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On the identity of the smallest random variable

YIGAL GERCHAK[1] and QI-MING HE[2]

[1] *Department of Industrial Engineering, Tel-Aviv University, Ramat-Aviv, Tel-Aviv, 69978, Israel and Department of Management Sciences, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1*
E-mail: ygerchak@eng.tau.ac.il
[2] *Department of Industrial Engineering, DalTech, Dalhousie University, Halifax, Nova Scotia, Canada B3J 2X4*
E-mail: Qi-Ming.He@dal.ca

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We compute and analyze the probability that a particular random variable will assume the smallest value among a set of random variables. This work was motivated by wanting to predict the "winner" in R&D and patent "races". Some general results and comparative statics are provided. For symmetric distributions we derive bounds on the probabilities of interest. We compute the probabilities of who will be the smallest (fastest) for normal, lognormal, Weibull, Pareto, binomial and PH-distributions. We also analyze several models of multivariate exponential distributions.

1. Introduction

Who will be the first to reach some goal? Whose achievement will be the best? Denoting the times it will take the n contestants to reach the goal by $X_1, \ldots, X_n$, the quantities we are interested in are

$$P\{X_j = \min(X_1, \ldots, X_n)\} \equiv P_{j|n}, \quad j = 1, \ldots, n. \qquad (1)$$

It is well-known that if the X_i's are independent exponential random variables with respective parameters $\lambda_1, \ldots, \lambda_n$, then

$$P_{j|n} = \lambda_j \Big/ \sum_{i=1}^{n} \lambda_i, \quad j = 1, \ldots, n. \qquad (2)$$

This observation is central in the analysis of exponential queueing systems as there it represents the probability of service completion preceding the next arrival or *vice versa*, and some reliability models (Ross, 1996).

However, the quantity $P_{j|n}$ has not been explored much for non-exponential distributions even for $n = 2$. This is understandable in dynamic contexts (like queues), since if the relevant distributions are not memoryless this probability will depend on how much time has passed since last arrival/service completion. $P_{n|n}$ plays an important role in the context of "record values" (Resnick, 1987), but there the X_i's are assumed identically distributed and continuous, so $P_{n|n} = 1/n$. But $P_{j|n}$ is a very important quantity in some "static" application areas where the relevant distributions are unequal and not necessarily exponential.

In many R&D, product development or patent "races" among firms, the question of who will be the first to achieve a certain breakthrough or attain a threshold is often of key importance (Tirole, 1988; Reinganum, 1989). The random time it will take a firm to reach such a threshold may depend on the resources it devotes to this activity. Gerchak and Parlar (1999) analyzed an R&D model where $P_{j|2}$ was a key quantity (see also Nti (1997)).

The quantities given in (1) provide information about the possibility that a particular component is the first one to fail in many reliability systems. Such information can be used in the design and maintenance of these reliability systems. For instance, it may be useful in identifying the breakdown component. It may also be useful in determining how many spare parts of a component should be kept in case of a breakdown.

The extensive literature on distributions (for example Johnson *et al.* (1995) and references therein) does not say much about $P_{j|n}$, or even $P_{j|2}$, for non-exponential distributions. One exception is the case of a Pareto distribution Pederzoli and Rathie, 1980; Johnson *et al.* (1995, Ch. 20. 10), where the distribution of a ratio of independent Pareto variables is given; one can deduce $P_{j|2}$ from this distribution. There is also a discussion of quotient and differences of Beta variables (Johnson *et al.*, 1995, Ch. 25.8), but the results are not really operational.

We shall first explore some general properties of $P_{j|n}$. In particular, we shall be interested in the relative magnitude of $P_{i|n}$ and $P_{j|n}$ when X_i and X_j are stochastically ordered. We also show that the $P_{j|n}$'s are invariant under monotone transformations. Some properties of $P_{j|n}$ for symmetric distributions are then explored and bounds for the probabilities of interest provided. Then we derive $P_{j|n}$ for several important distributions: normal, Weibull, Pareto, binomial and PH. We then focus on multi-variate exponentials, where the X_i's are not independent.

