 0\\ 0 & 0\end{pmatrix},\ A(1,11)=\begin{pmatrix}0.1 & 0.1\\ 0.5 & 0\end{pmatrix},\ A(1,11)=\begin{pmatrix}0.1 & 0.1\\ 0.2 & 0\end{pmatrix};$$
$$A(2,0)=\begin{pmatrix}0.1 & 0.2\\ 0.4 & 0.1\end{pmatrix},\ A(2,2)=\begin{pmatrix}0.2 & 0\\ 0.1 & 0.2\end{pmatrix},\ A(2,21)=\begin{pmatrix}0.1 & 0.2\\ 0 & 0\end{pmatrix},\ A(2,22)=\begin{pmatrix}0 & 0.2\\ 0.1 & 0.1\end{pmatrix};$$

For this Markov chain, matrices $\{G^*(1),\ G^*(2)\}$ are not stochastic and $sp(P(\mathbf{G}))$ = 1.0294. Thus, the Markov chain is transient, but $sp(P)$=0.9861<1. Therefore, $sp(P)$ fails to provide correct information for a classification of the Markov chain.