A LINEAR PROGRAM APPROACH TO ERGODICITY OF *M/G/*1 TYPE MARKOV CHAINS WITH A TREE STRUCTURE

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It has been shown recently that the Perron-Frobenius eigenvalue of a nonnegative matrix provides information for a complete classification of *M/G/*1 type Markov chains with a tree structure. The use of that ergodicity condition depends largely on the computation of a set of nonnegative matrices, which can be quite challenging. In this paper, without using a set of nonnegative matrices, we develop two linear programs whose solutions provide sufficient conditions for ergodicity of the Markov chains of interest. We also introduce a simple approximation to the ergodicity problem. Numerical examples demonstrate that the linear program approach, as well as the approximation approach, can be quite useful.

1 Introduction

Markov chains with a tree structure, introduced by Takine, Sengupta, and Yeung [12], have broad applications in stochastic modeling, especially in queueing theory. For instance, the queueing processes of a number of queueing systems with a last-come-first-served (LCFS) service discipline can be formulated into Markov chains with a tree structure (see HE and Alfa [5] and Takine, Sengupta, and Yeung [12], and references therein). In Van Houdt and Blondia [7], the data transmission process of a random access system is formulated as a Markov chain with a tree structure. As a result, the stability of these stochastic systems is closely related to the ergodicity of the corresponding Markov chains with a tree structure.

The ergodicity of Markov chains with a tree structure has attracted considerable attention recently. In HE [3, 4], it has been shown that the Perron-Frobenius eigenvalue of a nonnegative matrix provides information for a complete classification of *M/G/*1 type Markov chains with a tree structure. Unfortunately, the ergodicity condition is based on a set of nonnegative matrices that are the fixed points of certain matrix equations. When the number of phases involved is large, the computations required for calculating those matrices are quite demanding and, in some cases, impossible to implement because of computer space limitations (e.g., the random access memory of a computer). Therefore, other simpler conditions (sufficient or necessary) can be quite useful in practice.

In HE and Li [6], a linear program approach is used to find sufficient conditions for stability of a queueing system with multiple types of customers and a last-come-first-served preemptive repeat service discipline. In this paper, we generalize this linear program approach to *M/G*/1 type Markov chains with a tree structure. We develop two linear programs whose solutions provide information about ergodicity of the Markov chain of interest. The two linear programs are formulated using only original system parameters. Since efficient algorithms have been developed for solving linear programs, information for ergodicity can be obtained efficiently even when the number of phases is large. This is the main contribution of this paper. In addition, we also introduce a simple (approximation) condition for ergodicity.

In queueing theory and queueing networks, stability has been an important issue. Various approaches have been explored (Chen and Zhang [1], Kumar and Meyn [8]). In fact, the linear program approach has been used to find stability conditions for queueing networks with reentry (Kumar and Meyn [8]). Our work shows that the ergodicity problem of complicated Markov chains can be transformed into a linear program, if the Markov chains possess a certain structure.

Our work is based on matrix analytic methods and Foster's criteria for Markov chains. Latouche and Ramaswami [9] and Neuts [10, 11] provide an introduction to matrix analytic methods. Fayolle, et al. [2] gives an introduction to the classification of Markov chains, including Foster's criteria.

The rest of the paper is organized as follows. In Section 2, we introduce *M/G*/1 type Markov chains with a tree structure. In Section 3, we introduce three existing approaches to the ergodicity problem. In Section 4, we present two linear programming formulations whose solutions give sufficient conditions for ergodicity. In Section 5, we give some details about the implementation of numerical algorithms. In Section 6, we present some numerical examples to gain insight into the methods introduced in this paper and to draw general conclusions about the usefulness of the methods.

2 Markov Chain of Matrix *M/G/*1 Type with a Tree Structure

The following discrete time Markov process of matrix *M/G/*1 type with a tree structure was first introduced in Takine, Sengupta, and Yeung [12]. Consider a discrete time two-dimensional Markov chain $\{(C_n, \eta_n), n\geq 0\}$ in which the values of C_n are represented by the nodes of a K-ary tree, and η_n takes integer values between 1 and m, where m is a positive integer. C_n is referred to as the node variable and η_n the auxiliary (phase) variable of the Markov chain at time n.

The K-ary tree of interest is a tree for which each node has a parent and K children, except the root node of the tree. The root node is denoted as 0. Strings of integers between 1 and K are used to represent nodes of the tree. For instance, the kth child of the root node is represented by k, the lth child of node k is represented by kl, and so on.

Let $\aleph = \{J: J=k_1k_2\cdots k_n, 1\le k_i \le K, 1\le i \le n, n>0\}\cup\{0\}$. Any string $J\in\aleph$ represents a node in the K-ary tree. The length of a string J is defined as the number of integers in the string and is denoted by $|J|$. When $J = 0$, $|J| = 0$. The *addition operation* and the *subtraction operation* for strings in $\aleph$ are defined as follows: if $J = k_1\cdots k_n \in\aleph$, $J\neq 0$, $H = h_1\cdots h_i \in\aleph$, and $H\neq 0$, then $J+H = k_1\cdots k_nh_1\cdots h_i \in\aleph$; if $J\in\aleph$, then $J+0 = 0+J = J$; if $J = k_1\cdots k_n \in\aleph$ and $H = k_i\cdots k_n \in\aleph$, $i>0$, then $J-H = k_1\cdots k_{i-1} \in\aleph$.

The Markov chain $\{(C_n, \eta_n), n\ge 0\}$ takes values in $\aleph\times\{1, 2, \cdots, m\}$. To be called a homogenous Markov chain of matrix *M/G/*1 type with a tree structure, (C_n, η_n) transits at each step either to its parent node or to a descendent of its parent node. Assuming that $(C_n, \eta_n) = (H+k, i)$ for $k>0$ and $1\le i, i'\le m$, then $(C_{n+1}, \eta_{n+1}) = (H+J, i')$ with probability $a_{(i,i')}(k, J)$ for $J\in\aleph$. If $(C_n, \eta_n) = (0, i)$, then $(C_{n+1}, \eta_{n+1}) = (J, i')$ with probability $b_{(i,i')}(J)$ for $J\in\aleph$.