2. The model and general properties

Let $X_j \sim F_j(x) = P\{X_j < x\}, j = 1, \ldots, n$, and assume for now that $X_1, \ldots, X_n$ are independent. Then by conditioning on the value of X_j, we can show that

$$P_{j|n} \equiv P\{X_j = \min(X_1, \ldots, X_n)\} = \int_{-\infty}^{\infty} \prod_{i \neq j} \bar{F}_i(x) \mathrm{d}F_j(x), \tag{3}$$

where $\bar{F}_i(x) = 1 - F_i(x)$.

Note that if the X's are discrete with some common values, more than one of them can become the smallest. Thus in general $P_{j|n}$, $j = 1, \ldots, n$, need not constitute a proper probability mass function, i.e., in general, $\sum_{j=1}^{n} P_{j|n} \geq 1$.

Several important properties follow from (3).

Property 1. *For a given set of random variables* $(X_1, \ldots, X_n)$, $P_{j|n} \leq P_{j|(n-1)}$, $j = 1, \ldots, n-1$. *That agrees with intuition.*

Property 2. *For fixed i, consider another random variable* X_i' *that is independent of* $\{X_j, j \neq i\}$. *If* $X_i' \geq_{st} X_i$, *i.e., if* $\bar{F}_{X_i'}(x) \geq \bar{F}_{X_i}(x), \forall x$, *then* $P'_{j|n} \geq P_{j|n}$, $\forall j \neq i$, *and* $P'_{i|n} \leq P_{i|n}$. *That is, if one of the variables is replaced by a stochastically larger random variable, the chances that another variable is the smallest increase, and the chances that the changed variable is the smallest decrease, which is intuitive.*

The next property will be stated as a proposition and proved.

Proposition 1. *If* $X_i \leq_{st} X_j$, *then* $P_{i|n} \geq P_{j|n}$, $i, j = 1, \ldots, n$.

Proof. For any non-increasing function h defined over the appropriate range, $X_i \leq_{st} X_j$ implies (Ross, 1996) that

$$\int_{-\infty}^{\infty} h(x) \mathrm{d}F_i(x) \geq \int_{-\infty}^{\infty} h(x) \mathrm{d}F_j(x).$$

Now,

$$P_{i|n} = \int_{-\infty}^{\infty} \bar{F}_1(x) \ldots \bar{F}_{i-1}(x) \bar{F}_{i+1}(x) \ldots \bar{F}_n(x) \mathrm{d}F_i(x).$$

Since $\bar{F}_1(x) \ldots \bar{F}_{i-1}(x) \bar{F}_{i+1}(x) \ldots \bar{F}_n(x)$ is a non-increasing function, it follows that

$$\begin{aligned} P_{i|n} &\geq \int_{-\infty}^{\infty} \bar{F}_1(x) \ldots \bar{F}_{i-1}(x) \bar{F}_{i+1}(x) \ldots \bar{F}_n(x) \mathrm{d}F_j(x), \\ &\geq \int_{-\infty}^{\infty} \bar{F}_1(x) \ldots \bar{F}_{j-1}(x) \bar{F}_{j+1}(x) \ldots \bar{F}_n(x) \mathrm{d}F_j(x), \\ &= P_{j|n}, \end{aligned}$$

where the last inequality follows from the fact that $\bar{F}_j(x) \geq \bar{F}_i(x), \forall x$. ■

In some applications, random variables are transformed in various ways. Consider such a transformation $Y_j = h(X_j)$, $j = 1, \ldots, n$. We shall now show that the $P_{j|n}$'s are invariant under monotone transformations. Let ${}_hP_{j|n} \equiv P\{Y_j = \min(Y_1, \ldots, Y_n)\}$.

Proposition 2.

a) *If* $h(x)$ *is non-decreasing on the appropriate range, then* ${}_hP_{j|n} \geq P_{j|n}, 1 \leq j \leq n$, *while if* $h(x)$ *is strictly increasing, then* ${}_hP_{j|n} = P_{j|n}, 1 \leq j \leq n$. *On the other hand, if* ${}_hP_{j|n} \geq P_{j|n}$, $1 \leq j \leq n$, *for any set* $\{X_1, \ldots, X_n\}$, *then* $h(x)$ *is non-decreasing. If* ${}_hP_{j|n} = P_{j|n}$, $1 \leq j \leq n$, *for any set* $\{X_1, \ldots, X_n\}$, *then* $h(x)$ *is strictly increasing.*
b) *If* $h(x)$ *is non-increasing, then* ${}_hP_{j|n} \leq P^-_{j|n} \equiv P\{-X_j = \min(-X_1, \ldots, -X_n)\}$, $1 \leq j \leq n$, *while if* $h(x)$ *is strictly decreasing,* ${}_hP_{j|n} = P^-_{j|n}, 1 \leq j \leq n$.

