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ON THE CONVERGENCE AND LIMITS OF CERTAIN MATRIX SEQUENCES ARISING IN QUASI-BIRTH-AND-DEATH MARKOV CHAINS

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Abstract

We study the convergence of certain matrix sequences that arise in quasi-birth-and-death (QBD) Markov chains and we identify their limits. In particular, we focus on a sequence of matrices whose elements are absorption probabilities into some boundary states of the QBD. We prove that, under certain technical conditions, that sequence converges. Its limit is either the minimal nonnegative solution G of the standard nonlinear matrix equation, or it is a stochastic solution that can be explicitly expressed in terms of G. Similar results are obtained relative to the standard matrix R that arises in the matrix-geometric solution of the QBD. We present numerical examples that clarify some of the technical issues of interest.

Keywords: Matrix analytic methods; quasi-birth-and-death Markov chain; nonnegative matrix

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1. Introduction

Let Ω be the set of stochastic and substochastic matrices of order m (a positive integer). Let A_0, A_1, and A_2 be nonnegative matrices in Ω such that $A = A_0 + A_1 + A_2$ is stochastic, i.e. $A\boldsymbol{e} = \boldsymbol{e}$, where $\boldsymbol{e}$ is the column vector with all components one. To avoid insignificant cases, we assume that the matrices A_0 and A_2 are nonzero. For any matrix $X \in \Omega$, define $f(X)$ as the minimal nonnegative solution to the matrix equation:

$$f(X) = A_2 + (A_1 + A_0X)f(X). \tag{1.1}$$

Define a sequence of nonnegative matrices $\{Z(n), n \geq 0\}$ as follows: $Z(0) \in \Omega$ and $Z(n+1) = f(Z(n))$, $n \geq 0$. Our objective is to study the convergence of the sequence $\{Z(n), n \geq 0\}$ and to identify its limit (if it exists). The sequence $\{Z(n), n \geq 0\}$ is closely related to the minimal nonnegative solution G to the matrix equation:

$$G = A_2 + A_1G + A_0G^2. \tag{1.2}$$

It is shown in [16] that the matrix G is in Ω. Denote by $\mathrm{sp}(G)$ the Perron–Frobenius eigenvalue of the matrix G (i.e. the nonnegative eigenvalue of G with the largest modulus); $\mathrm{sp}(G)$ is also called the spectral radius of G. Denote by $\boldsymbol{g}$ the left eigenvector of the matrix G

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corresponding to $\mathrm{sp}(G)$, i.e. $\boldsymbol{g}G = \mathrm{sp}(G)\boldsymbol{g}$ and $\boldsymbol{g}\boldsymbol{e} = 1$. In this paper, we prove that, under certain conditions, the sequence $\{Z(n), n \geq 0\}$ converges either to G or $G + (I - G)\boldsymbol{e}\boldsymbol{g}$, where I is the identity matrix. We also identify conditions for the limit to be G or $G + (I - G)\boldsymbol{e}\boldsymbol{g}$ respectively.

The matrix G has much to do with the steady state distribution and fundamental periods of quasi-birth-and-death Markov chains with transition blocks $\{A_0, A_1, A_2\}$ (see [16], [17], [18] and Section 3 of this paper). Because of its importance to queueing models, the matrix G has received major attention. Neuts [16] formally introduced the matrix G, gave its probabilistic interpretation, and used it in queueing analysis. Since then, a number of papers focused on the matrix G (or the equally important matrix R) have appeared. Among them are [2], [4], [5], [6], [7], [9], [10], [11], [12], [15], [19], [20], and [22].

In all problems studied, the power-bounded solutions to (1.2) and algorithms for computing the matrix G are closely related to our work. Gail *et al.* [6] identified all the power-bounded solutions to (1.2) that include the matrices G and $G + (I - G)\boldsymbol{e}\boldsymbol{g}$. A number of papers ([1], [2], [4], [12], [13], [21], etc.) developed various algorithms for computing the matrix G. For the iteration algorithms utilizing (1.1) with $Z(0) = 0$, it has been proved that $\{Z(n), n \geq 0\}$ converges to G. If G is stochastic, for $Z(0) = I$, it was shown in [13] that $\{Z(n), n \geq 0\}$ converges to G. However, the convergence of the sequence $\{Z(n), n \geq 0\}$ is still an open problem if $Z(0) \neq 0$ or $Z(0) \neq I$. In this paper, we identify conditions for the convergence of $\{Z(n), n \geq 0\}$ if $Z(0)$ is stochastic or substochastic. We also give a probabilistic interpretation of the matrix sequence $\{Z(n), n \geq 0\}$. In addition, if $\{Z(n), n \geq 0\}$ does converge, we shall find the limit.

The matrices $\{Z(n), n \geq 0\}$ can be interpreted as the absorption probability matrix of some boundary states of a sequence of finite Markov chains (see Section 3). Understanding the matrices $\{Z(n), n \geq 0\}$ helps us study the corresponding quasi-birth-and-death Markov chain constructed from $\{A_0, A_1, A_2\}$. This is another motivation of this research.

In a recent paper, Latouche and Taylor [14] considered a similar problem. They use probabilistic arguments to answer a (convergence) question more general than the one in this paper. Unlike the conditions of this paper, which are given in terms of the properties of the matrices $\{A_0, A_1, A_2\}$ and the solution to (1.2), their conditions for convergence are given in terms of the structure of the corresponding QBD Markov chain.

The remainder of the paper is organized as follows. In Section 2, we present some preliminary results from nonnegative matrix theory. In Section 3, we give probabilistic interpretations and some basic equations for the sequence $\{Z(n), n \geq 0\}$. In Section 4, we prove that, under some conditions, the sequence $\{Z(n), n \geq 0\}$ converges and the limit is either G or $G + (I - G)\boldsymbol{e}\boldsymbol{g}$. Conditions for the limit of $\{Z(n), n \geq 0\}$ to be G or $G + (I - G)\boldsymbol{e}\boldsymbol{g}$ are identified in Sections 5, 6, and 7. While Section 5 deals with the recurrent case, Sections 6 and 7 consider the transient case. In Section 8, extensions to two matrix sequences related to the matrix R are carried out. Finally, in Section 9, the results obtained in this paper are summarized.

2. Preliminaries

In this section, we review useful results about nonnegative matrices. These are treated in detail in [8] and [23]. A few results related to (1.1) and to the matrix G are also presented.

Suppose that X is a nonnegative square matrix. Let $\mathrm{sp}(X)$ be the largest modulus of the eigenvalues of X. Then $\mathrm{sp}(X)$ is an eigenvalue of X and $\mathrm{sp}(X)$ is nondecreasing with respect to any element of the matrix X. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be the left and right eigenvectors of X corresponding to $\mathrm{sp}(X)$, respectively. Then $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonnegative. The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be normalized by

$\boldsymbol{ue} = \boldsymbol{uv} = 1$. If there exists a nonnegative c such that $X\boldsymbol{e} \leq c\boldsymbol{e}$, then $\mathrm{sp}(X) \leq c$. If the matrix X is irreducible, $\mathrm{sp}(X) > 0$, then the geometric and the algebraic degrees of $\mathrm{sp}(X)$ are 1, and the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are positive (i.e. every element of $\boldsymbol{u}$ or $\boldsymbol{v}$ is positive) and unique. Furthermore, if the matrix X is irreducible, then $\mathrm{sp}(X)$ is strictly increasing with respect to every element of X. If X is primitive (i.e. irreducible and aperiodic), $\mathrm{sp}(X)$ is the only eigenvalue of X with a modulus $\mathrm{sp}(X)$. If X is primitive, a simple approximation $X^n = (\mathrm{sp}(X))^n \boldsymbol{vu} + o((\mathrm{sp}(X))^n)$ can be verified by using the Jordan canonical form of matrix. Clearly, $\mathrm{sp}(X) \leq 1$ if $X \in \Omega$, and $\mathrm{sp}(X) = 1$ if X is stochastic.

Throughout this paper, for any two vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$, $\boldsymbol{u} \leq \boldsymbol{v}$ means that $u_i \leq v_i$ for $1 \leq i \leq m$; $\boldsymbol{u} < \boldsymbol{v}$ means that $\boldsymbol{u} \leq \boldsymbol{v}$ and $u_i < v_i$ for at least one i, $1 \leq i \leq m$.

For later use, we first derive, for any $X \in \Omega$, an explicit expression for the minimal nonnegative solution to (1.1). Note that, since various conditions on the matrices A, A_0, A_1, A_2, and G are used in different parts of this paper, the conditions for each theorem, lemma, or corollary are stated individually.

Lemma 2.1. *For any matrix $X \in \Omega$, the minimal nonnegative solution to (1.1) is given as*

$$f(X) = \sum_{k=0}^{\infty} (A_1 + A_0 X)^k A_2. \tag{2.1}$$

The sum in (2.1) converges for any $X \in \Omega$, and $f(X) \in \Omega$. Furthermore, if $\mathrm{sp}(A_1 + A_0 X) < 1$, *$f(X) = (I - A_1 - A_0 X)^{-1} A_2$ is the unique solution to (1.1).*

Proof. By (1.1) and iteration, we obtain

$$f(X) = \sum_{k=0}^{n} (A_1 + A_0 X)^k A_2 + (A_1 + A_0 X)^{n+1} f(X), \qquad n \geq 0. \tag{2.2}$$

Since $X \in \Omega$, we have $X\boldsymbol{e} \leq \boldsymbol{e}$. Consequently, $(A_1 + A_0 X)\boldsymbol{e} \leq \boldsymbol{e}$ and $\mathrm{sp}(A_1 + A_0 X) \leq 1$. If $\mathrm{sp}(A_1 + A_0 X) < 1$, then $(A_1 + A_0 X)^{n+1}$ tends to 0 as n goes to infinity and the sum on the right-hand side of (2.2) converges. That leads to (2.1) and $f(X) = (I - A_1 - A_0 X)^{-1} A_2$ is the unique solution to (1.1). Since $\boldsymbol{e} = (A_2 + A_1 + A_0)\boldsymbol{e} \geq (A_2 + A_1 + A_0 X)\boldsymbol{e}$, we have $\boldsymbol{e} \geq (I - A_1 - A_0 X)^{-1} A_2 \boldsymbol{e} = f(X)\boldsymbol{e}$, that is, $f(X) \in \Omega$.

If $\mathrm{sp}(A_1 + A_0 X) = 1$, we first show that the matrix $A_1 + A_0 X$ must be reducible. If $A_1 + A_0 X$ were irreducible, then $\mathrm{sp}(A_2 + A_1 + A_0 X) > \mathrm{sp}(A_1 + A_0 X) = 1$, since the matrix A_2 is nonzero. That contradicts $\mathrm{sp}(A_2 + A_1 + A_0 X) \leq 1$ since $(A_2 + A_1 + A_0 X)\boldsymbol{e} \leq \boldsymbol{e}$. Thus, the matrix $A_1 + A_0 X$ is reducible. After (possible) rearrangements of the rows and columns of the matrices $A_1 + A_0 X$, A_2, A_1, A_0, and A, we can obtain

$$A_1 + A_0 X = \begin{pmatrix} P_1 & P_2 \\ 0 & P_3 \end{pmatrix}$$

with $P_3 \boldsymbol{e} = \boldsymbol{e}$ and $\mathrm{sp}(P_1) < 1$. Since $(A_2 + A_1 + A_0 X)\boldsymbol{e} \leq \boldsymbol{e}$, it is easy to see that

$$A_2 = \begin{pmatrix} A_{2,11} & A_{2,12} \\ 0 & 0 \end{pmatrix},$$

where $A_{2,11}$ is of the same size as the matrix P_1. The equation (2.2) then becomes

$$f(X) = \sum_{k=0}^{n} \begin{pmatrix} P_1^k A_{2,11} & P_1^k A_{2,12} \\ 0 & 0 \end{pmatrix} + (A_1 + A_0 X)^{n+1} f(X)$$
$$\to \begin{pmatrix} (I - P_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} A_2, \quad \text{as } n \to \infty,$$

since $f(X)$ is the minimal nonnegative solution to (1.1). Since $(A_{2,11} + P_1)\boldsymbol{e} + (A_{2,12} + P_2)\boldsymbol{e} \le \boldsymbol{e}$, we find that $(I - P_1)^{-1}(A_{2,11}\boldsymbol{e} + A_{2,12}\boldsymbol{e}) \le \boldsymbol{e}$, and that leads to $f(X)\boldsymbol{e} \le \boldsymbol{e}$ and $f(X) \in \Omega$.