Note that transition probabilities depend only on the last integer in the string representing the current node $H+k$. If we call the node $J+k$ a type k node, it is clear that all type k nodes have the same transition probabilities, $1\le k\le K$. In matrix form, transition probabilities are represented as:

$A(k, J)$ is an $m\times m$ matrix with elements $a_{(i,i')}(k, J)$, $1\le k \le K$, for $J\in\aleph$;
$B(J)$ is an $m\times m$ matrix with elements $b_{(i,i')}(J)$ for $J\in\aleph$.

Let

$$\begin{aligned} A(k) &= \sum_{J\in\aleph} A(k, J), \quad 1\le k\le K; \\ B^{*(1)}(k) &= \sum_{J\in\aleph} B(J)N(J,k), \; 1\le k\le K, \end{aligned} \tag{2.1}$$

where $N(J, k)$ is the number of appearances of integer k in the string J. By the law of total probability, we must have $A(k)\mathbf{e} = \mathbf{e}$, $1\le k\le K$, and $\left(\sum_{J\in\aleph} B(J)\right)\mathbf{e} = \mathbf{e}$, where $\mathbf{e}$ is the column vector with all components being one.

3 Three Existing Approaches to Ergodicity

In this section, we introduce three approaches to establish ergodicity conditions of the Markov chain $\{(C_n, \eta_n), n\geq 0\}$ defined in Section 2.

1 *The Perron-Frobenius Eigenvalue (PEF) Approach*

Let $\mathbf{X} = \{X_1, \cdots, X_K\}$, where $X_1, X_2, \cdots$, and X_K are $m\times m$ stochastic matrices, i.e., X_k is nonnegative and $X_k\mathbf{e} = \mathbf{e}$, $1\leq k\leq K$. Let $\Re$ be a set of all $\mathbf{X}$ for which $X_1, \cdots$, and X_K are stochastic matrices and satisfy the following equations, for $1\leq k\leq K$, ($J = k_1\cdots k_{|J|}$)

$$X_k = A(k,0) + \sum_{J\in\aleph,\, J\neq 0} A(k,J) X_{k_{|J|}} X_{k_{|J|-1}} \cdots X_{k_1}. \tag{3.1}$$

By the well-known Brouwer's fixed point theorem, it was shown in HE [4] that the set $\Re$ is nonempty. For any fixed point $\mathbf{X} = \{X_1, \cdots, X_K\}\in\Re$, define $X^{(J)} = X_{k_{|J|}} X_{k_{|J|-1}} \cdots X_{k_1}$ for all $J=k_1\cdots k_{|J|}\in\aleph$ and define the following $m\times m$ matrices, for $J=k_1\cdots k_{|J|}\in\aleph$,

$$\begin{aligned} &N(0, j, \mathbf{X}) = 0, \quad 1\leq j\leq K; \\ &N(J, j, \mathbf{X}) = I\delta(k_{|J|}, j) + \sum_{n=1}^{|J|-1} X_{k_{|J|}} X_{k_{|J|-1}} \cdots X_{k_{n+1}} \delta(k_n, j), \;\; 1\leq j\leq K; \\ &p(k, j, \mathbf{X}) = \sum_{J\in\aleph} A(k,J)\, N(J, j, \mathbf{X}), \quad 1\leq k,\; j\leq K, \end{aligned} \tag{3.2}$$

where $\delta(k, j) = 1$, if $k=j$; 0, otherwise, and I is the identity matrix. Note that the matrix $N(J, j, \mathbf{X})$ counts the number of appearances of integer j in the string J and keeps track of the phase changes in the transition process. The matrix $p(k, j, \mathbf{X})$ can be interpreted as the average number of appearances of integer j in the next transition, given that the Markov chain is currently in node $H+k$ for $H\in\aleph$. Define an $mK\times mK$ matrix $P(\mathbf{X})$ by

$$P(\mathbf{X}) = \begin{pmatrix} p(1,1,\mathbf{X}) & \cdots & p(1,K,\mathbf{X}) \\ \vdots & \vdots & \vdots \\ p(K,1,\mathbf{X}) & \cdots & p(K,K,\mathbf{X}) \end{pmatrix}.$$

(3.3)

Let $sp(P(\mathbf{X}))$ be the Perron-Frobinus eigenvalue of the matrix $P(\mathbf{X})$ (i.e., the eigenvalue with the largest real part).

Theorem 3.1 (Theorem 3.2, HE [4]) Assume that the Markov chain $\{(C_n, \eta_n), n\geq 0\}$ is irreducible and aperiodic and that $B^{*(1)}(k)$ is finite, $1\leq k\leq K$. For any element $\mathbf{X}\in\Re$, if the matrix $P(\mathbf{X})$ is irreducible, then the Markov chain of matrix *M/G/*1 type with a tree structure $\{(C_n, \eta_n), n\geq 0\}$ is

1) positive recurrent if and only if $sp(P(\mathbf{X})) < 1$;
2) null recurrent if and only if $sp(P(\mathbf{X})) = 1$;
3) transient if and only if $sp(P(\mathbf{X})) > 1$.

If m=1, $\mathbf{X}$ is reduced to $\mathbf{X} = \{1, 1, \ldots, 1\}$. Then Theorem 3.1 gives an explicit ergodicity condition. If m>1, since the matrix set $\mathbf{X}$ has to be calculated in order to construct the matrix $P(\mathbf{X})$, the usefulness of Theorem 3.1 is compromised. Thus, there is a need to find ergodicity conditions without the presence of $\mathbf{X}$.