Proof. This proposition is proved by using a sample path method. It is easy to see that

$$\begin{aligned} &\{w : Y_j = \min(Y_1, \ldots Y_n)\} \\ &= \{w : Y_j \leq Y_1\} \cap \{w : Y_j \leq Y_2\} \cap \ldots \cap \{w : Y_j \leq Y_n\}, \\ &= \{w : h(X_j) \leq h(X_1)\} \cap \ldots \cap \{w : h(X_j) \leq h(X_n)\}. \end{aligned}$$

a) If $h(x)$ is non-decreasing, $\{w : h(X_j) \leq h(X_i)\} \supset \{w : X_j \leq X_i\}, 1 \leq i \leq n$. Then ${}_hP_{j|n} \geq P\{X_j = \min(X_1, \ldots, X_n)\} = P_{j|n}$. If $h(x)$ is strictly increasing, we have $\{w : h(X_j) \leq h(X_i)\} = \{w : X_j \leq X_i\}$, $1 \leq i \leq n$. Then ${}_hP_{j|n} = P\{X_j = \min(X_1, \ldots, X_n)\} = P_{j|n}$.
   Suppose that ${}_hP_{j|n} \geq P_{j|n}$ for any set $\{X_1, \ldots, X_n\}$. If $h(x)$ is not non-decreasing, then there exist x_1 and x_2 such that $x_1 < x_2$ and $h(x_1) > h(x_2)$. Consider two random variables X_1 and X_2 such that $X_1 \equiv x_1$ and $X_2 \equiv x_2$. Then $P_{1|2} = 1 > P_{2|2} = 0$. But ${}_hP_{1|2} = 0 < {}_hP_{2|2} = 1$, which is a contradiction. Therefore, $h(x)$ is non-decreasing. The strictly increasing part can be proved similarly.
b) If $h(x)$ is strictly decreasing, $\{w : h(X_j) \leq h(X_i)\} = \{w : X_j \geq X_i\} = \{w : -X_j \leq -X_i\}$. Then we have ${}_hP_{j|n} = P^-_{j|n}$. The non-increasing part follows. ■

Note that Proposition 2 does not require independence of the X_j's.

2.1. *Symmetric distributions*

When the X's are identically distributed then clearly $P_{1|n} = P_{2|n} = \ldots = P_{n|n}$. A natural question is whether that also holds for some non-identical distributions. Consider ***symmetric*** distributions, i.e., those where $P\{X - E(X) < x\} = P\{X - E(X) > -x\}, \forall x$. What can be said about $P_{j|n}$ for symmetric distributions with equal means?

Proposition 3.
If X_1 and X_2 are symmetric with $E(X_1) = E(X_2)$ then $P_{1|2} = P_{2|2} \geq 0.5$.

Proof. Since X_1 and X_2 are symmetric and have the same mean, then

$$P_{1|2} = P\{X_1 \leq X_2\} = P\{X_1 - E(X_1) \leq X_2 - E(X_2)\}, \\ = P\{X_1 - E(X_1) \geq X_2 - E(X_2)\} = P\{X_1 \geq X_2\} = P_{2/2}.$$

Since $P_{1|2} + P_{2|2} \geq 1$, we must have $P_{1|2} = P_{2|2} \geq 0.5$. ■

But the result does *not* extend to $n > 2$.
Example

$$X_1 = \begin{cases} -1 & w.p. \quad 0.5; \\ 1 & w.p. \quad 0.5. \end{cases}$$

$$X_2 = \begin{cases} -2 & w.p. \quad 0.5; \\ 2 & w.p. \quad 0.5. \end{cases}$$

$$X_3 = \begin{cases} -3 & w.p. \quad 0.5; \\ 3 & w.p. \quad 0.5. \end{cases}$$

Then $P_{1|3} = P_{2|3} = 0.25$, $P_{3|3} = 0.5$. ■

Nonetheless, some nice bounds can be found for $P_{j|n}$ for the symmetric case. To find these bounds, the following lemma is useful.