Lemma 2.2. *If* $\text{sp}(A_1 + A_0 X) < 1$ *and* $X\boldsymbol{e} = \boldsymbol{e}$*, then* $f(X)$ *is stochastic.*

Proof. Since $\text{sp}(A_1 + A_0 X) < 1$, the matrix $I - A_1 - A_0 X$ is invertible. Since $\boldsymbol{e} = (A_2 + A_1 + A_0)\boldsymbol{e} = (A_2 + A_1 + A_0 X)\boldsymbol{e}$, we obtain $(I - A_1 - A_0 X)\boldsymbol{e} = A_2\boldsymbol{e}$, which leads to $\boldsymbol{e} = (I - A_1 - A_0 X)^{-1} A_2 \boldsymbol{e} = f(X)\boldsymbol{e}$.

Let $G(0) = 0$ and $G(n+1) = f(G(n))$ for $n \ge 0$. The sequence $\{G(n), n \ge 0\}$ and the matrix G are related in a way that is important to our discussion.

Lemma 2.3. *The sequence* $\{G(n), n \ge 0\}$ *is nondecreasing and converges to the matrix* G.

Proof. Since $G(0) = 0$, $G(n)\boldsymbol{e} \le \boldsymbol{e}$ for all n by Lemma 2.1. Since $G(1) \ge G(0) = 0$, it follows by (2.1) and induction that $\{G(n), n \ge 0\}$ is nondecreasing. Thus, $\{G(n), n \ge 0\}$ converges and the limit is denoted by $\ddot{G}$. By (1.1), it is easy to see that $\ddot{G}$ satisfies (1.2). Since the matrix G is in Ω, by (1.2) and Lemma 2.1, $G \ge f(G)$. Since $G \ge G(0)$, by (2.1) and induction, it can be shown that $G \ge G(n)$ for all n, which implies that $G \ge \ddot{G}$. By definition, G is the minimal nonnegative solution to (1.2). Therefore, we must have $G = \ddot{G}$.

3. Probabilistic interpretations and some basic equations

In the context of quasi-birth-and-death (QBD) Markov chains, the matrices G, $\{G(n), n \ge 0\}$, and $\{Z(n), n \ge 0\}$ have explicit probabilistic interpretations. These are given in the first part of this section. Based on these probabilistic interpretations, we establish equations that are useful in the proofs of our main results.

At the center of attention is a QBD Markov chain $\{(q_k, J_k) : q_k \ge 0, 1 \le J_k \le m, k \ge 0\}$ with the transition probability matrix

$$Q = \begin{pmatrix} I & 0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Define level k as the set of states $\{(k, 1), (k, 2), \dots, (k, m)\}$ for $k \ge 0$. Note that all states of level 0 are absorption states. It is proved in [16] that the (i, j)th element of the matrix G is the conditional probability that the Markov chain Q reaches level k for the first time in the state (k, j), given that the Markov chain was initially in state $(k+1, i)$, for $1 \le i, j \le m$ and $k \ge 0$. We shall call the matrix G the absorption probability matrix of level 0. (Note that we shall use the term 'probability matrix' in a similar manner.)

In order to give probabilistic interpretations to the matrices $\{Z(n), n \geq 0\}$, we introduce the Markov chains $\{Q_n, n \geq 0\}$ defined by their transition probability matrices:

$$Q_0 = \begin{pmatrix} I & 0 & 0 \\ Z(0) & 0 & \boldsymbol{e} - Z(0)\boldsymbol{e} \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} I & 0 & & \\ A_2 & A_1 & A_0 & \\ & Z(0) & 0 & \boldsymbol{e} - Z(0)\boldsymbol{e} \\ & & 0 & 1 \end{pmatrix},$$

$$Q_n = \begin{pmatrix} I & 0 & & & & \\ A_2 & A_1 & A_0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & A_2 & A_1 & A_0 & \\ & & & Z(0) & 0 & \boldsymbol{e} - Z(0)\boldsymbol{e} \\ & & & & 0 & 1 \end{pmatrix}, \qquad n \geq 2.$$

The Markov chain Q_n has $n + 3$ levels of states. States $\{(0, j), 1 \leq j \leq m\}$ in level 0 and state $\{(n + 2, 1)\}$ in level $n + 2$ are *absorption states*. Level $n + 1$ can be considered as a *reflecting barrier*. When $Z(0)\boldsymbol{e} = \boldsymbol{e}$, we call level $n+1$ of Q_n a *solid reflecting barrier*. When $Z(0)\boldsymbol{e} \neq \boldsymbol{e}$, we call the level $n+1$ of Q_n a *leaking reflecting barrier*. The concept of reflecting barrier is useful in explaining the main results in Sections 4–7.

Lemma 3.1. *The matrix $Z(n)$ is the absorption probability matrix of level 0 of the Markov chain Q_n, given that the Markov chain starts in level 1. The matrix $G(n)$ is the absorption probability matrix of level 0 without visiting level $n+1$, given that the Markov chain Q_n starts in level 1.*

Proof. When $n = 0$, $Z(0)$ is clearly the absorption probability matrix of level 0 of the Markov chain Q_0. Suppose that Lemma 3.1 holds for Q_n. For Q_{n+1}, let $X(n + 1)$ be the absorption probability matrix of level 0, given that this Markov chain is initially in level 1. By spatial homogeneity, the first passage time from level 2 to level 1 for Q_{n+1} is equivalent to the first passage time from level 1 to level 0 for Q_n. Using that observation and conditioning on the first transition of Q_{n+1}, we obtain

$$\begin{aligned} X(n+1) &= A_2 + A_1 X(n+1) + A_0 Z(n) X(n+1) \\ &= \sum_{k=0}^{t} (A_1 + A_0 Z(n))^k A_2 + (A_1 + A_0 Z(n))^{t+1} X(n+1). \end{aligned} \tag{3.1}$$

Then we need to show that $X(n + 1)$ is the minimal nonnegative solution to (3.1). That can be done by considering the embedded Markov chain at the epochs when Q_{n+1} is in level 1. Then $\sum_{k=0}^{t}(A_1 + A_0 Z(n))^k A_2$ is the probability matrix that the embedded Markov chain reaches level 0 within t transitions. Since $X(n+1)$ is the probability matrix that Q_{n+1} reaches level 0 in finite time, we must have $X(n + 1) = \sum_{k=0}^{\infty}(A_1 + A_0 Z(n))^k A_2$, i.e. the minimal nonnegative solution to (3.1). By (1.1), (2.1), and (3.1), we have $X(n + 1) = f(Z(n)) = Z(n + 1)$.

The interpretation of the matrices $\{G(n), n \geq 0\}$ is obtained in a similar way.

Lemma 3.1 establishes the relationship between the matrices $\{Z(n), G(n), n \geq 0\}$ and the Markov chains $\{Q_n, n \geq 0\}$. From now on, we refer to the sequence $\{Z(n), n \geq 0\}$ as the absorption probabilities into level 0. This is another way to introduce the matrices $\{Z(n), n \geq 0\}$. In fact, the study of the solutions to (1.1) was originally motivated by consideration of these absorption probabilities. Next, we introduce two sets of matrices to elucidate further properties of the sequence $\{Z(n), n \geq 0\}$.

Definition 3.1. Let $D(n)$ be an $m \times m$ matrix whose (i, j)th element is the conditional probability that the Markov chain Q_n is eventually absorbed into level 0 in state $(0, j)$ and has visited level $n + 1$ at least once before, given that the Markov chain is in state $(1, i)$ initially, $1 \leq i, j \leq m$. We readily see that $D(0) = Z(0)$.

Definition 3.2. Let $W(n)$ be an $m \times m$ matrix whose (i, j)th element is the conditional probability that the Markov chain Q_n visits level $n + 1$ for the first time in state $(n + 1, j)$ before being absorbed into level 0 or level $n + 2$, given that the Markov chain is in state $(1, i)$ initially, $1 \leq i, j \leq m$. We readily see that $W(0) = I$.

The following relationships between the matrices $\{Z(n), n \geq 0\}$, $\{G(n), n \geq 0\}$, $\{D(n), n \geq 0\}$, and $\{W(n), n \geq 0\}$ are used in proving the main theorems in Sections 4–7.

Lemma 3.2. *For all* $n \geq 0$,

$$Z(n) = G(n) + D(n) \quad \text{and} \quad Z(n)\boldsymbol{e} \leq \boldsymbol{e}. \tag{3.2}$$

In addition, for any convergent subsequence $\{Z(n_t), n_t \geq 0\}$, *we have* $\lim_{n_t \to \infty} Z(n_t) \geq G$.

Proof. The results in (3.2) readily follow from Lemma 3.1 and the definition of $\{D(n), n \geq 0\}$. The rest follows from (3.2) and Lemma 2.3.

Lemma 3.2 shows that the limit of $\{Z(n), n \geq 0\}$ (if it exists) is in the convex subset $\{X : X \in \Omega \text{ and } X \geq G\}$ of Ω. Lemma 3.2 also implies that $\{Z(n), n \geq 0\}$ converges to the matrix G for any $Z(0)$ in Ω if G is stochastic (see Theorem 4.2). Thus, much effort in this paper is concentrated on dealing with cases where the matrix G is not stochastic.

Equation (3.2) shows that there are two possible paths to absorption in level 0: (i) without hitting the reflecting barrier ($G(n)$) and (ii) hitting the reflecting barrier at least once ($D(n)$). Since $\{G(n), n \geq 0\}$ converges to the matrix G monotonically, the convergence of $\{Z(n), n \geq 0\}$ is determined by that of $\{D(n), n \geq 0\}$. The following two expressions for $\{D(n), n \geq 0\}$ are useful in studying the convergence of $\{D(n), n \geq 0\}$ in Sections 4, 6, and 7.

Lemma 3.3. *For all* $n \geq 1$, *if* $\mathrm{sp}(A_1 + A_0 Z(n-1)) < 1$, *then*

$$D(n) = (I - A_1 - A_0 Z(n-1))^{-1} A_0 D(n-1) G(n). \tag{3.3}$$

If $\mathrm{sp}(A_1 + A_0 Z(k)) < 1$ *for* $0 \leq k \leq n-1$, *then*

$$D(n) = (I - A_1 - A_0 Z(n-1))^{-1} A_0 \cdots (I - A_1 - A_0 Z(0))^{-1} A_0 Z(0) G(1) \cdots G(n). \tag{3.4}$$

Proof. Let the Markov chain Q_n start in level 1. Conditioning on the first transition and by Lemma 3.2, we obtain, for all $n \geq 1$,

$$\begin{aligned} D(n) &= A_1 D(n) + A_0[D(n-1)Z(n) + G(n-1)D(n)] \\ &= A_1 D(n) + A_0[D(n-1)(D(n) + G(n)) + G(n-1)D(n)] \\ &= A_1 D(n) + A_0[Z(n-1)D(n) + D(n-1)G(n)]. \end{aligned}$$

That leads to $[1 - A_1 - A_0 Z(n-1)]D(n) = A_0 D(n-1) G(n)$. If $\mathrm{sp}(A_1 + A_0 Z(n-1)) < 1$, the matrix $I - A_1 - A_0 Z(n-1)$ is invertible and (3.3) follows. Equation (3.4) is readily obtained from (3.3).

Lemma 3.3 is important to the theorems in Section 6, but the condition $\{\mathrm{sp}(A_1 + A_0 Z(n)) < 1,\ n \geq 0\}$ does not always hold. In Section 6, we shall identify conditions under which it holds. Also, in Sections 6 and 7, counterexamples are presented to demonstrate the complexity of the issues related to $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$.

Lemma 3.4. *For all $n \geq 0$,*

$$Z(n) = G(n) + W(n)Z(0)Z(1)\cdots Z(n). \tag{3.5}$$

Proof. Conditioning on whether or not the Markov chain Q_n visits level $n+1$ prior to absorption into level 0, we obtain $Z(n) = G(n) + W(n)Z(0)Y(n)$, where $Y(n)$ is the probability matrix that Q_n transits from level n to level 0. The special structure of Q_n implies that the transitions from level n to level 0 can be divided into n stages: the first passage times from level n to level $n-1$, $n-1$ to $n-2, \ldots, 2$ to 1, and 1 to 0. It is easy to see that the transition probability matrix for the first passage time from level k to $k-1$ is $Z(n+1-k)$, $1 \leq k \leq n$, which leads to $Y(n) = Z(1)\cdots Z(n)$.

In order to deal with the sequence $\{W(n), n \geq 0\}$, we introduce a matrix sequence $\{\hat{G}(n), n \geq 0\}$ as follows. For any matrix $X \in \Omega$, define $\hat{f}(X)$ as the minimal nonnegative solution to the matrix equation:

$$\hat{f}(X) = A_0 + (A_1 + A_2 X)\hat{f}(X).$$

Let $\hat{G}(0) = 0$ and $\hat{G}(n+1) = \hat{f}(\hat{G}(n))$ for $n \geq 0$. According to Lemma 3.1, the matrix $\hat{G}(n)$ can be interpreted as the probability matrix that the Markov chain Q_n transits from level n to level $n+1$ *for the first time* before being absorbed into level 0, given that the Markov chain Q_n starts in level n. It is clear that $\{\hat{G}(n), n \geq 0\}$ and $\{G(n), n \geq 0\}$ have similar properties. For instance, $\{\hat{G}(n), n \geq 0\}$ is a nondecreasing matrix sequence and its limit exists. We denote the limit by $\hat{G}$, which is the minimal nonnegative solution to the matrix equation

$$\hat{G} = A_0 + A_1\hat{G} + A_2\hat{G}^2. \tag{3.6}$$

Lemma 3.5. *For the Markov chain Q_n, $W(n) = I$ if $n = 0$ and $W(n) = \hat{G}(1)\hat{G}(2)\cdots\hat{G}(n)$ if $n \geq 1$.*

Proof. The proof is similar to that of Lemma 3.4.