Remark: Let $\mathbf{G} = \{G_1, \cdots, G_K\}$ be the minimal nonnegative solutions to equation (3.1). It has been shown in HE [4] that $sp(P(\mathbf{G})) \leq 1$, a fact that is quite useful in accuracy check and validating algorithms.

2 The Perron-Frobenius Eigenvalue sp(Q)

In this section, we introduce a descriptor for ergodicity without using any fixed point $\mathbf{X}$ in $\Re$. It is easy to calculate the descriptor, though it may not provide correct information about the ergodicity of the Markov chain of interest.

Let $\theta(k)$ be the left invariant vector of the stochastic matrix $A(k)$, where $\theta(k)$ is nonnegative and is normalized by $\theta(k)\mathbf{e} = 1$, $1\leq k\leq K$. Let

$$q(k, j) = \theta(k)\sum_{J\in\aleph} A(k, J)\, N(J, j)\mathbf{e}, \quad 1\leq k,\ j\leq K. \tag{3.4}$$

Let Q be a $K\times K$ matrix with the (k, j)th element being $q(k, j)$. Denote by $sp(Q)$ the Perron-Frobenius eigenvalue of the matrix Q.

Intuitively, $sp(Q)$, similar to $sp(P(\mathbf{X}))$, measures the average magnitude of an one-step movement of the Markov chain $\{(C_n, \eta_n), n\geq 0\}$. Thus, $sp(Q)$ should have a close relationship with ergodicity of the Markov chain. Furthermore, the computations of the matrix Q and of $sp(Q)$ are straightforward. The size of the matrix Q is smaller than that of the matrix $P(\mathbf{X})$. Therefore, it would be ideal if $sp(Q)$ could replace $sp(P(\mathbf{X}))$ for ergodicity (i.e., if $sp(Q) < 1$, the Markov chain is positive recurrent; if $sp(Q) > 1$, the Markov chain is transient.) Unfortunately, $sp(Q)$ may not provide correct information for ergodicity of the Markov chain. The change of the phase variable η_n depends on the type of node C_n. Therefore, $\theta(k)$ may not provide accurate information about the steady state distribution of the phase η_n. Consequently, $sp(Q)$ may not accurately measure the average magnitude of the one-step movement of the Markov chain.

Nonetheless, our numerical examples show that $sp(Q)$ is close to $sp(P(\mathbf{X}))$ and can be useful in practice since its computation is much easier than that of $sp(P(\mathbf{X}))$. In Section 6, we shall present a large number of examples to show the relationship between $sp(Q)$ and $sp(P(\mathbf{X}))$.

3 *Sufficient Conditions for Ergodicity*

The following sufficient conditions for ergodicity have been obtained in HE [4]. Denote by R_+ the set of nonnegative real numbers. Let $\mathbf{z} = (z_1, \cdots, z_K) \in R_+^K$ and define, for $1 \le k \le K$,

$$A^*(k, \mathbf{z}) = \sum_{J \in \aleph} \mathbf{z}^{(J)} A(k, J), \quad B^*(\mathbf{z}) = \sum_{J \in \aleph} \mathbf{z}^{(J)} B(J), \tag{3.5}$$

where $\mathbf{z}^{(J)} = z_{j_{|J|}} \cdots z_{j_1}$ if $J \in \aleph$ and $J \ne 0$, and $\mathbf{z}^{(J)} = 1$ if $J=0$.

Lemma 3.2 (Lemma 6.1, HE [4]) Assume that the Markov chain $\{(C_n, \eta_n), n \ge 0\}$ is irreducible and aperiodic and that $B^{*(1)}(k)$ is finite, $1 \le k \le K$. If there exists an $m \times 1$ positive vector $\mathbf{u}$ such that $A^*(k, \mathbf{z})\mathbf{u} < z_k \mathbf{u}$, (i.e., every element of $A^*(k, \mathbf{z})\mathbf{u}$ is strictly smaller than its counterpart in $z_k \mathbf{u}$,) and $B^*(\mathbf{z})\mathbf{u} < \infty$ for some $\mathbf{z}$ satisfying $1 < z_k < \infty$, $1 \le k \le K$, then the Markov chain $\{(C_n, \eta_n), n \ge 0\}$ is positive recurrent.

Lemma 3.3 (Lemma 6.2, HE [4]) Assume that the Markov chain $\{(C_n, \eta_n), n \ge 0\}$ is irreducible and aperiodic and that $B^{*(1)}(k)$ is finite, $1 \le k \le K$. If there exists an $m \times 1$ positive vector $\mathbf{u}$ such that $A^*(k, \mathbf{z})\mathbf{u} \le z_k \mathbf{u}$ for some $\mathbf{z}$ satisfying $z_k > 0$, $1 \le k \le K$, and $0 < z_k < 1$ for at least one k $(1 \le k \le K)$, then the Markov chain $\{(C_n, \eta_n), n \ge 0\}$ is transient.

These two sufficient conditions do not make use of any fixed point $\mathbf{X} = \{X_1, \cdots, X_K\}$ in $\Re$ and they provide correct information about ergodicity. However, it is not straightforward to verify the conditions. In Section 4, based on Lemmas 3.2 and 3.3, we develop two linear programs for the ergodicity problem.