Lemma 1. *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $0 \leq x_i \leq 0.5$, $1 \leq i \leq n$. Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ with $\delta_i \geq 0$, $1 \leq i \leq n$. Define a function $g(\boldsymbol{x})$ as*

$$g(\boldsymbol{x}) = \prod_{i=1}^{n}(1 + \delta_i - x_i) + \prod_{i=1}^{n} x_i. \tag{4}$$

Then the function $g(\boldsymbol{x})$ is decreasing with respect to x_i when $0 \leq x_i \leq 0.5$, $1 \leq i \leq n$. Furthermore, we have

$$(0.5)^{n-1} \leq \prod_{i=1}^{n}(0.5 + \delta_i) + (0.5)^n \\ \leq g(\boldsymbol{x}) \leq \prod_{i=1}^{n}(1 + \delta_i). \tag{5}$$

Proof. Since $0 \leq x_i \leq 0.5$ for all $1 \leq i \leq n$, it is easy to verify that the partial derivative of $g(\boldsymbol{x})$ with respect to x_i is negative. Therefore, $g(\boldsymbol{x})$ is decreasing with respect to x_i, $1 \leq i \leq n$. The lower and upper bound are then obtained by setting $\boldsymbol{x} = (0.5, \ldots, 0.5)$ and $\boldsymbol{x} = (0, \ldots, 0)$ respectively. ■

Proposition 4. *Assume that random variables $\{X_1, \ldots, X_n\}$ are symmetric and have the same mean. Let $\delta_j = max_{x>0} P\{X_j = x\}$, $1 \leq j \leq n$. Then we have, for $1 \leq j \leq n$,*

$$P_{j|n} \leq \left(\frac{1 + P\{X_j = 0\}}{2}\right) \prod_{i \neq j}(1 + \delta_i),$$

$$P_{j|n} \geq \prod_{i=1}^{n}\left(\frac{1 - P\{X_i = 0\}}{2}\right) + \prod_{i=1}^{n}\left(\frac{1 + P\{X_i = 0\}}{2}\right) \geq (0.5)^{n-1}. \tag{6}$$

Furthermore, if $\{X_1, \ldots, X_n\}$ are all continuous, then $(0.5)^{n-1} \leq P_{j|n} \leq 0.5$.

Proof. Without loss of generality, we assume that $E(X_1) = \ldots = E(X_n) = 0$. Since all the distributions are symmetric and independent, we have the following evaluation:

$$P_{j|n} = P\{X_j \leq X_i, 1 \leq i \leq n\} = \int_{-\infty}^{0-} \prod_{i \neq j} P\{X_i \geq x\} \mathrm{d}F_j(x) \\ + P\{X_j = 0\} \prod_{i \neq j} P\{X_i \geq 0\} + \int_{0+}^{\infty} \prod_{i \neq j} P\{X_i \geq x\} \mathrm{d}F_j(x), \\ = P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) + \int_{0+}^{\infty} \prod_{i \neq j} P\{X_i \leq x\} \mathrm{d}F_j(x) \\ + \int_{0+}^{\infty} \prod_{i \neq j} P\{X_i \geq x\} \mathrm{d}F_j(x), \\ = P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) \\ + \int_{0+}^{\infty} \left[\prod_{i \neq j} F_i(x+) + \prod_{i \neq j} \bar{F}_i(x)\right] \mathrm{d}F_j(x),$$

where $F_i(x+) = F_i(x) + P\{X_i = x\}$, $1 \leq i \leq n$.