4. Convergence of $\{Z(n), n \geq 0\}$

In general, establishing the convergence of $\{Z(n), n \geq 0\}$ is complicated. We begin by discussing some easy cases, where fewer conditions are needed.

Theorem 4.1. *Consider the sequence $\{Z(n), n \geq 0\}$ introduced in Section 1. The limits of convergent subsequences of $\{Z(n), n \geq 0\}$ have the same row sums, i.e. the sequence $\{Z(n)\boldsymbol{e}, n \geq 0\}$ converges.*

Proof. Suppose that $\{Z(n_t)\boldsymbol{e}\}$ and $\{Z(k_t)\boldsymbol{e}\}$ are convergent subsequences with limits $Z_1\boldsymbol{e}$ and $Z_2\boldsymbol{e}$ respectively. We want to show that $Z_1\boldsymbol{e} = Z_2\boldsymbol{e}$. Without loss of generality, we assume that $n_t < k_t < n_{t+1}$. Postmultiplying by $\boldsymbol{e}$ on both sides of (3.5), since $Z(n)\boldsymbol{e} \leq \boldsymbol{e}$ for $n \geq 0$, we have

$$
\begin{aligned}
Z(k_t)\mathbf{e} &= G(k_t)\mathbf{e} + W(k_t)Z(0)Z(1)\cdots Z(k_t)\mathbf{e} \\
&\le G(k_t)\mathbf{e} + W(k_t)Z(0)Z(1)\cdots Z(n_t)\mathbf{e} \\
&= [G(k_t) - G(n_t)]\mathbf{e} + [W(k_t) - W(n_t)]Z(0)Z(1)\cdots Z(n_t)\mathbf{e} \\
&\quad + G(n_t)\mathbf{e} + W(n_t)Z(0)Z(1)\cdots Z(n_t)\mathbf{e} \\
&\le [G(k_t) - G(n_t)]\mathbf{e} + [W(k_t) - W(n_t)]\mathbf{e} + Z(n_t)\mathbf{e} \\
&\le [G(k_t) - G(n_t)]\mathbf{e} + Z(n_t)\mathbf{e}. \qquad (4.1)
\end{aligned}
$$

The last inequality is due to Lemma 3.5 and $\hat{G}(n)\mathbf{e} \le \mathbf{e}, n \ge 0$, which implies that $[W(k_t) - W(n_t)]\mathbf{e} \le 0$ if $k_t > n_t$. By Lemma 2.3, (4.1) implies that $Z_2\mathbf{e} \le Z_1\mathbf{e}$. Similarly, it can be shown that $Z_1\mathbf{e} \le Z_2\mathbf{e}$. Therefore, $Z_2\mathbf{e} = Z_1\mathbf{e}$. Thus, all the convergent subsequences of $\{Z(n)\mathbf{e}, n \ge 0\}$ have the same limit.

Since the sequence $\{Z(n)\mathbf{e}, n \ge 0\}$ is uniformly bounded, any subsequence of $\{Z(n)\mathbf{e}, n \ge 0\}$ must have a convergent subsequence with a finite limit. According to the above proof, all these limits must be the same. Therefore, we conclude that $\{Z(n)\mathbf{e}, n \ge 0\}$ converges.

Theorem 4.2. *Consider the sequence $\{Z(n), n \ge 0\}$ introduced in Section 1.*

(i) *If the matrix G is stochastic, then $\{Z(n), n \ge 0\}$ converges to G for any $Z(0)$ in Ω.*

(ii) *If $Z(0) \le G$ and $Z(0) \in \Omega$, then $\{Z(n), n \ge 0\}$ converges to G. Consequently, if $Z(0) \le (I - A_1)^{-1}A_2$, then $\{Z(n), n \ge 0\}$ converges to G.*

Proof. Part (i) follows immediately from Lemma 3.2. To prove (ii), notice that $G \ge f(G)$ by Lemma 2.1. Then it is easy to see that, if $Z(0) \le G$, then $Z(n) \le G$ for all $n \ge 0$. By Lemma 3.2, $\{Z(n), n \ge 0\}$ must converge and the limit is G. By (1.2), it is easy to verify that $(\sum_{n=0}^{\infty} A_1^n)A_2 = (I - A_1)^{-1}A_2 \le G$. The matrix $I - A_1$ is invertible since A_0 and A_2 are nonzero and A is irreducible. The second part of (ii) follows.

Theorem 4.2 implies that the convergence of $\{Z(n), n \ge 0\}$ can be verified if $Z(0)$ is in some special subset of Ω. Following this direction, we prove that $\{Z(n), n \ge 0\}$ converges if $Z(0)$ has the form $G + \mathbf{v}_0\mathbf{g}$ with $0 \le \mathbf{v}_0 \le (I - G)\mathbf{e}$ and G is irreducible. First, some results related to the matrix G are presented.

Lemma 4.1. *If the matrix G is irreducible, the matrix A is irreducible.*

Proof. Suppose that G is irreducible. If A is reducible, then the matrices $\{A_2, A_1, A_0\}$ are reducible in a similar manner. Consequently, the minimal nonnegative solution to (1.2), G, is reducible, which is a contradiction.

The converse of Lemma 4.1 is not always true. A counterexample is given as follows.

Example 4.1. Set

$$
A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = 0, \qquad A_0 = \begin{pmatrix} 0 & 0 \\ 1-p & p \end{pmatrix},
$$

with $0 < p < 1$. There are only two solutions to (1.2) in Ω, namely,

$$
G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.
$$

It is easy to see that A is primitive. But the matrix G is reducible.

Lemma 4.2. *Assume that the matrix A and the matrix G are irreducible. Let $\boldsymbol{\theta}$ be the (unique) left eigenvector of A corresponding to the Perron–Frobenius eigenvalue 1, that is, $\boldsymbol{\theta} A = \boldsymbol{\theta}$ and $\boldsymbol{\theta}\boldsymbol{e} = 1$. Then we have*

(i) $(I - A_1 - A_0G)^{-1}A_0(I - G)\boldsymbol{e} = (I - G)\boldsymbol{e}$;

(ii) *if* $\mathrm{sp}(G) < 1$, *then* $\boldsymbol{\theta} A_0(I - A_1 - A_0G)^{-1} = \boldsymbol{\theta}$; *and*

(iii) $\mathrm{sp}((I - A_1 - A_0G)^{-1}A_0) = \mathrm{sp}(A_0(I - A_1 - A_0G)^{-1}) = 1$.

Proof. Since $G \in \Omega$, according to the proof of Lemma 2.1, either the matrix $A_1 + A_0G$ is reducible with $\mathrm{sp}(A_1 + A_0G) = 1$ or it has $\mathrm{sp}(A_1 + A_0G) < 1$. If $A_1 + A_0G$ is reducible with $\mathrm{sp}(A_1 + A_0G) = 1$, then by the proof of Lemma 2.1, we must have

$$A_1 + A_0G = \begin{pmatrix} P_1 & P_2 \\ 0 & P_3 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} A_{2,11} & A_{2,12} \\ 0 & 0 \end{pmatrix},$$

which implies that the matrix G is reducible. That is a contradiction. Therefore, if G is irreducible, we must have $\mathrm{sp}(A_1 + A_0G) < 1$, i.e. the matrix $I - A_1 - A_0G$ is invertible if G is irreducible.

Postmultiply by $\boldsymbol{e}$ on both sides of the equation $G = A_2 + A_1G + A_0G^2$. Replacing $A_2\boldsymbol{e}$ by $\boldsymbol{e} - A_1\boldsymbol{e} - A_0\boldsymbol{e}$ and after some algebra, we obtain $A_0(I - G)\boldsymbol{e} = (I - A_1 - A_0G)(I - G)\boldsymbol{e}$, which leads to (i). Part (ii) is obtained similarly by premultiplying by $\boldsymbol{\theta}$ on both sides of $G = A_2 + A_1G + A_0G^2$.

By (ii), $\boldsymbol{\theta}[A_0(I - A_1 - A_0G)^{-1}]^n = \boldsymbol{\theta}$ for $n \geq 0$. Since A is irreducible, $\boldsymbol{\theta}$ is positive. Thus, the eigenvalue of $A_0(I - A_1 - A_0G)^{-1}$ with the largest modulus is 1, in other words, $\mathrm{sp}(A_0(I - A_1 - A_0G)^{-1}) = 1$. Since the matrix $I - A_1 - A_0G$ is invertible, the matrix $A_0(I - A_1 - A_0G)^{-1}$ and the matrix $(I - A_1 - A_0G)^{-1}A_0$ are similar. Therefore, $\mathrm{sp}((I - A_1 - A_0G)^{-1}A_0) = \mathrm{sp}(A_0(I - A_1 - A_0G)^{-1}) = 1$.

Theorem 4.3. *Consider the sequence $\{Z(n), n \geq 0\}$ introduced in Section 1. Assume that the matrix G is irreducible. Let $Z(0) = G + \boldsymbol{v}_0\boldsymbol{g}$ with $0 \leq \boldsymbol{v}_0 \leq (I - G)\boldsymbol{e}$. If $(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0 < (I - G)\boldsymbol{e}$, then $Z(n)$ converges to G. If $(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0 = (I - G)\boldsymbol{e}$, then $Z(n)$ converges to $G + (I - G)\boldsymbol{e}\boldsymbol{g}$.*

Proof. If $\mathrm{sp}(G) = 1$, then the matrix G must be stochastic since it is irreducible. By Theorem 4.2, $Z(n)$ converges to G for any $Z(0)$ and Theorem 4.3 follows. In the rest of the proof, we focus on the case where $\mathrm{sp}(G) < 1$.

By Lemma 4.1, A is irreducible if G is irreducible. From (1.1), we have $[I - A_1 - A_0(G + \boldsymbol{v}_0\boldsymbol{g})]Z(1) = A_2$. By the proof of Lemma 4.2, $I - A_1 - A_0G$ is invertible if G is irreducible. Since $I - A_1 - A_0G$ is invertible, we obtain $[I - (I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0\boldsymbol{g}]Z(1) = (I - A_1 - A_0G)^{-1}A_2 = G$, which leads to

$$\begin{aligned} Z(1) &= G + (I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0\boldsymbol{g}Z(1) \\ &= G + \sum_{k=0}^{\infty}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0[\boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0]^k\boldsymbol{g}\,\mathrm{sp}(G) \\ &= G + \frac{\mathrm{sp}(G)}{1 - \boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0\boldsymbol{g} \\ &\equiv G + \boldsymbol{v}_1\boldsymbol{g}, \end{aligned} \tag{4.2}$$

where $\boldsymbol{v}_1 = t_1(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0$ and $t_1 = \mathrm{sp}(G)/[1 - \boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0]$. Since $\boldsymbol{v}_0 \le (I - G)\boldsymbol{e}$, $\boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0 \le \boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0(I - G)\boldsymbol{e} = \boldsymbol{g}(I - G)\boldsymbol{e} = 1 - \mathrm{sp}(G) < 1$. (Note that $\mathrm{sp}(G) > 0$ since G is irreducible.) Thus, the sum on the second line of (4.2) converges. Further, we have $t_1 \le \mathrm{sp}(G)/[1 - (1 - \mathrm{sp}(G))] = 1(\equiv t_0)$. Inductively, we can show that $Z(n) = G + \boldsymbol{v}_n\boldsymbol{g}$, where

$$t_n = \frac{\mathrm{sp}(G)t_{n-1}}{1 - \boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_{n-1}}\ (\le t_{n-1}), \qquad n > 0;$$
$$\boldsymbol{v}_n = t_n(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_{n-1} = t_nt_{n-1}\cdots t_0[(I - A_1 - A_0G)^{-1}A_0]^n\boldsymbol{v}_0, \qquad n > 0. \tag{4.3}$$

By Lemma 4.2, $[(I - A_1 - A_0G)^{-1}A_0]^n\boldsymbol{v}_0 \le (I - G)\boldsymbol{e}$. Thus, $\boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_{n-1} \le 1 - \mathrm{sp}(G) < 1$.