4 Linear Programs for Ergodicity Conditions

Let $\delta = (\delta_1, \delta_2, \ldots, \delta_K)^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_m)^T$, where superscript "T" represents matrix transpose. Define a linear system with variables (δ, $\mathbf{v}$, ε) as follows:

$$\sum_{j=1}^{K} \delta_j \mathbf{d}(k, j) + \big(A(k) - I\big)\mathbf{v} + \varepsilon \mathbf{e} \le 0, \quad 1 \le k \le K, \tag{4.1}$$

where $\delta \ge 0$, $-\infty < v_j < \infty$, $1 \le j \le m$, $\varepsilon \ge 0$, and

$$\mathbf{d}(k, j) = \begin{cases} \sum_{J \in \aleph} A(k, J) N(J, j)\mathbf{e}, & 1 \le j \le K,\ j \ne k; \\ \sum_{J \in \aleph} A(k, J) N(J, j)\mathbf{e} - \mathbf{e}, & j = k. \end{cases} \tag{4.2}$$

Lemma 4.1 Consider the Markov chain $\{(C_n, \eta_n), n \ge 0\}$ defined in Section 3. We assume that $B^{*(1)}(k)$ is finite, $1 \le k \le K$, and $B^*(\mathbf{z}) < \infty$ for some $\mathbf{z}$ satisfying $1 < z_k < \infty$, $1 \le k \le K$. If the linear system (4.1) has a solution (δ, $\mathbf{v}$, ε) with positive δ_k, $1 \le k \le K$, and positive ε, then the Markov chain $\{(C_n, \eta_n), n \ge 0\}$ is positive recurrent.

Proof. We use Lemma 3.2 to prove Lemma 4.1. The idea is to choose a direction $\delta = (\delta_1, \delta_2, \ldots, \delta_K)^T$ with $\delta_k \ge 0$, $1 \le k \le K$, such that, when $z_k(t) = 1 + \delta_k t$, $1 \le k \le K$, (i.e., $\mathbf{z}(t) = \mathbf{e} + \delta t$) and $\mathbf{u}(t) = \mathbf{e} + t\mathbf{v}$, we have, for $1 \le k \le K$,

$$A^*(k, \mathbf{z}(t))\mathbf{u}(t) \le z_k(t)\mathbf{u}(t) - t\varepsilon \mathbf{e}, \tag{4.3}$$

with positive $\mathbf{u}(t)$ and $\delta \mathbf{e} = 1$ for some positive ε and positive t. By using the Taylor expansion of $A^*(k, \mathbf{z}(t))$ with respect to the variable t, the problem can be transformed into the linear system (4.1) in the following way:

$$A^*(k,\mathbf{z}(t))\mathbf{u}(t) = \left(\sum_{J\in\aleph} A(k,J)(\mathbf{z}(t))^{(J)}\right)\mathbf{u}(t)$$

$$= \left[\sum_{J\in\aleph} A(k,J) + \left(\sum_{J\in\aleph}\sum_{j=1}^{K} A(k,J)N(J,j)\delta_j\right)t + o(t)\right](\mathbf{e}+t\mathbf{v}) \tag{4.4}$$

$$= \left(\sum_{J\in\aleph} A(k,J)\right)\mathbf{e} + t\left[\sum_{J\in\aleph} A(k,J)\mathbf{v} + \sum_{j=1}^{K}\delta_j \sum_{J\in\aleph} A(k,J)N(J,j)\mathbf{e}\right] + o(t).$$

Then, for $1\le k\le K$, inequality (4.3) becomes,

$$A(k)\mathbf{e} + t\left[A(k)\mathbf{v} + \sum_{j=1}^{K}\delta_j \sum_{J\in\aleph} A(k,J)N(J,j)\mathbf{e}\right] + o(t). \tag{4.5}$$
$$\le (1+\delta_k t)(\mathbf{e}+t\mathbf{v}) - t\varepsilon\mathbf{e} = \mathbf{e} + t(\delta_k\mathbf{e} + \mathbf{v} - \varepsilon\mathbf{e}) + o(t).$$

Canceling the vector **e** and letting $t \to 0$ on both sides of the inequality in equation (4.5), we obtain

$$\sum_{j=1}^{K}\delta_j\left(\sum_{J\in\aleph} A(k,J)N(J,j)\right)\mathbf{e} + A(k)\mathbf{v} \le \delta_k\mathbf{e} + \mathbf{v} - \varepsilon\mathbf{e} \tag{4.6}$$
$$\Leftrightarrow \quad \sum_{j=1}^{K}\delta_j\left(\sum_{J\in\aleph} A(k,J)N(J,j)\right)\mathbf{e} - \delta_k\mathbf{e} + \big(A(k)-I\big)\mathbf{v} + \varepsilon\mathbf{e} \le 0.$$

It is easy to see that inequality (4.6) is equivalent to the linear system (4.1). If the linear system (4.1) has a solution (δ, **v**, ε) with positive δ_k, $1\le k\le K$, and positive ε, then equation (4.3) holds for small enough positive t and some positive $\varepsilon' \le \varepsilon$. Note that we choose positive t small enough to ensure that 1) inequality (4.3) is satisfied, 2) $z_k(t)>1$, $1\le k\le K$, 3) $\mathbf{u}(t)$ is positive, and 4) $B^*(\mathbf{z}(t))$ is finite. By Lemma 3.2, the Markov chain is positive recurrent.

To find a condition for transient Markov chains, we define the following linear system for (δ, **v**, ε):

$$-\sum_{j=1}^{K}\delta_j\mathbf{d}(k,j) + \big(A(k)-I\big)\mathbf{v} + \varepsilon\mathbf{e} \le 0, \quad 1\le k\le K. \tag{4.7}$$

with $\delta \geq 0$, $-\infty < v_j < \infty$, $1 \leq j \leq m$, and $\varepsilon \geq 0$.

Lemma 4.2 Consider the Markov chain $\{(C_n, \eta_n), n\geq 0\}$ defined in Section 3. If the linear system (4.7) has a solution $(\delta, \mathbf{v}, \varepsilon)$ with nonzero nonnegative vector δ and positive ε, then the Markov chain $\{(C_n, \eta_n), n\geq 0\}$ is transient.