Note that $\bar{F}_i(x)$ is non-increasing in x and $\bar{F}_i(x) \leq 0.5$ when $x > 0$. Thus, by Lemma 1, the function $\prod_{i \neq j}(F_i(x) + \delta_i) + \prod_{i \neq j} \bar{F}_i(x)$ is non-decreasing in x when $x > 0$. Using Lemma 1, the upper bound is obtained as follows.

$$P_{j|n} \leq P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) \\ + \int_{0+}^{\infty} \left[\prod_{i \neq j}(F_i(x) + \delta_i) + \prod_{i \neq j} \bar{F}_i(x)\right] \mathrm{d}F_j(x),$$

$$\leq P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) + \int_{0+}^{\infty} \lim_{t \to \infty} \left[\prod_{i \neq j} (F_i(t) + \delta_i) + \prod_{i \neq j} \bar{F}_i(t) \right] \mathrm{d}F_j(x), = P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) + \left(\frac{1 - P\{X_j = 0\}}{2} \right) \prod_{i \neq j} (1 + \delta_i),$$

$$\leq \left(\frac{1 + P\{X_j = 0\}}{2} \right) \prod_{i \neq j} (1 + \delta_i).$$

The last inequality is due to $\prod_{i \neq j} \bar{F}_i(0) \leq \prod_{i \neq j} (1 + \delta_i)$. Also using Lemma 1, the lower bounds are obtained as follows.

$$P_{j|n} \geq P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) + \int_{0+}^{\infty} \left[\prod_{i \neq j} F_i(x) + \prod_{i \neq j} \bar{F}_i(x) \right] \mathrm{d}F_j(x),$$

$$\geq P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) + \int_{0+}^{\infty} \lim_{t \to 0} \left[\prod_{i \neq j} F_i(t) + \prod_{i \neq j} \bar{F}_i(t) \right] \mathrm{d}F_j(x),$$

$$= P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) + \int_{0+}^{\infty} \left[\prod_{i \neq j} F_i(0+) + \prod_{i \neq j} \bar{F}_i(0+) \right] \mathrm{d}F_j(x),$$

$$= P\{X_j = 0\} \prod_{i \neq j} \bar{F}_i(0) + \bar{F}_j(0+) \left[\prod_{i \neq j} F_i(0) + \prod_{i \neq j} \bar{F}_i(0+) \right],$$

$$= P\{X_j = 0\} \prod_{i \neq j} \left(\frac{1 + P\{X_i = 0\}}{2} \right) + \frac{(1 - P\{X_j = 0\})}{2} \cdot \times \left[\prod_{i \neq j} \left(\frac{1 + P\{X_i = 0\}}{2} \right) + \prod_{i \neq j} \left(\frac{1 - P\{X_i = 0\}}{2} \right) \right],$$

$$= \prod_{i=1}^{n} \left(\frac{1 - P\{X_i = 0\}}{2} \right) + \prod_{i=1}^{n} \left(\frac{1 + P\{X_i = 0\}}{2} \right) \geq (0.5)^{n-1}.$$

The bounds for the continuous case are obtained by setting $P\{X_i = 0\} = \delta_i = 0$, $1 \leq i \leq n$, in the above formulas.

3. Specific distributions

3.1. *Normal/lognormal*

Consider normal random variables X_i with mean μ_i and standard deviation σ_i, $1 \leq i \leq n$. Generally, we have

$$P_{j|n} = \int_{-\infty}^{\infty} \prod_{i \neq j} P\left\{ Z \geq \frac{x - \mu_i}{\sigma_i} \right\} \frac{1}{\sigma_j \sqrt{2\pi}} \exp\left(\frac{-(x - \mu_j)^2}{2\sigma_j^2} \right) \mathrm{d}x,$$

where Z is the generic random variable for normal distributions with mean zero and variance one. Since the normal is a symmetric distribution, a number of interesting results can be obtained. First, Proposition 3 applies to the case of equal means and $n = 2$. Second, when $\mu_1 = \ldots = \mu_n$, we have $(0.5)^{n-1} \leq P_{j|n} \leq 0.5$ by Proposition 4. Third, if $\sigma_1 = \sigma_2 = \ldots = \sigma_n$ and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$, then $X_1 \leq_{st} X_2 \leq_{st} \ldots \leq_{st} X_n$, so, by Proposition 1, $P_{1|n} \geq P_{2|n} \geq \ldots \geq P_{n|n}$.