Since G is irreducible, $\boldsymbol{g}$ is positive. If $(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0 < (I - G)\boldsymbol{e}$, then $\boldsymbol{g}(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0 < 1 - \mathrm{sp}(G)$ and $t_1 < \mathrm{sp}(G)/[1 - (1 - \mathrm{sp}(G))] = 1$. Thus, $t_n \le t_1 < 1$ for $n > 0$. By the second equality in (4.3), $\boldsymbol{v}_n$ converges to zero, which implies that $Z(n)$ converges to G. If $(I - A_1 - A_0G)^{-1}A_0\boldsymbol{v}_0 = (I - G)\boldsymbol{e}$, then $t_n = 1$ and $Z(n) = G + (I - G)\boldsymbol{eg}$ for $n \ge 1$. Therefore, $Z(n)$ converges to $G + (I - G)\boldsymbol{eg}$.

Theorem 4.3 shows that if $Z(0)$ is in the convex subset $\{G + \boldsymbol{v}_0\boldsymbol{g} : 0 \le \boldsymbol{v}_0 \le (I - G)\boldsymbol{e}\}$ of Ω and G is irreducible, then the sequence $\{Z(n), n \ge 0\}$ converges and there are only two possible limits: G and $G + (I - G)\boldsymbol{eg}$. That fact is used later in this paper (e.g. in the proof of Theorem 4.4).

Note 4.1. It seems possible to generalize Theorem 4.3 to cases with $Z(0)$ in the convex subset $\{G + \Lambda : \Lambda$ is nonnegative and $\Lambda\boldsymbol{e} \le (I - G)\boldsymbol{e}\}$ of Ω. That approach is left open for future research.

If G is not stochastic, $Z(0) \le G$ does not hold, or $Z(0) \ne G + \boldsymbol{v}_0\boldsymbol{g}$ with $0 \le \boldsymbol{v}_0 \le (I - G)\boldsymbol{e}$, then convergence of $\{Z(n), n \ge 0\}$ can be a complicated problem (see examples in Sections 6 and 7). Based on our experience, the complexity of the problem comes from the reducibility of the matrix A, the multiplicity of nonnegative solutions to (1.2), the reducibility of the matrix G, and the selection of the matrix $Z(0)$. For instance, according to Gail *et al.* [6], there can be many substochastic or stochastic solutions to (1.2) if G is not stochastic. Many of these solutions to (1.2) can be the limit of a sequence $\{Z(n), n \ge 0\}$ with some carefully chosen $Z(0)$ in Ω. In order to reduce the complexity of the problem and to identify the limit explicitly, we assume that the matrices G and $\hat{G}$ (see (3.6)) are primitive and show that the sequence $\{Z(n), n \ge 0\}$ converges to either G or $G + (I - G)\boldsymbol{eg}$. We shall then discuss briefly how to ensure that both G and $\hat{G}$ are primitive. In Sections 5–7, we shall identify conditions on $\{A_2, A_1, A_0\}$ for the limit to be G or $G + (I - G)\boldsymbol{eg}$.

Lemma 4.3. *If* $\mathrm{sp}(\hat{G}) < 1$, *then the sequence* $\{W(n), n \ge 0\}$ *converges to zero. If* $\mathrm{sp}(\hat{G}) = 1$, *we assume that* $\hat{G}$ *is primitive. Denote by* $\hat{\boldsymbol{g}}$ *the left eigenvector corresponding to the Perron–Frobenius eigenvalue of* $\hat{G}$. *Then the sequence* $\{W(n), n \ge 0\}$ *converges to* $\boldsymbol{v}_0\hat{\boldsymbol{g}}$, *where* $\boldsymbol{v}_0$ *is a nonnegative vector satisfying* $\boldsymbol{v}_0 \le (I - G)\boldsymbol{e}$.

Proof. If $\mathrm{sp}(\hat{G}) < 1$, then $\mathrm{sp}(\hat{G}^n) = (\mathrm{sp}(\hat{G}))^n \to 0$ and $\hat{G}^n \to 0$ when n goes to infinity. Since the sequence $\{\hat{G}(n), n \ge 0\}$ is nondecreasing, we have $W(n) \le \hat{G}(1)\cdots\hat{G}(n) \le \hat{G}^n \to 0$.

If $\mathrm{sp}(\hat{G}) = 1$ and $\hat{G}$ is primitive, then the left and right eigenvectors corresponding to $\mathrm{sp}(\hat{G}) = 1$ are unique. Since $\hat{G}\boldsymbol{e} \leq \boldsymbol{e}$ and $\hat{G}$ is irreducible, we must have $\hat{G}\boldsymbol{e} = \boldsymbol{e}$, i.e. $\hat{G}$ is a stochastic matrix. Since the sequence $\{W(n), n \geq 0\}$ is uniformly bounded, there must be a convergent subsequence, say $W(n_t) \to W_0$. Using the same argument, it can be concluded that there is a convergent subsequence of $\{W(n_t - 1)\}$ converging to a matrix W_1. Then we have $W_0 = W_1\hat{G}$ since $W(n_t) = W(n_t - 1)\hat{G}(n_t)$. Furthermore, we must have a convergent subsequence of $\{W(n_t - k)\}$ converging to a matrix W_k, and $W_0 = W_k\hat{G}^k$ for any $k > 0$. Since $\{W_k, k \geq 0\}$ is uniformly bounded, it has a convergent subsequence. Denote by W_∞ the limit of the convergent subsequence of $\{W_k, k \geq 0\}$. Since $\mathrm{sp}(\hat{G}) = 1$ is the only eigenvalue of $\hat{G}$ with modulus 1, we have $\hat{G}^k = \boldsymbol{e}\hat{\boldsymbol{g}} + o(1)$, which implies that $W_0 = W_k\hat{G}^k \to W_\infty\boldsymbol{e}\hat{\boldsymbol{g}} \equiv \boldsymbol{v}_0\hat{\boldsymbol{g}}$. Thus, we have proved that the limit of any convergent subsequence $\{W(n_t)\}$ of $\{W(n), n \geq 0\}$ has the form $\boldsymbol{v}_0\hat{\boldsymbol{g}}$.

Since $\hat{G}(n)\boldsymbol{e} \leq \boldsymbol{e}$, the sequence $\{W(n)\boldsymbol{e}, n \geq 0\}$ is nonincreasing. Thus, $\{W(n)\boldsymbol{e}, n \geq 0\}$ converges. This implies that $\boldsymbol{v}_0$ is the same for any convergent subsequence of $\{W(n), n \geq 0\}$. Since $\{W(n), n \geq 0\}$ is uniformly bounded, it converges and the limit has the form $\boldsymbol{v}_0\hat{\boldsymbol{g}}$, where $\boldsymbol{v}_0$ is a nonnegative vector. Since $[G(n) + W(n)]\boldsymbol{e} \leq \boldsymbol{e}$, we must have $\boldsymbol{v}_0 \leq (I - G)\boldsymbol{e}$.

Theorem 4.4. *Consider the sequence $\{Z(n), n \geq 0\}$ introduced in Section 1. Assume that the matrices G and $\hat{G}$ are primitive. Then $\{Z(n), n \geq 0\}$ converges and, for any $Z(0) \in \Omega$, the limit is either G or $G + (I - G)\boldsymbol{e}\boldsymbol{g}$.*

Proof. Under the assumptions, by Lemmas 3.4 and 4.3, the limit of any convergent subsequence of $\{Z(n), n \geq 0\}$ has the form $Z^* = G + \boldsymbol{v}_0\hat{\boldsymbol{g}}Z(0)Y$, where Y is the limit of a convergent subsequence of $\{Z(1)\cdots Z(n), n \geq 0\}$. Consider a convergent subsequence $\{Z(n_t), n_t \geq 0\}$ and denote its corresponding limit by $G + \boldsymbol{v}_0\boldsymbol{u}_0$ (where $\boldsymbol{u}_0 = \hat{\boldsymbol{g}}Z(0)Y$). Rewrite (3.5) as

$$
\begin{aligned}
Z(n_t) &= G(n_t) + W(n_t)Z(0)Z(1)\cdots Z(n_t - 1)Z(n_t) \\
&= G(n_t) + [W(n_t) - W(n_t - 1)]Z(0)Z(1)\cdots Z(n_t - 1)Z(n_t) \\
&\quad + [W(n_t - 1)Z(0)Z(1)\cdots Z(n_t - 1)]Z(n_t) \\
&= G(n_t) + [W(n_t) - W(n_t - 1)]Z(0)Z(1)\cdots Z(n_t - 1)Z(n_t) \\
&\quad + [Z(n_t - 1) - G(n_t - 1)]Z(n_t). \qquad (4.4)
\end{aligned}
$$

Denote the limit of a convergent subsequence of $\{Z(n_t - 1)\}$ by $G + \boldsymbol{v}_0\boldsymbol{u}_1$. By (4.4), we have $Z^* = G + (\boldsymbol{v}_0\boldsymbol{u}_1)Z^*$. Note that $W(n)$ converges by Lemma 4.3. Since $\mathrm{sp}(\boldsymbol{v}_0\boldsymbol{u}_1) < \mathrm{sp}(G + \boldsymbol{v}_0\boldsymbol{u}_1) \leq 1$ and G is irreducible, we have $\boldsymbol{u}_1\boldsymbol{v}_0 = \mathrm{sp}(\boldsymbol{v}_0\boldsymbol{u}_1) < 1$. It is then easy to see that $Z^* = G + \boldsymbol{v}_0\boldsymbol{u}_1G/(1 - \boldsymbol{u}_1\boldsymbol{v}_0) \equiv G + \boldsymbol{v}_0\boldsymbol{u}_2G$, where $\boldsymbol{u}_2 = \boldsymbol{u}_1/(1 - \boldsymbol{u}_1\boldsymbol{v}_0)$. It is clear that the same conclusion holds for any convergent subsequence of $\{Z(n_t - 2)\}$, i.e. the limit has the form $G + \boldsymbol{v}_0\boldsymbol{u}_2G$. Using $\boldsymbol{v}_0\boldsymbol{u}_2G$ for the limit of any convergent subsequence of $Z(n_t - 1) - G(n_t - 1)$ in (4.4), we obtain $Z^* = G + \boldsymbol{v}_0\boldsymbol{u}_3G^2$, where $\boldsymbol{u}_3$ is a vector obtained from the convergent subsequence of $\{Z(n_t - 2)\}$. In general, suppose that the limit of $\{Z(n_t) - G(n_t)\}$ has the form $\boldsymbol{v}_0\boldsymbol{u}_kG^{k-1}$ for k ($k > 1$). In the same spirit, there exists a convergent subsequence of $\{Z(n_t - 1) - G(n_t - 1)\}$ with a limit of the form $\boldsymbol{v}_0\boldsymbol{u}_kG^{k-1}$. Replacing $Z(n_t - 1) - G(n_t - 1)$ in (4.4) by $\boldsymbol{v}_0\boldsymbol{u}_kG^{k-1}$ when $n_t \to \infty$, we obtain $Z^* = G + \boldsymbol{v}_0\boldsymbol{u}_kG^{k-1}Z^*$, which implies that

$$
Z^* = G + \boldsymbol{v}_0\left(\sum_{t=0}^{\infty}(\boldsymbol{u}_kG^{k-1}\boldsymbol{v}_0)^t\right)\boldsymbol{u}_kG^k = G + \frac{\boldsymbol{v}_0\boldsymbol{u}_kG^k}{1 - \boldsymbol{u}_kG^{k-1}\boldsymbol{v}_0}. \qquad (4.5)
$$

Note that $\boldsymbol{u}_kG^{k-1}\boldsymbol{v}_0 = \mathrm{sp}(\boldsymbol{v}_0\boldsymbol{u}_kG^{k-1}) < \mathrm{sp}(G + \boldsymbol{v}_0\boldsymbol{u}_kG^{k-1}) \leq 1$. By (4.5), with a mild abuse of notation, we denote $Z^* = G + \boldsymbol{v}_0\boldsymbol{u}_kG^k$, where $\boldsymbol{u}_k$ is a nonnegative vector, for $k > 0$. If

$\boldsymbol{v}_0 = \mathbf{0}$, then $Z^* = G = G + \boldsymbol{v}_0\boldsymbol{g}$. Suppose that $\boldsymbol{v}_0$ is nonzero. Since $Z^*\boldsymbol{e} \leq \boldsymbol{e}$, $\boldsymbol{u}_k G^k$ is uniformly bounded. Since G is primitive, we have $G^k = (\mathrm{sp}(G))^k \boldsymbol{v}\boldsymbol{g} + o((\mathrm{sp}(G))^k)$, where $\boldsymbol{v}$ is the left eigenvector of G corresponding to $\mathrm{sp}(G)$. Thus, $\boldsymbol{u}_k G^k$ converges to $c\boldsymbol{g}$, where c is a nonnegative constant. This implies that $Z^* = G + \boldsymbol{v}_0\boldsymbol{g}$ (where $\boldsymbol{v}_0$ is for $c\boldsymbol{v}_0$). In summary, we have proved that $Z^* = G + \boldsymbol{v}_0\boldsymbol{g}$ with $\mathbf{0} \leq \boldsymbol{v}_0 \leq (I - G)\boldsymbol{e}$ for any convergent subsequence of $\{Z(n), n \geq 0\}$.