Proof. The proof is similar to that of Lemma 4.1. Choose $z_k(t) = 1-\delta_k t$, $1\leq k\leq K$, i.e., $\mathbf{z}(t) = \mathbf{e} - \delta t$, and $\mathbf{u}(t) = \mathbf{e} + t\mathbf{v}$. By Lemma 3.3, we need to find δ and $\mathbf{v}$ such that $A^*(k, \mathbf{z}(t))\mathbf{u}(t) \leq z_k(t)\mathbf{u}(t)$ holds for some positive t. As in the proof of Lemma 4.1, we expand $A^*(k, \mathbf{z}(t))\mathbf{u}(t) \leq z_k(t)\mathbf{u}(t)$ and evaluate the expanded expressions in the following way:

$$\begin{aligned} &A^*(k, \mathbf{z}(t))\mathbf{u}(t) \\ &= A(k)\mathbf{e} + t\left[A(k)\mathbf{v} - \sum_{j=1}^{K}\delta_j \sum_{J\in\aleph} A(k,J)N(J,j)\mathbf{e}\right] + o(t) \\ &\leq \mathbf{e} + t(\mathbf{v} - \delta_k \mathbf{e}) + o(t). \end{aligned} \tag{4.8}$$

Canceling the vector $\mathbf{e}$ and letting $t \to 0$ on both sides of the inequality in equation (4.8), we obtain

$$-\sum_{j=1}^{K}\delta_j\left(\sum_{J\in\aleph} A(k,J)N(J,j)\right)\mathbf{e} + \delta_k\mathbf{e} + \big(A(k) - I\big)\mathbf{v} \leq 0. \tag{4.9}$$

We add the vector $\varepsilon\mathbf{e}$ to the left hand side of equation (4.9) to obtain equation (4.7). If the linear system (4.7) has a solution $(\delta, \mathbf{v}, \varepsilon)$ with a nonzero nonnegative vector δ and positive ε, inequality (4.9) is satisfied in strict sense. That implies that inequality (4.8) holds for small enough positive t. Thus the conditions given in Lemma 3.3 are satisfied for small enough positive t. Therefore, the Markov chain is transient.

It is easy to see that the key step in the application of Lemmas 4.1 and 4.2 is to show the existence of the required solutions to the linear systems (4.1) and (4.7). For that purpose, we introduce the following linear programs. First, we define a linear program from linear system (4.1) to get a sufficient condition for an ergodic Markov chain:

$$
\begin{aligned}
&\xi_1 = \max\{\varepsilon\} \\
&s.t. \begin{pmatrix} \mathbf{d}(1,1) & \cdots & \mathbf{d}(1,K) & A(1)-I & \mathbf{e} \\ \mathbf{d}(2,1) & \cdots & \mathbf{d}(2,K) & A(2)-I & \mathbf{e} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{d}(K,1) & \cdots & \mathbf{d}(K,K) & A(K)-I & \mathbf{e} \\ 1 & \cdots & 1 & 0 \ \cdots \ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \mathbf{v} \\ \varepsilon \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}; \\
&\quad \delta \ge 0, \ \ \varepsilon \ge 0, \quad -\infty < v_j < \infty, 1 \le j \le m.
\end{aligned}
\tag{4.10}
$$

Note that the constraints of the above linear program are from linear system (4.1), except that the constraint $\delta\mathbf{e} = 1$ is added to ensure a finite optimal solution (which may not be unique). Next, we define a linear program from linear system (4.7) to get a sufficient condition for a transient Markov chain:

$$
\begin{aligned}
&\xi_2 = \max\{\varepsilon\} \\
&s.t. \begin{pmatrix} -\mathbf{d}(1,1) & \cdots & -\mathbf{d}(1,K) & A(1)-I & \mathbf{e} \\ -\mathbf{d}(2,1) & \cdots & -\mathbf{d}(2,K) & A(2)-I & \mathbf{e} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{d}(K,1) & \cdots & -\mathbf{d}(K,K) & A(K)-I & \mathbf{e} \\ 1 & \cdots & 1 & 0 \ \cdots \ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \mathbf{v} \\ \varepsilon \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \\
&\quad \delta \ge 0, \ \ \varepsilon \ge 0, \quad -\infty < v_j < \infty, 1 \le j \le m.
\end{aligned}
\tag{4.11}
$$

Now, we are ready to present the main theorem of this paper.

Theorem 4.3 Consider the Markov chain $\{(C_n, \eta_n), n \ge 0\}$ defined in Section 3. The linear system (4.1) has a solution with positive (δ, ε) if and only if $\xi_1 > 0$. The linear system (4.8) has a solution with nonzero δ and positive ε if and only if $\xi_2 > 0$. Consequently, if $\xi_1 > 0$, the Markov chain is positive recurrent (provided that other assumptions of Lemma 4.1 are satisfied); if $\xi_2 > 0$, the Markov chain is transient.

Proof. First, we note that both (4.10) and (4.11) have a feasible solution $(\delta, \mathbf{v}, \varepsilon) = (0, \ldots, 0)$. Therefore, optimal solutions exist for both problems and $\xi_1, \xi_2 \ge 0$.

If the linear system (4.1) has a solution with positive (δ, ε), then we have $\xi_1 > \varepsilon > 0$. On the other hand, if the objective function of the optimal solution of (4.10) is positive, then the linear system (4.1) has a solution $(\delta, \mathbf{v}, \varepsilon)$ with a positive ε. Next, we show that δ is also positive. If δ=0, then the constraints in equation (4.10) (except the last line) become $(A(k)-I)\mathbf{v} + \varepsilon\mathbf{e} \le 0$, $1 \le k \le K$. Multiplying both sides of these inequalities by the nonnegative and nonzero vector $\theta(k)$ yields $\varepsilon\theta(k)\mathbf{e} = \varepsilon \le 0$, $1 \le k \le K$, which is a contradiction. (Note that $\theta(k)$ is the left invariant vector of $A(k)$ defined in Subsection 3.2.) Therefore, δ is nonzero. If the vector δ is not positive, i.e., if there exists k such that $\delta_k=0$, then we multiply $\theta(k)$ on both sides of the kth inequality in (4.10) to obtain

$$\sum_{j=1, j\neq k}^{K} \delta_j \theta(k)\mathbf{d}(k, j) + \varepsilon\theta(k)\mathbf{e} \le 0. \tag{4.12}$$

Since all of the components of the above vectors are nonnegative and some of them are positive ($\varepsilon>0$), the inequality cannot hold. Therefore, the vector δ is positive. According to Lemma 4.1, the Markov chain is positive recurrent.