Finally, if $\mu_1 = \mu_2 = \ldots = \mu_n = \mu$ and $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n$, we would like to show that $P_{1|n} \leq P_{2|n} \leq \ldots \leq P_{n|n}$. This is intuitive since σ_i represents standard deviation. To prove $P_{1|n} \leq P_{2|n} \leq \ldots \leq P_{n|n}$, let $\{Z_1, \ldots, Z_n\}$ be n independent normal random variables with mean zero and variance one. Then we can write $X_i = \sigma_i Z_i + \mu$, $1 \leq i \leq n$. Then

$$P_{j|n} = P\{X_j \leq X_i, i \neq j\} = P\left\{ Z_j \leq \frac{\sigma_i}{\sigma_j} Z_i, i \neq j \right\},$$

$$= \int_0^{\infty} \left[\prod_{i \neq j} P\left\{ Z_i \leq x \frac{\sigma_j}{\sigma_i} \right\} + \prod_{i \neq j} \left(1 - P\left\{ Z_i \leq x \frac{\sigma_j}{\sigma_i} \right\} \right) \right] \mathrm{d}F_{Z_j}(x),$$

$$= \int_0^{\infty} \left[\prod_{i \neq j} P\left\{ Z \leq x \frac{\sigma_j}{\sigma_i} \right\} + \prod_{i \neq j} \left(1 - P\left\{ Z \leq x \frac{\sigma_j}{\sigma_i} \right\} \right) \right] \mathrm{d}F_Z(x).$$

By Lemma 1, the integrand in the last line of the above equation is increasing with respect to σ_j. Therefore, $P_{j|n}$ is non-decreasing in j.

If X_j is normal, then $Y_j = e^{X_j}$ is lognormal. Since $h(x) = e^x$ is strictly increasing, it follows from Proposition 2 that ${}_hP_{j|n} = P_{j|n}$, $j = 1, \ldots, n$.

3.2. *Weibull*

Here $\bar{F}(x) = \exp(-\lambda x)^\alpha$, $\lambda > 0$, $\alpha > 0$, $x \geq 0$. Thus if $X_j \sim$ Weibull (α, λ_j), $j = 1, \ldots, n$ and the X_j's are independent,

$$P_{j|n} = \int_0^{\infty} \lambda_j \alpha (\lambda_j x)^{\alpha - 1} \exp(-\lambda_j x)^\alpha \prod_{i \neq j} \exp(-\lambda_i x)^\alpha \mathrm{d}x,$$

$$= \lambda_j^\alpha \Big/ \sum_{i=1}^{n} \lambda_i^\alpha. \qquad (7)$$

This generalization of the exponential case was also observed by Gerchak and Parlar (1999).

3.3. *Pareto*

Here $\bar{F}(x) = (k/x)^a$, $k > 0$, $a > 0$, $x \geq k$. Note that X is stochastically increasing in k, and stochastically decreasing in a.

Let $X_j \sim$ Pareto (k_j, a_j), $j = 1, \ldots, n$, and the X_j's are independent. While it is possible to obtain $P_{j|2}$ from the distribution of X_1/X_2 derived by Pederzoli and Rathi (1980), interested readers should also refer to Johnson *et al.* (1995, Ch. 20.10), it is also possible to obtain the more general quantity $P_{j|n}$ directly from (3):

$$P_{j|n} = \int_{k_j}^{\infty} a_j k_j^{a_j} x^{-(a_j+1)} \prod_{i \neq j} (k_i/x)^{a_i} \mathrm{d}x,$$

$$= a_j \prod_{i=1}^{n} k_i^{a_i} \Bigg/ \left(\sum_{i=1}^{n} a_i \right) k_j^{\sum_{i=1}^{n} a_i}. \tag{8}$$

3.4. *Binomial*

If $X_j \sim \text{bin}\,(m, p_j)$, $j = 1, \ldots, n$, and $p_1 \geq p_2 \geq \ldots \geq p_n$ then $X_1 \geq_{st} X_2 \geq \ldots \geq_{st} X_n$. Thus by Proposition 1, $P_{1|n} \leq P_{2|n} \leq \ldots \leq P_{n|n}$. If $m = 1$ then $P_{j|n} = 1 - p_j + \prod_{i=1}^{n} p_i$, $j = 1, \ldots, n$.