By Theorem 4.1, $Z^*\boldsymbol{e} = G\boldsymbol{e} + \boldsymbol{v}_0$ is the same for all convergent subsequences of $\{Z(n), n \geq 0\}$. This implies that $\boldsymbol{v}_0$ is the same for all convergent subsequences. Thus, the sequence $\{Z(n), n \geq 0\}$ converges and the limit has the form $Z^* = G + \boldsymbol{v}_0\boldsymbol{g}$ with $0 \leq \boldsymbol{v}_0 \leq (I - G)\boldsymbol{e}$ under the assumptions of Theorem 4.4. Since $Z(n+1) = f(Z(n))$, we obtain $Z^* = f(Z^*)$, which is equivalent to (1.2). Since G is primitive and Z^* has the form $G + \boldsymbol{v}_0\boldsymbol{g}$, Theorem 4.3 implies that Z^* is either G or $G + (I - G)\boldsymbol{e}\boldsymbol{g}$.

The proof of Theorem 4.4 (as well as that of Theorems 6.1 and 7.1) shows that the validity of the approximation $G^k = (\mathrm{sp}(G))^k \boldsymbol{v}\boldsymbol{g} + o((\mathrm{sp}(G))^k)$ for G^k (and for $\hat{G}^k$), as $k \to \infty$, is critical to the convergence of $\{Z(n), n \geq 0\}$. In order to get that approximation, we assumed that both G and $\hat{G}$ are primitive. But the conditions are restrictive and difficult to check directly.

Alternatively, we can impose appropriate conditions on the matrices $\{A, A_0, A_1, A_2\}$. Let $A^*(z) = A_0 + zA_1 + z^2A_2$. Assume that A is irreducible and the function $zI - A^*(z)$ is nonsingular for $|z| = 1$ except $z = 1$. By Theorem 4 in [6], the matrices G and $G + (I - G)\boldsymbol{e}\boldsymbol{g}$ are the only solutions to (1.2) in Ω. Furthermore, $\mathrm{sp}(G)$ is the only eigenvalue of G of modulus $\mathrm{sp}(G)$ and its geometric and algebraic degrees are one. Thus, the approximation for G^k is valid. The same results hold for the matrix $\hat{G}$. Therefore, the results in Theorem 4.4 (and Theorems 6.1 and 7.1) follow under these conditions on A and $zI - A^*(z)$.

5. The recurrent case

Section 4 shows that under certain conditions, $Z(n)$ converges to either G or $G + (I - G)\boldsymbol{e}\boldsymbol{g}$. We still need to distinguish between these two alternatives. Here, and in Sections 6 and 7, we identify conditions for either limit. Using examples, we show why our conditions are necessary.

Mainly, we distinguish between the case $\boldsymbol{\theta}A_2\boldsymbol{e} \geq \boldsymbol{\theta}A_0\boldsymbol{e}$ and the case $\boldsymbol{\theta}A_2\boldsymbol{e} < \boldsymbol{\theta}A_0\boldsymbol{e}$. The conditions $\boldsymbol{\theta}A_2\boldsymbol{e} > \boldsymbol{\theta}A_0\boldsymbol{e}$, $\boldsymbol{\theta}A_2\boldsymbol{e} = \boldsymbol{\theta}A_0\boldsymbol{e}$, and $\boldsymbol{\theta}A_2\boldsymbol{e} < \boldsymbol{\theta}A_0\boldsymbol{e}$ are used to classify the QBD Markov chain Q defined in Section 3: $\boldsymbol{\theta}A_2\boldsymbol{e} > \boldsymbol{\theta}A_0\boldsymbol{e}$ for positive recurrence, $\boldsymbol{\theta}A_2\boldsymbol{e} = \boldsymbol{\theta}A_0\boldsymbol{e}$ for null recurrence, and $\boldsymbol{\theta}A_2\boldsymbol{e} < \boldsymbol{\theta}A_0\boldsymbol{e}$ for transience. The condition $\boldsymbol{\theta}A_2\boldsymbol{e} > \boldsymbol{\theta}A_0\boldsymbol{e}$ is called Neuts' drift condition for the positive recurrence of QBD Markov chains.

The case $\boldsymbol{\theta}A_2\boldsymbol{e} > \boldsymbol{\theta}A_0\boldsymbol{e}$ can easily be dealt with. If $\boldsymbol{\theta}A_2\boldsymbol{e} > \boldsymbol{\theta}A_0\boldsymbol{e}$, then the Markov chain Q_n is, in general, drifting towards level zero. Thus, the possibility of Q_n hitting the reflecting barrier (given that Q_n starts in level 1) tends to zero when n goes to infinity, i.e. $\lim_{n\to\infty} D(n) = 0$. Thus, we expect that $\{Z(n), n \geq 0\}$ converges to G in this case. The second case ($\boldsymbol{\theta}A_2\boldsymbol{e} = \boldsymbol{\theta}A_0\boldsymbol{e}$) differs from the first in the behavior of the Markov chains Q_n and Q, but the same result holds for $\{Z(n), n \geq 0\}$. The results are summarized in the following theorem.

Theorem 5.1. *If $\boldsymbol{\theta}A_2\boldsymbol{e} \geq \boldsymbol{\theta}A_0\boldsymbol{e}$, then the matrix G is stochastic and is the unique solution in Ω to (1.2). For any $Z(0) \in \Omega$, the sequence $\{Z(n), n \geq 0\}$ converges to G.*

Proof. It is proved in [18] that, under the stated conditions, the matrix G is stochastic and it is the unique solution to (1.2) in Ω. By Theorem 4.2, the sequence $\{Z(n), n \geq 0\}$ converges to G for any $Z(0) \in \Omega$.

For the case $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$, the problem is more involved. Some conditions on the matrices A, G, and $Z(0)$ are necessary to identify the limit. Since the limit has much to do with $Z(0)$, we shall distinguish three cases according to $Z(0)$: (i) $Z(0)$ is stochastic, (ii) $Z(0)$ is *strictly substochastic* (i.e. $Z(0)\boldsymbol{e} \le \varepsilon \boldsymbol{e}$ for some positive $\varepsilon < 1$), and (iii) other cases (i.e. some components of $Z(0)\boldsymbol{e}$ are 1 while others are less than 1). Note that we use the term 'strictly substochastic' with a slightly different meaning than in the general literature. The first case is dealt with in Section 6 and the second and third in Section 7.

6. The stochastic case: $Z(0)\boldsymbol{e} = \boldsymbol{e}$

If $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$, then the Markov chain Q_n is, in general, drifting away from level 0. So the matrix G is not stochastic. On the other hand, if $Z(0)\boldsymbol{e} = \boldsymbol{e}$, then level $n+1$ is a solid reflecting barrier for Q_n. The Markov chain Q_n should eventually be absorbed into level 0 for any finite n. Thus, it is expected that $\{Z(n), n \ge 0\}$ converges to a stochastic matrix. Unfortunately, Example 6.1(a) demonstrates that $\{Z(n), n \ge 0\}$ does not have to converge to a stochastic matrix if $Z(0)$ is stochastic, even though the matrix G is primitive. Therefore, other conditions are required to guarantee that $\{Z(n), n \ge 0\}$ converges to a stochastic matrix. Some of these conditions are identified in this section.

Theorem 6.1. *Assume that the matrix G is primitive, that $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$, and that $Z(0)\boldsymbol{e} = \boldsymbol{e}$. If* $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$, *for all $n \ge 0$, then the sequence $\{Z(n), n \ge 0\}$ converges to the matrix $G + (I - G)\boldsymbol{e}\boldsymbol{g}$. On the other hand, if $\{Z(n), n \ge 0\}$ converges to $G + (I - G)\boldsymbol{e}\boldsymbol{g}$ for some stochastic matrix $Z(0)$, then* $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ *for all sufficiently large n.*

Proof. The proof is given in Appendix A.

Note 6.1. In this section and in Section 7, the matrix $\hat{G}$ is not used. But the proof of Theorem 6.1 can be significantly reduced if we assume that the matrix $\hat{G}$ is also primitive. In that case, Theorem 4.4 is used. The proof is as follows. Since $Z(0)\boldsymbol{e} = \boldsymbol{e}$ and $\mathrm{sp}(A_1 + A_0 Z(0)) < 1$, we must have $Z(1)\boldsymbol{e} = \boldsymbol{e}$ by Lemma 2.2. By induction, it is easy to prove that $Z(n)\boldsymbol{e} = \boldsymbol{e}$ for all n since $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ for all n. Thus, any convergent subsequence of $\{Z(n), n \ge 0\}$ converges to a stochastic matrix. Therefore, by Theorem 4.4, $\{Z(n), n \ge 0\}$ converges to $G + (I - G)\boldsymbol{e}\boldsymbol{g}$. The proof of necessity is the same as that in Appendix A.

Intuitively, the condition $\{\mathrm{sp}(A_1 + A_0 Z(n)) < 1, n \ge 0\}$ ensures that there is no closed subset of states (except levels 0 and $n+2$) for all $\{Q_n, n \ge 0\}$. The condition $Z(0)\boldsymbol{e} = \boldsymbol{e}$ ensures that Q_n will never be absorbed into level $n+2$ for all n. Thus, $\{Z(n), n \ge 0\}$ are all stochastic. The sequence converges to a stochastic matrix. To explain why the limiting matrix is $G + (I - G)\boldsymbol{e}\boldsymbol{g}$, we introduce the (fictitious) level ∞ (infinity) as a reflecting barrier for the Markov chain Q. Level ∞ is a solid reflecting barrier since $Z(0)$ is stochastic. The first part of the limit, G, represents the absorption into level 0 without hitting the reflecting barrier. The second part, $(I - G)\boldsymbol{e}\boldsymbol{g}$, is interpreted as follows. Elements of $(I - G)\boldsymbol{e}$ are the probabilities that the Markov chain Q hits the reflecting barrier (level ∞). The vector $\boldsymbol{g}$, which is the quasistationary distribution of G, is the distribution of states when the Markov chain Q eventually enters level 0 after being forced back from the *remote* solid reflecting barrier (level ∞).

Theorem 6.1 shows that, if $Z(0)$ is stochastic, the condition $\{\mathrm{sp}(A_1 + A_0 Z(n)) < 1, n \ge 0\}$ is a sufficient condition and is 'almost' a necessary condition for $\{Z(n), n \ge 0\}$ to converge to $G + (I - G)\boldsymbol{e}\boldsymbol{g}$. However, the condition $\{\mathrm{sp}(A_1 + A_0 Z(n)) < 1, n \ge 0\}$ is not necessary for the convergence of $\{Z(n), n \ge 0\}$. If the condition $\{\mathrm{sp}(A_1 + A_0 Z(n)) < 1, n \ge 0\}$ does not hold, Example 6.1 shows that the matrix $Z(0)$ can affect the limit of $\{Z(n), n \geq 0\}$ in a major way.

Example 6.1. Set

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0.8 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}.$$

Clearly, $A = A_0 + A_1 + A_2$ is irreducible and aperiodic. It can be shown that $\boldsymbol{\theta} = (1/1.9, 0.9/1.9)$, $\boldsymbol{\theta} A_2 \boldsymbol{e} = 0.2/1.9 < \boldsymbol{\theta} A_0 \boldsymbol{e} = 0.9/1.9$, and

$$G = \begin{pmatrix} 0.111111 & 0.111111 \\ 0.013889 & 0.013889 \end{pmatrix},$$

which is irreducible and aperiodic.

(a) Choose

$$Z(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathrm{sp}(A_1 + A_0 Z(0)) = 1$,

$$Z(1) = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix},$$

and $\lim_{n\to\infty} Z(n) = G$. For $n > 0$, $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$.

(b) Choose

$$Z(0) = \begin{pmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{pmatrix},$$

for which $\mathrm{sp}(A_1 + A_0 Z(n)) < 1, n \geq 0$. Then

$$\lim_{n\to\infty} Z(n) = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} = G + (I - G)\boldsymbol{e}\boldsymbol{g}.$$

Carefully examining Example 6.1(a), we find that a combination of matrices A_0 and $Z(0)$ may create closed subsets in the state space of the Markov chain $Q_n, n \geq 0$. This implies that $\mathrm{sp}(A_1 + A_0 Z(0)) = 1$. This is why $\{Z(n), n \geq 0\}$ may not converge to a stochastic matrix even when there is a solid reflecting barrier.

In the remainder of this section, we shall focus on the matrices A_1 and A_2 in order to find simple conditions that ensure that $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ for $n \geq 0$ and G is primitive.