The second part of the theorem about equation (4.11) can be proved similarly, except that the vector δ only has to be nonzero for this case. Details are omitted.

We note that, if m=1, information provided by the solutions of (4.10) and (4.11) for ergodicity of Markov chains is sufficient and necessary (provided that the other assumptions of Lemma 4.1 are satisfied). That is, the Markov chain is positive recurrent if and only if (4.10) has a positive optimal objective value; the Markov chain is transient if and only if (4.11) has a positive optimal objective value. Consequently, if neither (4.10) nor (4.11) has a positive optimal objective value, then the Markov chain of interest is null recurrent. Unfortunately, it is not easy to check how accurate the information provided by equations (4.10) and (4.11) is if m>1. In Section 6, a numerical analysis will be carried out to analyze the usefulness of Theorem 4.3.

Remark: The sufficient conditions given by Theorem 4.3 are closely related to $P(\mathbf{X})$. To see the relationship, let $\mathbf{v}$=0. Then the constraints of equation (4.10) becomes $P(\mathbf{X})\Delta - \Delta + \varepsilon\mathbf{e} \le 0$, where the vector $\Delta = (\delta_1\mathbf{e}^T, \delta_2\mathbf{e}^T, \ldots, \delta_K\mathbf{e}^T)^T$. Thus, finding a solution to (4.10) is equivalent to finding a special type of subinvariant measure of $P(\mathbf{X})$. However, $P(\mathbf{X})$ may not have such a subinvariant vector. Thus, Theorem 4.3 may fail to provide information for ergodicity.

It can be shown that (4.10) and (4.11) provide consistent information about ergodicity. If $\xi_1 > 0$, then $\xi_2 = 0$ and the Markov chain is positive recurrent. On the other hand, if $\xi_2 > 0$, then $\xi_1 = 0$ and the Markov chain is transient. It is possible that $\xi_1 = \xi_2 = 0$. For this case, ξ_1 and ξ_2 provide no information about the ergodicity of the Markov chain. These results can be proved in a way similar to that of Property 5.5 in HE and Li [6]. Details are omitted.

To end this section, we outline a computational scheme to check whether or not the Markov chain defined in Section 2 is ergodic.

Step 1. Calculate $\{A(k), 1\le k\le K\}$.
Step 2. Calculate $\{\mathbf{d}(k, j), 1\le k, j\le K\}$ by equation (4.2).
Step 3. Solve linear programs (4.10) and (4.11).
Step 4. If neither (4.10) nor (4.11) provides information about system stability, use the PFE method given in Section 3.1.

Compared to the methods introduced in Section 3.1, the linear program approach has a larger matrix (the constraints of equations (4.10) and (4.11)) to deal with. In fact, the space complexity of the PFE approach is $O(Km^2)$ and the space complexity of the linear program approach is $O(K+m+1)(Km+1) = O(K^2m+Km^2)$. If m is much larger than K, then the space complexity of the two methods is more or less the same. On the other hand, the matrix iterations for $\mathbf{X} = \{X_1, \ldots, X_K\}$ that are necessary for the PFE approach are avoided for the linear program approach so that the time complexity of the linear program approach is low and numerical precision can be ensured. Furthermore, there are well-developed algorithms and software that can solve linear programs efficiently, even when the number of phases is large. Therefore, the linear program approach has its advantages over the other approach.

5 Computational Details

In order to use the methods introduced in Sections 3 and 4, we have to compute summations of matrices over string $J\in\aleph$. Examples of such summations can be found in equations (2.1), (3.1), (3.2), (3.4), and (4.2). However, the actual implementation of such a summation is not straightforward. We introduce the following transformations that transform the summations over $J\in\aleph$ into summations over two indices. The latter can be implemented easily in computation.

For any string $J=k_1k_2\ldots k_{|J|}\in\aleph$ and $J\neq 0$, we introduce a pair of integers (n, t) as follows:

$$n = |J|, \quad t = \sum_{i=1}^{n}(k_i - 1)K^{n-i}. \tag{5.1}$$

On the other hand, for any pair of integers (n, t) with $n>0$ and $0\le t\le K^n - 1$, we introduce a string $J = k_1\ldots k_n$ as follows:

$$k_1 = \left\lfloor \frac{t}{K^{n-1}} \right\rfloor, \quad k_2 = \left\lfloor \frac{t - k_1 K^{n-1}}{K^{n-2}} \right\rfloor, \quad \cdots, \quad k_n = \left\lfloor \frac{t - \sum_{i=1}^{n-1} k_i K^{n-i}}{K^{n-n}} \right\rfloor, \tag{5.2}$$

where $\lfloor x \rfloor$ represents the largest integer that is smaller than or equal to x.

Lemma 5.1. Assume that the string J=0 corresponds to the pair $(n, t) = (0, 0)$. Then the transformations defined by equations (5.1) and (5.2) are two one-to-one transforms between the two sets $\aleph$ and $\{(n, t)\text{: } 0 \le t \le K^n - 1,\ n \ge 0\}$. Furthermore, define $\aleph_n = \{J\text{: } J \in \aleph \text{ and } |J| = n\}$ for $n \ge 0$. Then the transformations defined by equations (5.1) and (5.2) are one-to-one transformations between the two sets $\aleph_n$ and $\{(n, t)\text{: } 0 \le t \le K^n - 1\}$ for all $n \ge 0$.