3.5. *PH-distributions*

Following Neuts (1981), let $X \sim (\boldsymbol{\alpha}, \boldsymbol{T})$ where $\boldsymbol{\alpha}$ is a row vector of size m such that $\boldsymbol{\alpha e} = 1$, where $\boldsymbol{e}$ is a vector of ones, and $\boldsymbol{T}$ is an $m \times m$ matrix. All diagonal elements of $\boldsymbol{T}$ are negative and all off-diagonal elements of $\boldsymbol{T}$ are non-negative. Let $\boldsymbol{T}^0 = -\boldsymbol{Te}$.

Assume that $\{X_1, \ldots, X_n\}$ are independent and $X_j \sim (\boldsymbol{\alpha}_j, \boldsymbol{T}_j)(m_j)$.

Construct the following absorbing Markov process:

$$\boldsymbol{Q} = \begin{pmatrix} \hat{\boldsymbol{T}} & \hat{\boldsymbol{T}}_1^0 & \ldots & \hat{\boldsymbol{T}}_n^0 \\ 0 & & \boldsymbol{I} & \end{pmatrix},$$

where

$$\hat{\boldsymbol{T}} = \boldsymbol{T}_1 \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T}_2 \otimes \boldsymbol{I} + \ldots + \boldsymbol{I} \otimes \boldsymbol{T}_n,$$

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2 \otimes \ldots \otimes \boldsymbol{\alpha}_n,$$

$$\hat{\boldsymbol{T}}_j^0 = \boldsymbol{e} \otimes \boldsymbol{T}_j^0 \otimes \boldsymbol{e}, \quad 1 \leq j \leq n,$$

$\boldsymbol{I}$ is the identity matrix.

Proposition 5.

$$P_{j|n} = -\hat{\boldsymbol{\alpha}} \hat{\boldsymbol{T}}^{-1} \hat{\boldsymbol{T}}_j^0, \quad 1 \leq j \leq n. \tag{9}$$

Proof. The formula is obtained by definitions. ■

Note: For discrete time case,

$$P_{j|n} = \hat{\boldsymbol{\alpha}} (\boldsymbol{I} - \hat{\boldsymbol{T}})^{-1} \hat{\boldsymbol{T}}_j^0, \quad 1 \leq j \leq n, \tag{10}$$

where

$$\boldsymbol{T}_i^0 = \boldsymbol{e} - \boldsymbol{T}_i \boldsymbol{e}. \tag{11}$$

4. Implications of dependence

4.1. *Bi(Multi)variate Exponentials*

One of the BiVariate Exponentials (BVE) proposed by Gumble (1960), interested readers should also refer to Johnson and Kotz (1972, Ch. 41.3), has the distribution

$$P\{X_1 \leq x_1, X_2 \leq x_2\} = (1 - \exp(-\lambda_1 x_1))(1 - \exp(-\lambda_2 x_2)) \times (1 + c\exp(-\lambda_1 x_1 - \lambda_2 x_2)), \quad x_1, x_2 \geq 0, \tag{12}$$

where $-1 \leq c \leq 1$ (the resulting correlation coefficient ranges from $-1/4$ to $1/4$). (This distribution has no known multivariate extension.)

Let $f_{X_1,X_2}(x_1, x_2)$ be the corresponding joint density. Then

$$P\{X_1 \leq X_2\} = \int_{x_1=0}^{\infty} \int_{x_2=x_1}^{\infty} f_{X_1,X_2}(x_1, x_2) \mathrm{d}x_2 \, \mathrm{d}x_1,$$

$$= \lambda_1/(\lambda_1 + \lambda_2) + c\lambda_1\{2/(\lambda_1 + \lambda_2) - 2/(2\lambda_1 + \lambda_2) - 1/(\lambda_1 + 2\lambda_2)\}. \tag{13}$$

It can be shown that $P(X_1 \leq X_2)$ is increasing in the correlation coefficient c iff $\lambda_1 > \lambda_2$.

Another well-known bivariate exponential is the one due to Marshall and Olkin (1967a) that is also discussed by Johnson and Kotz (1972, Ch. 41.3) and Barlow and Proschan (1975). Here

$$P(X_1 > x_1, X_2 > x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)). \tag{14}$$

The correlation coefficient is $\lambda_{12}/(\lambda_1 + \lambda_2 + \lambda_{12})$.