Corollary 6.1. *Assume that the matrix G is primitive and that $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$. If for some positive ε, $A_2 \boldsymbol{e} \geq \varepsilon \boldsymbol{e}$ holds, then the sequence $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{e}\boldsymbol{g}$ for any stochastic $Z(0)$.*

Proof. By Theorem 6.1, it is sufficient to prove that, under the current conditions, $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ for $n \geq 0$. For $n = 0$, if $\mathrm{sp}(A_1 + A_0 Z(0)) = 1$, there exists a nonzero, nonnegative vector $\boldsymbol{u}$ such that $\boldsymbol{u}(A_1 + A_0 Z(0)) = \boldsymbol{u}$ and $\boldsymbol{u}\boldsymbol{e} = 1$. Since $Z(0)\boldsymbol{e} = \boldsymbol{e}$, $1 = \boldsymbol{u}(A_1 + A_0 Z(0))\boldsymbol{e} = \boldsymbol{u}(A_1 + A_0)\boldsymbol{e} = \boldsymbol{u}(A_0 + A_1 + A_2)\boldsymbol{e} - \boldsymbol{u} A_2 \boldsymbol{e} = 1 - \boldsymbol{u} A_2 \boldsymbol{e}$. This implies that $\boldsymbol{u} A_2 \boldsymbol{e} = 0$, which contradicts $\boldsymbol{u} A_2 \boldsymbol{e} \geq \boldsymbol{u}\varepsilon\boldsymbol{e} = \varepsilon > 0$. Thus, $\mathrm{sp}(A_1 + A_0 Z(0)) < 1$. By Lemma 2.2, $Z(1)$ is stochastic. The proof can then be completed by induction.

Intuitively, the condition $A_2\boldsymbol{e} \geq \varepsilon\boldsymbol{e}$ for some positive ε ensures that the Markov chain Q_n can move towards level 0 (jump to the left) at any state. It implies that there will be no closed subset for any Q_n. Thus, $\{Z(n), n \geq 0\}$ converges to a stochastic matrix.

In order to use Corollary 6.1, we need to check whether the matrix G is primitive or not. Unfortunately, this cannot be verified directly. Thus, we present the following operational version of Corollary 6.1 (with possibly stronger conditions).

Corollary 6.2. (i) *If $\boldsymbol{\theta} A_2\boldsymbol{e} < \boldsymbol{\theta} A_0\boldsymbol{e}$, the matrix A_2 is irreducible, and the matrix G is aperiodic, then the sequence $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$ for any stochastic $Z(0)$.*

(ii) *If $\boldsymbol{\theta} A_2\boldsymbol{e} < \boldsymbol{\theta} A_0\boldsymbol{e}$, and the matrix A_2 is primitive, then the sequence $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$ for any stochastic $Z(0)$.*

Proof. First, the irreducibility of A_2 implies that G is irreducible since $G \geq A_2$. It also implies that $A_2\boldsymbol{e} \geq \varepsilon\boldsymbol{e}$ for some positive ε. Thus, all the conditions in Corollary 6.1 are satisfied. Second, if A_2 is primitive, G is primitive since $G \geq A_2$. Thus, all the conditions in Corollary 6.1 are satisfied.

Example 6.1(b) shows that conditions on the matrix A_2 given in Corollary 6.1 and Corollary 6.2 are not necessary for $\{Z(n), n \geq 0\}$ to converge to $G + (I - G)\boldsymbol{eg}$. Next, we identify another condition to ensure that $\text{sp}(A_1 + A_0Z(n)) < 1$ for $n \geq 0$.

Corollary 6.3. *Assume that G is primitive and that $\boldsymbol{\theta} A_2\boldsymbol{e} < \boldsymbol{\theta} A_0\boldsymbol{e}$. If the matrix $A_1 + A_2$ is irreducible, then the sequence $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$ for any stochastic $Z(0)$.*

Proof. Similar to Corollary 6.1, it is sufficient to prove that, under all these conditions, $\text{sp}(A_1 + A_0Z(n)) < 1, n \geq 0$. For $n = 0$, suppose that $\text{sp}(A_1 + A_0Z(0)) = 1$. Since $A_1 + A_2$ is irreducible and A_2 is nonzero, we must have $\text{sp}(A_2 + A_1 + A_0Z(0)) > \text{sp}(A_1 + A_0Z(0)) = 1$. But $(A_2 + A_1 + A_0Z(0))\boldsymbol{e} = (A_2 + A_1 + A_0)\boldsymbol{e} = \boldsymbol{e}$, which implies that $\text{sp}(A_2 + A_1 + A_0 Z(0)) = 1$. This is a contradiction. Therefore, $\text{sp}(A_1 + A_0Z(0)) < 1$ and $Z(1)$ is stochastic. The corollary is proved by induction.

Again, Example 6.1(b) shows that the condition that $A_1 + A_2$ is irreducible is not necessary for $\text{sp}(A_1 + A_0Z(n)) < 1, n \geq 0$. However, if A_0 has a row of zeros or $A_1 + A_2$ is reducible, the problem becomes more complicated. We use the following example to further demonstrate the impact of the structure of the matrices A_1 and A_2 on the convergence of $\{Z(n), n \geq 0\}$.

Example 6.2. Set

$$A_0 = \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.5 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{pmatrix}.$$

Corollary 6.1 proves that for any stochastic $Z(0)$, $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$. However, small changes in A_1 and A_2 can affect the result dramatically.

Reset

$$A_0 = \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.5 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}.$$

In this case, $A_1 + A_2$ is reducible and $\text{sp}(A_1 + A_0Z(0)) = 1$ when

$$Z(0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Nonetheless, it can be shown that

$$Z(1) = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ 0 & 0 \end{pmatrix}$$

and $\{Z(n), n \geq 0\}$ converges to

$$G = \begin{pmatrix} 0.11699 & 0.11699 \\ 0.00692 & 0.00692 \end{pmatrix}.$$

It is also easy to verify that $\{Z(n), n \geq 0\}$ converges to

$$G + (I - G)\boldsymbol{eg} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

if

$$Z(0) = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}.$$

Note 6.2. The condition that $A_1 + A_0 Z(n)$ is irreducible for all n guarantees that $\mathrm{sp}(A_1 + A_0 Z(n)) < 1,\ n \geq 0$. But checking this condition is as hard as checking that $\mathrm{sp}(A_1 + A_0 Z(n)) < 1,\ n \geq 0$, directly.

To end this section, we give an operational version of Corollary 6.3.

Corollary 6.4. (i) *If $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$, the matrix $(I - A_1)^{-1} A_2$ is irreducible, and the matrix G is aperiodic, then the sequence $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$ for any stochastic $Z(0)$.*

(ii) *If $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$ and the matrix $(I - A_1)^{-1} A_2$ is primitive, then the sequence $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$ for any stochastic $Z(0)$.*

Proof. Since $G \geq (I - A_1)^{-1} A_2$, all the conditions in Corollary 6.3 are satisfied. This proves the first part of the corollary. The second part is obtained similarly.

7. The substochastic case: $Z(0)\boldsymbol{e} \neq \boldsymbol{e}$

If $Z(0)$ is not stochastic, then the reflecting barrier is leaking, i.e. Q_n can go from level $n + 1$ to level $n + 2$. Consequently, Q_n can go from level 1 to level $n + 2$. Thus, if $Z(0)$ is not stochastic, we expect $\{Z(n), n \geq 0\}$ to converge to the matrix G. However, Example 7.1 shows that $\{Z(n), n \geq 0\}$ may converge to the matrix $G + (I - G)\boldsymbol{eg}$ even if $Z(0)$ is not stochastic. In this section, we identify conditions guaranteeing the convergence of $\{Z(n), n \geq 0\}$ to G if $Z(0)$ is not stochastic. The main result is given in the following theorem.

Theorem 7.1. *Assume that the matrix G is primitive and that $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$. The sequence $\{Z(n), n \geq 0\}$ converges to G if and only if* $\limsup_{n\to\infty} \mathrm{sp}(Z(n)) < 1$. *Furthermore, if the sequence $\{Z(n), n \geq 0\}$ converges to G,* $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ *for large enough n.*

Proof. The proof is given in Appendix B.

Note 7.1. The proof of Theorem 7.1 can be significantly reduced if the matrix $\hat{G}$ is also primitive. The proof is as follows. According to Theorem 4.4, $\{Z(n), n \geq 0\}$ converges to either G or $G + (I - G)\boldsymbol{eg}$. If $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$, we must have $\limsup_{n\to\infty} \mathrm{sp}(Z(n)) = 1$, which contradicts the assumption. Therefore, $\{Z(n), n \geq 0\}$ converges to G. The proof of necessity is the same as that in Appendix B.

The following example shows that $\{Z(n), n \geq 0\}$ may converge to $G + (I - G)\boldsymbol{eg}$ even if $Z(0)$ is not stochastic.

Example 7.1. Set

$$A_0 = \begin{pmatrix} 0.2 & 0 \\ 0.9 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0.3 & 0.3 \\ 0 & 0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}.$$

(a) If

$$Z(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then $\{Z(n), n \geq 0\}$ converges to

$$G = \begin{pmatrix} 0.16667 & 0.16667 \\ 0.03333 & 0.03333 \end{pmatrix}.$$

For this case, $\text{sp}(A_1 + A_0 Z(n)) < 1$ holds only for $n \geq 1$.

(b) If

$$Z(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then, even though $Z(0)$ is not stochastic,

$$Z(n) = G + (I - G)\boldsymbol{eg} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \qquad n \geq 1.$$

For this case, $\text{sp}(A_1 + A_0 Z(n)) < 1$ holds for $n \geq 0$.

Example 7.1 shows that the condition that $\text{sp}(A_1 + A_0 Z(n)) < 1$ holds for $n \geq 1$ does not guarantee that the sequence $\{Z(n), n \geq 0\}$ converges to either G or $G+(I-G)\boldsymbol{eg}$ if $Z(0)$ is not stochastic. However, $\text{sp}(A_1 + A_0 Z(n)) < 1$ for large enough n is necessary for $\{Z(n), n \geq 0\}$ to converge to G.

Two conditions are identified to ensure $\limsup_{n\to\infty} \text{sp}(Z(n)) < 1$ if $Z(0)$ is not stochastic. First, we focus on the case where $Z(0)$ is strictly substochastic.

Corollary 7.1. *Assume that the matrix G is primitive and that $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$. If $Z(0)\boldsymbol{e} \leq \varepsilon \boldsymbol{e}$ for some positive $\varepsilon < 1$, then the sequence $\{Z(n), n \geq 0\}$ converges to G.*

Proof. By Theorem 7.1, it is sufficient to show that $\limsup_{n\to\infty} \text{sp}(Z(n)) < 1$. If $\limsup_{n\to\infty} \text{sp}(Z(n)) = 1$, there exists a convergent subsequence $\{\text{sp}(Z(n_t)), n_t \geq 0\}$ such that $\lim_{n_t\to\infty} \text{sp}(Z(n_t)) = 1$. Since $\{Z(n_t), n_t \geq 0\}$ is uniformly bounded, there exists a convergent subsequence $\{Z(n_{t'}), n_{t'} \geq 0\}$ that converges to $\bar{Z}$, say, with $\bar{Z}\boldsymbol{e} \leq \boldsymbol{e}$. It is easy to see that $\text{sp}(\bar{Z}) = 1$. Since $\bar{Z} \geq G$ and G is irreducible, the matrix $\bar{Z}$ is irreducible. Then we must have $\bar{Z}\boldsymbol{e} = \boldsymbol{e}$, otherwise $\text{sp}(\bar{Z}) < 1$. Since $Z(n)\boldsymbol{e} \leq \boldsymbol{e}$ for $n \geq 0$ and $W(n)\boldsymbol{e} \leq \boldsymbol{e} - G(n)\boldsymbol{e}$,

$$\boldsymbol{e} = \bar{Z}\boldsymbol{e} \leq \lim_{n\to\infty} [G(n)\boldsymbol{e} + W(n)Z(0)\boldsymbol{e}] \leq G\boldsymbol{e} + \lim_{n\to\infty} W(n)\varepsilon\boldsymbol{e} \leq \boldsymbol{e} - \varepsilon(I - G)\boldsymbol{e}. \tag{7.1}$$

Since $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$, $(I - G)\boldsymbol{e} \neq \boldsymbol{0}$, which contradicts (7.1).

Intuitively, $Z(0)\boldsymbol{e} \leq \varepsilon\boldsymbol{e}$ implies that every state of the reflecting barrier is leaking. Under the drift condition $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$, there is always a chance for Q_n to hit the leaking reflecting barrier and be absorbed into level $n + 2$. Thus, we expect that the probability that the Markov chain will be absorbed into level 0 is less than 1.

Next, we relax the condition on $Z(0)$ by assuming only that $Z(0)$ is not stochastic. On the other hand, we add the condition that the matrix A_0 is primitive. This condition is contrasted to the condition that $A_1 + A_2$ is irreducible given in Corollary 6.3.