Proof. The conclusion is obtained from the fact that the transformation defined by equation (5.1) is the inverse of the transformation defined by equation (5.2) and vice versa.

6 Numerical Examples

To study the usefulness of the linear program approach and $sp(Q)$, we have run and analyzed a large number of numerical examples. In this section, we present the results of our numerical analysis. First, we show in Example 6.1 that the PFE, $sp(Q)$, and linear program methods may provide different information for ergodicity. Second, in Example 6.2, we show how good $sp(Q)$ and the linear program approach can be by summarizing the results for a large number of randomly chosen examples.

Example 6.1 Consider an *M/G/*1 type Markov chain with a tree structure with the following transition blocks: K=m=2,

$$A(1,0) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix},\ A(1,1) = \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0 \end{pmatrix},\ A(1,2) = \begin{pmatrix} 0 & 0.1 \\ 0.1 & 0 \end{pmatrix},\ A(1,11) = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \end{pmatrix},$$
$$A(1,12) = \begin{pmatrix} 0 & 0.1 \\ 0 & 0 \end{pmatrix},\ A(1,21) = \begin{pmatrix} 0.1 & 0 \\ 0.35 - \mu/2 & 0 \end{pmatrix},\ A(1,22) = \begin{pmatrix} 0 & 0.6 - \mu \\ 0 & 0.35 - \mu/2 \end{pmatrix},$$

$$A(2,0) = \begin{pmatrix} \frac{\mu}{2} & \frac{\mu}{2} \\ 0 & \mu \end{pmatrix},\ A(2,1) = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \end{pmatrix},\ A(2,2) = \begin{pmatrix} 0.2 & 0.1 \\ 0.2 & 0 \end{pmatrix}, A(2,11) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A(2,12) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ A(2,21) = \begin{pmatrix} 0 & 0.1 \\ 0.1 & 0 \end{pmatrix},\ A(2,22) = \begin{pmatrix} 0 & 0.6-\mu \\ 0.6-\mu & 0 \end{pmatrix},$$

where $0 < \mu < 0.6$. For μ goes from 0.2 to 0.5, the values of $sp(P(\mathbf{X}))$, $sp(Q)$, ξ_1, and ξ_2 are shown in Table 6.1. Note that, in Table 6.1, $sp(P(\mathbf{X}))$ is the only measurement that always provides correct information about ergodicity.

Table 6.1 Values of $sp(P(\mathbf{X}))$, $sp(Q)$, ξ_1, and ξ_2 for Example 6.1

μ	0.2	0.3	0.35	0.3597	0.37	0.4	0.5
$sp(P(\mathbf{X}))$	1.31422	1.1170	1.0188	0.999863	0.9883	0.9211	0.7278
$sp(Q)$	1.31421	1.1171	1.0190	1.000084	0.9809	0.9214	0.7285
ξ_1	0	0	0	0	0.0098	0.0370	0.1317
ξ_2	0.1530	0.0558	0.0089	0	0	0	0

As shown in Table 6.1, if $sp(P(\mathbf{X})) \approx 1$ ($\mu \approx 0.36$), ξ_1 and ξ_2 do not provide information about ergodicity since they are both zero. If $\mu < 0.35$, ξ_2 is positive so that the Markov chain is transient. If $\mu > 0.37$, ξ_1 is positive so that the Markov chain is positive recurrent. If $\mu = 0.3597$, $sp(P(\mathbf{X})) = 0.999863 < 1 < 1.000084 = sp(Q)$. For this case, $sp(Q)$ does not provide correct information about ergodicity. Nonetheless, if $\mu < 0.359$ or $\mu > 0.36$, information about ergodicity provided by $sp(Q)$ is consistent with that provided by $sp(P(\mathbf{X}))$.

Example 6.2 In this example, we plot $sp(P(\mathbf{X}))$ against $sp(Q)$, ξ_1, and ξ_2 respectively for a large number of randomly chosen examples. In Figure 6.1, we plot $(sp(P(\mathbf{X})), sp(Q))$. In Figure 6.2, we plot $(sp(P(\mathbf{X})), \xi_1)$ and $(sp(P(\mathbf{X})), \xi_2)$.

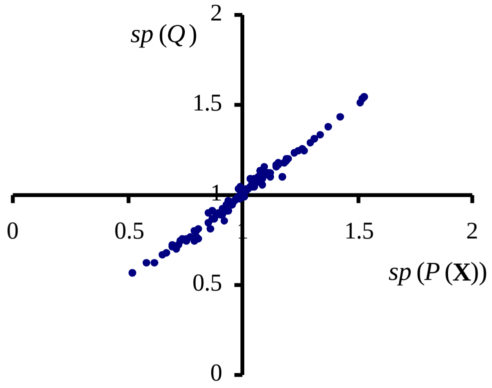

Figure 6.1 Plot of ($sp(P(\mathbf{X}))$, $sp(Q)$).

Figure 6.1 demonstrates clearly that $sp(Q)$ and $sp(P(\mathbf{X}))$ are very close for almost all the examples, though outliers do exist. Thus, $sp(Q)$ can be a useful indicator of ergodicity of the Markov chain.