The corresponding joint density can be shown to equal

$$f(x_1, x_2) = \begin{cases} \lambda_1(\lambda_2 + \lambda_{12})\exp(-\lambda_1 x_1 - (\lambda_2 + \lambda_{12})x_2), & 0 \leq x_1 < x_2, \\ \lambda_2(\lambda_1 + \lambda_{12})\exp(-(\lambda_1 + \lambda_{12})x_1 - \lambda_2 x_2), & 0 \leq x_2 < x_1, \\ \lambda_{12}\exp(-(\lambda_1 + \lambda_2 + \lambda_{12})x_1), & 0 \leq x_1 = x_2. \end{cases} \tag{15}$$

Using (15) one can then show that

$$P_{1|2} = \frac{\lambda_1 + \lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}, \qquad P_{2|2} = \frac{\lambda_2 + \lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}. \tag{16}$$

Note that even though this distribution is continuous, $P_{1|2} + P_{2|2} > 1$ when $\lambda_{12} > 0$. Also note that both $P_{1|2}$ and $P_{2|2}$ are increasing in λ_{12}.

An alternative proof of (16) is as follows: Let $Y_1 \sim 1 - e^{-\lambda_1 t}$, $Y_2 \sim 1 - e^{-\lambda_2 t}$, $Y_{12} \sim 1 - e^{-\lambda_{12} t}$, where Y_1, Y_2 and Y_{12} are independent. Then $X_1 = \min\{Y_1, Y_{12}\}$, $X_2 = \min\{Y_2, Y_{12}\}$, so

$$P_{1|2} = P\{Y_1 \le \min\{Y_2, Y_{12}\}\} + P\{Y_{12} \le \min\{Y_1, Y_2\}\}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_{12}} + \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}} = \frac{\lambda_1 + \lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}, \quad (17)$$

and similarly for $P_{2|2}$.

The Marshall and Olkin BVE distribution has a multivariate extension (Marshall and Olkin, 1967b; Johnson and Kotz, 1972; Barlow and Proschan, 1975):

$$P\{X_1 > x_1, \ldots, X_n > x_n\}$$

$$= \exp\left(-\sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \lambda_{i_1 i_2 \ldots i_k} \max\{x_{i_1}, \ldots, x_{i_k}\}\right)$$

Using the second method of proof, the result for the Marshall and Olkin BVE generalizes to the Marshall and Olkin MVE.

Proposition 6.

$$P_{j|n} = \left(\sum_{0 \le t < j \le k \le n} \sum_{1 \le i_1 < \ldots < i_t < j < i_{t+1} < \ldots < i_k \le n} \lambda_{i_1 \cdots i_t j i_{t+1} \ldots i_k}\right) \Big/ \left(\sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \lambda_{i_1 i_2 \ldots i_k}\right), \quad j = 1, \ldots, n. \quad (18)$$

5. Future research

While we found $P_{j|n}$ for several (classes of) distributions, its value and properties for other distributions remain to be investigated. For the Weibull, it is not clear what happens if the shape parameters are not equal. Logistic, gamma and hyper (mixed) exponentials should also be explored. Other multivariate distributions may also be worthy of investigation.

In the application areas which motivated this work $P_{j|n}$ is a key step towards finding the best way for competitive firms, say, to allocate resources among projects (Gerchak and Parlar, 1999). As such, it is important that $P_{j|n}$ will have a simple/workable form. It is reassuring that such turned out to be the case for some key distributions.

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Biographies

Yigal Gerchak is a Professor at the Department of Industrial Engineering, Tel-Aviv University and at the Department of Management Sciences, University of Waterloo, Canada. His current IE research interests include supply chain management in decentralized system, coordination in assembly systems and production control with random yields. He has published over 80 articles on these and other topics in leading academic journals. He serves on the Editorial Boards of *IIE Transactions* and the *MSOM Journal*.

Qi-Ming He is currently an Associate Professor in the Industrial Engineering Department of Dalhousie University. His main research areas are algorithmic methods in applied probability, queueing theory, inventory control, and production management. He has published articles in *Operations Research*, *Advances in Applied Probability*, *IIE Transactions*, and others. Recently, he has been working on queueing systems with multiple types of customers and inventory systems with multiple types of demands.

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