Corollary 7.2. *Assume that G is primitive and that* $\boldsymbol{\theta} A_2 \boldsymbol{e} < \boldsymbol{\theta} A_0 \boldsymbol{e}$. *If* A_0 *is primitive, then the sequence* $\{Z(n), n \geq 0\}$ *converges to G for any* $Z(0)\boldsymbol{e} \neq \boldsymbol{e}$ *and* $Z(0) \in \Omega$.

Proof. By Theorem 7.1, it is sufficient to show that $\limsup_{n\to\infty} \mathrm{sp}(Z(n)) < 1$. Since A_0 is irreducible and aperiodic, every element of A_0^m is positive (recall that A_0 is an $m \times m$ matrix). By definition, $Z(m)\boldsymbol{e} \leq \boldsymbol{e} - A_0^m(\boldsymbol{e} - Z(0)\boldsymbol{e})$. Since every element of A_0^m is positive and $\boldsymbol{e} - Z(0)\boldsymbol{e}$ is nonzero, the vector $A_0^m(\boldsymbol{e} - Z(0)\boldsymbol{e})$ is positive.

If $\limsup_{n\to\infty} \mathrm{sp}(Z(n)) = 1$, then there exists a convergent subsequence $\{\mathrm{sp}(Z(n_t)), t \geq 0\}$ such that $\lim_{n_t\to\infty} \mathrm{sp}(Z(n_t)) = 1$. Conditioning on the first passage time from level 1 to level $n-m$ for the Markov chain Q_n, we rewrite (3.5) as $Z(n) = G(n-m) + W(n-m)\, Z(m) \cdots Z(n)$. Then we obtain $Z(n)\boldsymbol{e} \leq G(n-m)\boldsymbol{e} + W(n-m)Z(m)\boldsymbol{e}$. For any convergent subsequence of $\{Z(n_t), t \geq 0\}$ and $\{W(n_t - m), t \geq 0\}$, we have (denote by $\bar{Z}$ and $\bar{W}$ their respective limits)

$$\bar{Z}\boldsymbol{e} \leq G\boldsymbol{e} + \bar{W}[\boldsymbol{e} - A_0^m(\boldsymbol{e} - Z(0)\boldsymbol{e})] < G\boldsymbol{e} + \bar{W}\boldsymbol{e} \leq \boldsymbol{e}.$$

Since $\bar{Z}$ $(\geq G)$ is irreducible, we have $\mathrm{sp}(\bar{Z}) < 1$, which contradicts $\lim_{n_t\to\infty} \mathrm{sp}(Z(n_t)) = 1$.

Note 7.2. The condition that $A_0\boldsymbol{e} \geq \varepsilon\boldsymbol{e}$ for some positive ε does not guarantee that the sequence $\{Z(n), n \geq 0\}$ converges to G for an arbitrary substochastic $Z(0)$. See Example 7.1 for a counterexample.

8. Generalizations to the matrix R

In this section, we extend the results for G to the matrix R that arises in the matrix-geometric solution ([16] and [24]). We do so by two methods. The first method consists in directly translating the results from G to the corresponding results for R. That method is based on the duality between G and R. The second method is based on the probabilistic interpretation of the matrices G and R. That method is interesting since it provides insight into the Markov chains $\{Q, Q_n, n \geq 0\}$ introduced in Section 3.

8.1. A duality approach

We consider a matrix sequence $\{L(n), n \geq 0\}$ generated as follows. Let $L(0)$ be a nonnegative matrix and $L(n+1)$ the minimal nonnegative solution to the equation

$$L(n+1) = A_0 + L(n+1)[A_1 + L(n)A_2], \qquad n \geq 0.$$

The matrix R is defined to be the minimal nonnegative solution to the equation

$$R = A_0 + RA_1 + R^2 A_2.$$

We are interested in the relationship between the sequence $\{L(n), n \geq 0\}$ and the matrix R, a problem that is analogous to that of $\{Z(n), n \geq 0\}$ and G. The dual relationship between the matrices G and R provides ready answers.

Let $\Lambda = \mathrm{diag}(\boldsymbol{\theta})$, $\tilde{A}_2 = \Lambda^{-1}A_0^\top\Lambda$, $\tilde{A}_1 = \Lambda^{-1}A_1^\top\Lambda$, and $\tilde{A}_0 = \Lambda^{-1}A_2^\top\Lambda$, where '$\top$' represents matrix transpose. Let $\tilde{G}$ be the minimal nonnegative solution to (1.2) when $\{\tilde{A}_0, \tilde{A}_1, \tilde{A}_2\}$ replaces $\{A_0, A_1, A_2\}$. For $\tilde{Z}(0) \in \Omega$, a sequence $\{\tilde{Z}(n), n \geq 0\}$ is generated by using (1.1) when $\{\tilde{A}_0, \tilde{A}_1, \tilde{A}_2\}$ replaces $\{A_0, A_1, A_2\}$. It was shown in [3] and [21] that the matrices R and $\tilde{G}$ have a dual relationship $\tilde{G} = \Lambda^{-1} R^{\top} \Lambda$. It is also easy to see that, if $\tilde{Z}(0) = \Lambda^{-1}(L(0))^{\top}\Lambda \in \Omega$, then $\tilde{Z}(n) = \Lambda^{-1}(L(n))^{\top}\Lambda, n \geq 0$. Thus, all the results obtained in Sections 2–7 can be passed on to the matrix R and the sequence $\{L(n), n \geq 0\}$ through the matrices $\{\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \tilde{G}\}$ and their corresponding sequence $\{\tilde{Z}(n) = \Lambda^{-1}(L(n))^{\top}\Lambda, n \geq 0\}$. Details are omitted.

8.2. The expected number of visits to level 2

In [16], the matrix R is interpreted as the expected number of visits to level $n+1$ before reaching level n or lower levels, given that the Markov chain Q is in level n initially. The above duality approach does not give similar probabilistic interpretations to $L(n)$. We now take a probabilistic approach and introduce a sequence $\{R(n), n \geq 0\}$ directly from Markov chain $\{Q_n, n \geq 0\}$.

In [10], the following relationships have been established for the Markov chain Q:

$$R = A_0 + RU; \qquad G = A_2 + UG; \qquad U = A_1 + RA_2 = A_1 + A_0 G,$$

where U is an $m \times m$ matrix whose (i,j)th element is the probability that the Markov chain Q reaches level 1 in state $(1, j)$ before visiting any other state in levels 0 or 1, given that the Markov chain Q is in state $(1, i)$ initially. Next, we introduce matrix sequences $\{R(n), n \geq 0\}$ and $\{U(n), n \geq 0\}$ for $\{Q_n, n \geq 0\}$ and establish similar relationships between $\{R(n), n \geq 0\}$, $\{U(n), n \geq 0\}$, and $\{Z(n), n \geq 0\}$.

Definition 8.1. The matrix $R(n)$ is an $m \times m$ matrix whose (i, j)th element is the expected number of visits to the state $(2, j)$ before reaching levels 1 or 0, given that the Markov chain Q_n is in state $(1, i)$ initially.

Definition 8.2. The matrix $U(n)$ is an $m \times m$ matrix whose (i, j)th element is the probability that the Markov chain Q_n reaches level 1 in state $(1, j)$ before visiting any other state in levels 0 or 1, given that the Markov chain Q_n is in state $(1, i)$ initially.

By the definitions and probabilistic arguments, it can be shown that (details are omitted)

$$R(n) = \begin{cases} 0, & n = 0, \\ A_0, & n = 1, \\ A_0 + R(2)A_1 + R(2)R(1)Z(0), & n = 2, \\ A_0 + R(n)A_1 + R(n)R(n-1)A_2, & n \geq 3, \end{cases}$$
$$U(n) = \begin{cases} 0, & n = 0, \\ A_1 + A_0 Z(0), & n = 1, \\ A_1 + R(n)A_2 = A_1 + A_0 Z(n-1), & n \geq 2. \end{cases} \tag{8.1}$$

Equation (8.1) leads to the following interesting relationships:

$$R(n) = \begin{cases} A_0 + R(n)U(n-1), & n \geq 1, \\ A_0 + R(n)A_1 + R(n)A_0 Z(n-2), & n \geq 2, \end{cases}$$
$$Z(n) = \begin{cases} A_2 + U(n)Z(n), & n \geq 1, \\ A_2 + A_1 Z(n) + R(n)A_2 Z(n), & n \geq 2. \end{cases} \tag{8.2}$$

Theorem 8.1. *Assume that G is irreducible. If $\{Z(n), n \geq 0\}$ converges to G, then $\{R(n), n \geq 0\}$ converges to $R = A_0(I - A_1 - A_0G)^{-1}$. If $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$, then $\{R(n), n \geq 0\}$ converges to $R + A_0(I - G)\boldsymbol{eg}(I - A_1 - A_0G)^{-1}/\mathrm{sp}(G)$.*

Proof. If $\{Z(n), n \geq 0\}$ converges to G, then $\mathrm{sp}(A_1 + A_0Z(n)) < 1$ for large enough n by Theorem 7.1. By (8.1) and Theorem 7.1, when n is large enough, $R(n) = A_0[I - A_1 - A_0Z(n-2)]^{-1}$. Letting n tend to infinity, $R(n)$ converges to $A_0(I - A_1 - A_0G)^{-1} = R$. The matrix $I - A_1 - A_0G$ is invertible since G is irreducible. If $\{Z(n), n \geq 0\}$ converges to $G + (I - G)\boldsymbol{eg}$, $\mathrm{sp}(A_1 + A_0Z(n)) < 1$ for large enough n by Theorem 6.1. Thus,

$$\lim_{n\to\infty} R(n) = R + \lim_{n\to\infty} R(n)[A_0(I - G)\boldsymbol{eg}(I - A_1 - A_0G)^{-1}].$$

Using $A_0(I - A_1 - A_0G)^{-1} = R$ and Lemma 4.2, similar to the proof of Theorem 4.3, we obtain the expected result.

9. Summary

In this paper, we studied the convergence of the sequence $\{Z(n), n \geq 0\}$ generated by using (1.1). We have shown that, if the matrix G is stochastic or if $Z(0) \leq G$, then the sequence $\{Z(n), n \geq 0\}$ converges to G. We have also shown that, if the matrix G and the matrix $\hat{G}$ are primitive, then the sequence $\{Z(n), n \geq 0\}$ converges to either G or $G + (I - G)\boldsymbol{eg}$.

We then identified conditions for the limit to be G or $G + (I - G)\boldsymbol{eg}$. Under the assumption that $\boldsymbol{\theta} A_2\boldsymbol{e} \geq \boldsymbol{\theta} A_0\boldsymbol{e}$, we have proved that the sequence $\{Z(n), n \geq 0\}$ converges to the stochastic matrix G for any stochastic or substochastic $Z(0)$. Assuming that G is primitive and $\boldsymbol{\theta} A_2\boldsymbol{e} < \boldsymbol{\theta} A_0\boldsymbol{e}$, we have proved that (i) if $Z(0)$ is stochastic, the sequence $\{Z(n), n \geq 0\}$ converges to the matrix $G + (I - G)\boldsymbol{eg}$ if 'and only if' $\mathrm{sp}(A_1 + A_0Z(n)) < 1, n \geq 0$; (ii) if $Z(0)$ is not stochastic, the sequence $\{Z(n), n \geq 0\}$ converges to G if and only if $\limsup_{n\to\infty} \mathrm{sp}\,(Z(n)) < 1$. Two conditions are identified to ensure that $\mathrm{sp}(A_1 + A_0Z(n)) < 1, n \geq 0$: (i) $A_2\boldsymbol{e} \geq \varepsilon\boldsymbol{e}$ for some positive ε, and (ii) the matrix $A_1 + A_2$ is irreducible. Two conditions are identified to ensure that $\limsup_{n\to\infty} \mathrm{sp}(Z(n)) < 1$: (i) $Z(0)\boldsymbol{e} \leq \varepsilon\boldsymbol{e}$ for some positive $\varepsilon < 1$, and (ii) the matrix A_0 is primitive.

The results obtained in this paper were extended to two matrix sequences related to the matrix R in the matrix-geometric solution by using some relationships between G and R.

Some open problems remain. For instance, if (1.2) has many solutions and the matrix G is not primitive, what will be the limit of $\{Z(n), n \geq 0\}$? Furthermore, it appears that the results for the QBD Markov chains can be generalized to the $M/G/1$ paradigm. But the problem becomes tedious and many more details have to be worked out.

Appendix A. Proof of Theorem 6.1

Assume that the matrix G is primitive. By Lemma 2.3 and Lemma 3.2, it is sufficient to prove that $\{D(n), n \geq 0\}$ converges to $(I - G)\boldsymbol{eg}$. We shall use Lemma 3.3 and the fact that $\{G(n), n \geq 0\}$ is nondecreasing in our proof.