By Theorem 4.3, ξ_1 may be useful if the Markov chain is positive recurrent and ξ_2 may be useful if the Markov chain is transient. Therefore, we plot ($sp(P(\mathbf{X}))$, ξ_1) only for $sp(P(\mathbf{X})) < 1$ and ($sp(P(\mathbf{X}))$, ξ_2) only for $sp(P(\mathbf{X})) > 1$. Figure 6.2 shows that both ξ_1 and ξ_2 may be zero and so fail to provide information about ergodicity if $sp(P(\mathbf{X})) \approx 1$ ($0.9 < sp(P(\mathbf{X})) < 1.1$). Nonetheless, Figure 6.2 does show that ξ_1 starts to provide information if $sp(P(\mathbf{X}))$ goes below 0.9 and ξ_2 starts to provide information if $sp(P(\mathbf{X}))$ goes beyond 1.1. Figure 6.2 also shows that ξ_1 and ξ_2 may provide useful information even if $sp(P(\mathbf{X})) \approx 1$ ($0.9 < sp(P(\mathbf{X})) < 1.1$). It is reasonable to conclude that if neither (4.10) nor (4.11) provides information about ergodicity, i.e., $\xi_1 = \xi_2 = 0$, then the Markov chain is "*close*" to null recurrent. More numerical examples for the relationship between $sp(P(\mathbf{X}))$, ξ_1, and ξ_2 can be found in HE and Li [6].

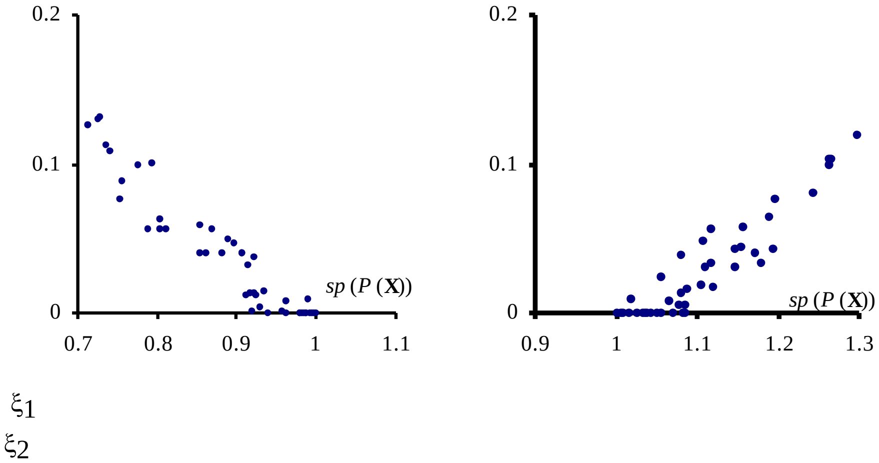

Figure 6.2 Plots of ($sp(P(\mathbf{X}))$, ξ_1) and ($sp(P(\mathbf{X}))$, ξ_2)

In summary, $sp(Q)$ is close to $sp(P(\mathbf{X}))$ and may provide correct information about ergodicity if $sp(P(\mathbf{X}))$ is not close to one. The linear program approach provides correct information about ergodicity if one of the two linear programs has a positive optimal value, which is usually true if the Markov chain is not on the border of being transient or recurrent. Without using $sp(P(\mathbf{X}))$, the following scheme can be used for the ergodicity problem: 1) If $\xi_1 > 0$, the Markov chain is positive recurrent; 2) If $\xi_2 > 0$, the Markov chain is transient; 3) If $\xi_1 = \xi_2 = 0$ and $sp(Q)$ is close to one, the Markov chain is on the border of being transient or recurrent; 4) Otherwise, the ergodicity of the Markov chain is unsure.

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References

1. Chen, H. and H. Zhang, Stabiliy of multiclass queueing networks under priority service disciplines, *Operations Research*, **Vol 48**, 26-37 (2000).
2. Fayolle, G., V.A Malyshev, and M.V. Menshikov, *Topics in the Constructive Theory of Countable Markov Chains*, Cambridge University Press (1995).

3. HE, Q-M, Classification of Markov processes of *M/G/*1 type with a tree structure and its applications to queueing systems, *O.R. Letters*, **Vol 26**, 67-80 (2000).
4. HE, Q-M, Classification of Markov processes of matrix *M/G/*1 type with a tree structure and its applications to the *MMAP*[*K*]/*PH*[*K*]/1 queue, *Stochastic Models*, **Vol 15**, 407-433 (2000).
5. HE, Q-M and A.S. Alfa, The discrete time *MMAP*[*K*]/*PH*[*K*]/1/LCFS-GPR queues and its variants, *Advances in Algorithmic Methods for Stochastic Models – Proceedings of the 3rd International conference on Matrix analytic Methods*, G. Latouche and P.G. Taylor (Editors), 2000, Notable Publications Inc., NJ., 167-190 (2000).
6. HE, Q-M and Hui Li, Stability conditions of the *MMAP*[*K*]/*G*[*K*]/1/LCFS preemptive repeat queue, Working paper #01-02, Department of Industrial Engineering, Dalhousie University (2001) (Submitted for publication).
7. Van Houdt, B. and C. Blondia, Stability and performance of stack algorithms for random access communication modeled as a tree structured QBD Markov chain, *Stochastic Models*, **Vol 17** (2001).
8. Kumar, P.R. and S.P. Meyn, Duality and linear programs for stability and performance analysis of queueing networks and scheduling policies, *IEEE Transactions on Automatic Control*, **Vol 41/**1, 4-17 (1996).
9. Latouche, G. and V. Ramaswami, *Introduction to Matrix Analytic Methods in Stochastic Modelling*, SIAM, Philadelphia, Pennsylvania (1999).
10. Neuts, M.F., *Matrix-Geometric Solutions in Stochastic Models: An algorithmic Approach*, The Johns Hopkins University Press, Baltimore (1981).
11. Neuts, M.F., *Structured Stochastic Matrices of M/G/*1 *type and Their Applications*, Marcel Dekker, New York (1989).
12. Takine, T., B. Sengupta, and R.W. Yeung, A generalization of the matrix *M/G/*1 paradigm for Markov chains with a tree structure, *Stochastic Models*, **Vol 11**, 411-421 (1995).