Let $\boldsymbol{v}$ be the right eigenvector of the matrix G corresponding to $\rho = \mathrm{sp}(G)$ with $\boldsymbol{gv} = 1$. Since G is primitive, all components of $\boldsymbol{v}$ are positive and all elements of G^m are positive (see [8]). Note that G is an $m \times m$ matrix. Thus, for small enough $\varepsilon (0 < \varepsilon < \rho^m)$, the matrix $G^m - \varepsilon\boldsymbol{vg}$ is positive. Since $\{G(n), n \geq 0\}$ converges monotonically to G, it is clear that when n is large enough, $G(n - m + 1) \cdots G(n) \geq G^m - \varepsilon\boldsymbol{vg}$. Since $\mathrm{sp}(A_1 + A_0Z(n)) < 1$ for $n \geq 0$, the matrix $I - A_1 - A_0Z(n)$ is invertible for $n \geq 0$. Therefore, (3.4) holds. For a positive integer $k < n/m$, let

$$H[n,k] = (I - A_1 - A_0 Z(n-1))^{-1} A_0 \cdots (I - A_1 - A_0 Z(0))^{-1} A_0 Z(0) G(1) \cdots G(n - mk).$$

For a fixed positive integer k, (3.4) leads to the inequality:

$$H[n,k](G^m - \varepsilon \boldsymbol{v}\boldsymbol{g})^k \le D(n) \le H[n,k] G^{mk}. \tag{A.1}$$

Postmultiplying by the vector $\boldsymbol{v}$ on both sides of the above inequalities yields

$$\begin{aligned} H[n,k](G^m - \varepsilon \boldsymbol{v}\boldsymbol{g})^k \boldsymbol{v} &\le D(n)\boldsymbol{v} \le H[n,k] G^{mk} \boldsymbol{v} \\ &\Rightarrow (\rho^m - \varepsilon)^k H[n,k]\boldsymbol{v} \le D(n)\boldsymbol{v} \le \rho^{mk} H[n,k]\boldsymbol{v} \\ &\Rightarrow \frac{D(n)\boldsymbol{v}}{\rho^{mk}} \le H[n,k]\boldsymbol{v} \le \frac{D(n)\boldsymbol{v}}{(\rho^m - \varepsilon)^k}. \end{aligned} \tag{A.2}$$

Since $\{D(n), n \ge 0\}$ is uniformly bounded and every element of the vector $\boldsymbol{v}$ is positive, the last line in (A.2) implies that for every fixed k, the sequence $\{H[n,k], n \ge 0\}$ is uniformly bounded.

Consider a convergent subsequence of $\{Z(n), n \ge 0\}$. Suppose that the limit of that subsequence $\{Z(n_t), t \ge 0\}(n_t \to \infty$ when $t \to \infty)$ is $\tilde{Z} = G + \tilde{D}$. Since the sequence $\{H[n_t, k], t \ge 0\}$ is uniformly bounded, there exists a subsequence $\{n_{t'}, t' \ge 0\}$ of $\{n_t, t \ge 0\}$ such that $\{H[n_{t'}, k], t' \ge 0\}$ converges and the limit is denoted by $H^*[k]$. Letting $n_{t'} \to \infty$, the inequalities in (A.1) imply that

$$H^*[k](G^m - \varepsilon \boldsymbol{v}\boldsymbol{g})^k \le \tilde{D} \le H^*[k] G^{mk}, \tag{A.3}$$

for small enough positive ε. Letting $\varepsilon \to 0$ in (A.3), we obtain $H^*[k]G^{mk} = \tilde{D}$ for any positive k. Since G is primitive, we have $G^{mk} = \rho^{mk}\boldsymbol{v}\boldsymbol{g} + o(\rho^{mk})$, which implies that

$$\tilde{D} = \rho^{mk} H^*[k]\boldsymbol{v}\boldsymbol{g} + h^*[k] o(\rho^{mk}).$$

Since $\tilde{D}\boldsymbol{v} = H^*[k]G^{mk}\boldsymbol{v} = \rho^{mk}H^*[k]\boldsymbol{v}$ for all k and all components of $\boldsymbol{v}$ are positive, the sequence $\{\rho^{mk}H^*[k], k \ge 0\}$ is uniformly bounded. Thus, there exists a convergent subsequence $\{\rho^{mk'}H^*[k'], k' \ge 0\}$ with its limit denoted by $\tilde{H}$. Then

$$\begin{aligned} \tilde{D} &= \lim_{k' \to \infty} \{\rho^{mk'} H^*[k']\boldsymbol{v}\boldsymbol{g} + H^*[k'] o(\rho^{mk'})\} \\ &= \lim_{k' \to \infty} \{\rho^{mk'} H^*[k']\boldsymbol{v}\boldsymbol{g} + \rho^{mk'} H^*[k'] o(1)\} = \tilde{H}\boldsymbol{v}\boldsymbol{g}. \end{aligned}$$

Therefore, $\tilde{Z} = G + \tilde{H}\boldsymbol{v}\boldsymbol{g}$. Since $Z(0)\boldsymbol{e} = \boldsymbol{e}$ and $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ for $n \ge 0$, by Lemma 2.2, $Z(n)\boldsymbol{e} = \boldsymbol{e}$ for all n. Therefore, $\tilde{Z}\tilde{\boldsymbol{e}} = \boldsymbol{e}$. Since $\boldsymbol{g}\boldsymbol{e} = 1$, we have $\tilde{H}\boldsymbol{v} = (I - G)\boldsymbol{e}$. Finally, we obtain that $\tilde{Z} = G + (I - G)\boldsymbol{e}\boldsymbol{g}$.

Since for any convergent subsequence of $\{Z(n), n \ge 0\}$, the equality $\tilde{Z} = G + (I - G)\boldsymbol{e}\boldsymbol{g}$ holds and $\{Z(n), n \ge 0\}$ is uniformly bounded, we conclude that the sequence $\{Z(n), n \ge 0\}$ converges to $G + (I - G)\boldsymbol{e}\boldsymbol{g}$ if $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ for $n \ge 0$.

Now, suppose that $\{Z(n), n \ge 0\}$ converges to $G + (I - G)\boldsymbol{e}\boldsymbol{g}$. If there exists an infinite subsequence such that $\mathrm{sp}(A_1 + A_0 Z(n_t)) = 1$ for $n_t \ge 0$, by the proof of Lemma 2.1, $Z(n_t + 1)$ has the form

$$\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix},$$

i.e. some rows of $Z(n_t + 1)$ are zero. Thus, there exists a subsequence of $\{Z(n_t + 1)\}$ that converges and the limit matrix has at least one zero row. Clearly, that contradicts the fact that the limit matrix of $\{Z(n), n \ge 0\}$ is stochastic.

Appendix B. Proof of Theorem 7.1

Assume that the matrix G is primitive. Since $\mathrm{sp}(G) < 1$, the necessity of the condition is obvious. The method of proving sufficiency of the condition is to use (3.5) and to show that $Y(n) = Z(1) \cdots Z(n)$ converges to 0 when n goes to infinity. For that purpose, we prove that any convergent subsequence of $\{Y(n), n \geq 1\}$ converges to 0.

Since $Z(n), n \geq 0$, are stochastic or substochastic matrices, the sequence $\{Y(n), n \geq 1\}$ is uniformly bounded. Suppose that a subsequence $\{Y(n_t), n_t \geq 0\}$ converges to a matrix Y. We choose $n_{t+1} > n_t + m$. Set $Y(n_{t+1}) = Y(n_t)U(t)$. It is easy to see that $\{U(t), t \geq 0\}$ is uniformly bounded and it has convergent subsequences. Suppose that a subsequence of $\{U(t), t \geq 0\}$ converges to U. Then $Y = YU$. This implies that $Y = YU^k$ for any k. If $\mathrm{sp}(U) < 1$, it is clear that $Y = 0$. Suppose that $\mathrm{sp}(U) = 1$. Since $n_{t+1} > n_t + m$ and $Z(n) \geq G(n)$, we have

$$(G(n_t))^m Z(n_t + m + 1) \cdots Z(n_{t+1}) \leq U(t) \quad \Rightarrow \quad G^m \hat{U} \leq U; \tag{B.1}$$

$$Z(n_t + 1) \cdots Z(n_t + m) Z(n_t + m + 1) \cdots Z(n_{t+1}) = U(t) \quad \Rightarrow \quad \dot{U}\hat{U} = U, \tag{B.2}$$

for some subsequence of $\{n_t\}$ and some matrices $\dot{U}$ (the limit of a subsequence of $\{Z(n_t + 1) \cdots Z(n_t + m)\}$ and $\hat{U}$ (the limit of a subsequence of $\{Z(n_t + m + 1) \cdots Z(n_{t+1})\}$ in Ω. We also have

$$Z(n_t + 1) \cdots Z(n_{t+1} - m)(G(n_t))^m \leq U(t) \quad \Rightarrow \quad \tilde{U} G^m \leq U; \tag{B.3}$$

$$Z(n_t + 1) \cdots Z(n_{t+1} - m) Z(n_{t+1} - m + 1) \cdots Z(N_{t+1}) = U(t) \quad \Rightarrow \quad \tilde{U}\ddot{U} = U, \tag{B.4}$$

for some subsequence of $\{n_t\}$ and some matrices $\ddot{U}$ (the limit of a convergent subsequence of $\{Z(n_{t+1} - m + 1) \cdots Z(n_{t+1})\}$ and $\tilde{U}$ (the limit of a convergent subsequence of $\{Z(n_t + 1) \cdots Z(n_{t+1} - m)\}$ in Ω. If $\hat{U}$ is a zero matrix, the matrix U is 0 by (B.2). This is impossible, since we assumed that $\mathrm{sp}(U) = 1$. Therefore, $\hat{U}$ is not a zero matrix. Since every element of the matrix G^m is positive, the matrix U has no zero row by (B.1). Similarly, $\tilde{U}$ is not a zero matrix by (B.4) and U has no zero column by (B.3). Since U has no zero column, $\hat{U}$ has no zero column by (B.2). Since $\hat{U}$ has no zero column, every element of U is positive by (B.1). Thus, the matrix U is primitive. Let $\boldsymbol{u}$ be the left eigenvector of the matrix U corresponding to $\mathrm{sp}(U) = 1$. The vector $\boldsymbol{u}$ is positive. By $Y = YU^k$ for any k, it can be shown that $Y = \mathrm{sp}(Y)\boldsymbol{x}\boldsymbol{u}$, where $\boldsymbol{u}\boldsymbol{x} = 1$ and the vector $\boldsymbol{x}$ is nonzero. In order to prove that $Y = 0$, we only need to prove that $\mathrm{sp}(Y) = 0$.

The sequence $\{Y(n_t)Z(n_t + 1), n_t \geq 0\}$ is uniformly bounded and must have a convergent subsequence. Denote the limit by YZ. Since $Y(n_t)\boldsymbol{e} \geq Y(n_t)Z(n_t + 1)\boldsymbol{e} \geq Y(n_{t+1})\boldsymbol{e}$, we have $YZ\boldsymbol{e} = Y\boldsymbol{e}$. This implies that $\mathrm{sp}(Y)\boldsymbol{x}\boldsymbol{u}Z\boldsymbol{e} = \mathrm{sp}(Y)\boldsymbol{x}\boldsymbol{u}\boldsymbol{e}$, that is $\mathrm{sp}(Y)\boldsymbol{x}\boldsymbol{u}Z\boldsymbol{e} = \mathrm{sp}(Y)\boldsymbol{x}$. If $\mathrm{sp}(Y) > 0$, we obtain $\boldsymbol{x}\boldsymbol{u}Z\boldsymbol{e} = \boldsymbol{x}$, which implies that $\boldsymbol{u}Z\boldsymbol{e} = 1$. This implies that $\boldsymbol{u}(\boldsymbol{e} - Z\boldsymbol{e}) = 0$. Since $\limsup_{n\to\infty} \mathrm{sp}(Z(n)) < 1$, we have $\mathrm{sp}(Z) < 1$ and $Z\boldsymbol{e} \neq \boldsymbol{e}$, which implies that $\boldsymbol{e} - Z\boldsymbol{e} \neq 0$ and $\boldsymbol{e} - Z\boldsymbol{e} \geq 0$. This implies that $\boldsymbol{u}(\boldsymbol{e} - Z\boldsymbol{e}) > 0$ since every element of $\boldsymbol{u}$ is positive. This is a contradiction. Therefore, $\mathrm{sp}(Y) = 0$, that is $Y = 0$.

Since any convergent subsequence of $\{Y(n), n \geq 0\}$ converges to 0, we conclude that $\{Y(n), n \geq 0\}$ converges 0 and $\{Z(n), n \geq 0\}$ converges to G.

If $\{Z(n), n \geq 0\}$ converges to G, $\mathrm{sp}(A_1 + A_0 Z(n)) < 1$ for large enough n must be true. Otherwise, G should have zero row(s), which contradicts the fact that G is primitive.

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