

On the Relation Between the Benefits of Risk Pooling and the Variability of Demand

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Abstract

The benefits of pooling risks, manifested in inventory management by consolidating multiple random demands in one location, are well known. What is less well understood are the determinants of the magnitude of the savings. Recently several researchers speculated about the impact of demands' variabilities on the benefits of risk-pooling. We provide an example where increased variability of the individual demands actually reduces the benefits of risk pooling. We prove, however, that if we restrict increased variability to a common linear transformation, the greater the demand variabilities the larger the benefits of consolidating them, in agreement with intuition. We also provide bounds on the benefits of the consolidation of demands. Our results do not require independence of the demands, apply to any number of pooled demands, and are obtained in pure-cost driven model.

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1 Introduction

Risk pooling, or statistical economies of scale, is a pervasive phenomenon which underlies major economic activities like banking and insurance. In operations management, it is often achieved by consolidating a product with random demand into one location, which is long known to be beneficial (e.g. Eppen 1979, Cherikh 2000). Though the potential for such benefit is being repeatedly (re)discovered in many contexts, a more interesting set of questions pertain to the *manner* in which the risk-pooled inventory policy differs from the non-pooled solutions and their aggregate.

These issues were explored both in the “minimalistic” setting of aggregating several demands in newsvendor-like models (Gerchak and Mossman 1992, Gerchak and Wang 1997, Anupindi and Bassok 1999, Yang and Schrage 2000), as well as in assemble-to-order models with component commonality (Gerchak and Henig 1989, Eynan 1996, Eynan and Rosenblatt 1996, Hillier 1999, Dong and Lee 2000) and delayed differentiation and substitution problems (Lee and Tang 1997, Rao, Swaminathan and Zhang 2000).

In particular, Gerchak and Mossman (1992) showed that aggregation of independent demands in pure-cost driven newsvendor models, while beneficial, need not reduce* the optimal *inventory* nor move it closer to the mean or relevant median demand. Gerchak and Wang (1997) showed that such is still the case in a newsvendor model with quadratic shortage costs (a cost structure which better reflects the costs of liquidating increasingly less liquid assets to meet claims). Yang and Schrage (2000) analyzed the implications of partial substitutions.

Anupindi and Bassok (1999) raised the question whether the expected sales always

*Partly since that observation is not well known, and partly since some researchers derive required inventory amounts from service level constraints (in which case risk pooling always reduces inventory), one occasionally encounters unqualified statements about risk pooling always resulting in smaller stocks (e.g., Simchi-Levi et al. 2000, p. 60).

increase with aggregation (one of the parties in their supply chain, the manufacturer, would then benefit). The answer, again, is that such will not always be the case.

What seems to have not yet been seriously explored, however, are the exact determinants of the savings due to pooling. In particular, one would like to know how demand *variability* affects the consequences and benefits of risk-pooling. Intuition seems to suggest that the benefits of risk-pooling will increase with respect to the original demands' variabilities; after all, it has no (statistical) benefits if demands are known. Benjaafar and Kim (2001) show, however, that in make-to-stock queues, high variability of original demands renders risk-pooling of little benefit. See also Benjaafar et al. (2002). Simchi-Levi et al. (2000, p. 60) state that an increase in benefits due to randomness always holds, which is indeed true in a service-level model which they use. The situation is less obvious in a setting where trades-off between overage and underage costs are considered (pure-cost driven model), which is the one of interest to us. Pertaining to that question, Rao et al. (2000, Section 5.3.3) state that "We found that proving this observation is difficult in general", and then show that it holds for a particular example. We provide an example which shows that increasing the variability of one of the individual demands may reduce the benefit of combining it with another. We do however, prove that within the more economically meaningful framework of mean-preserving variability-increasing transformations (see definitions and references in Section 4), the intuition is right: increased demand variability always results in larger benefits. Our result does not require independence and applies to any number of pooled demands. Other related insights are also derived.

2 The Model and its Basic Properties

Let X be a non-negative random demand with distribution function $F(x)$ (denoted as $X \sim F$). Let c be the unit ordering/production cost, net of salvage value; the unit opportunity loss for missing a sale is scaled to one, and we assume that $c < 1$. Then the basic newsvendor

problem is:

$$\Phi^*(X) \equiv \min_{0 \leq Q < \infty} \left\{ cQ + \int_Q^\infty (x - Q) dF(x) \right\} = \min_{0 \leq Q < \infty} \{ cQ + E(X - Q)^+ \}, \quad (1)$$

where $(x - Q)^+ = x - Q$ if $x \geq Q$; 0 if $x < Q$.

The solution to equation (1) is $Q_X^* = F^{-1}(1 - c)$, where $F^{-1}(x)$ is the inverse function of $F(x)$ and the optimal objective value can be written as $\Phi^*(X) = \int_{F^{-1}(1-c)}^\infty x dF(x)$.

If another random demand $Y \sim G$ is served by a separate stock, the total optimal inventory will be $Q_X^* + Q_Y^* = F^{-1}(1 - c) + G^{-1}(1 - c)$, and the sum of expected costs $\Phi^*(X) + \Phi^*(Y) = \int_{F^{-1}(1-c)}^\infty x dF(x) + \int_{G^{-1}(1-c)}^\infty x dG(x)$.

If the two demands are consolidated, then the relevant random demand becomes $X + Y \equiv Z$. We shall *not* assume that X and Y are independent and thus Z 's distribution is generally not the convolution of X and Y 's distributions.

The basic advantage of risk-pooling is captured by the fact that the optimal risk-pooled costs $\Phi^*(X + Y)$ are always lower than or equal to the sum of the unpooled costs $\Phi^*(X) + \Phi^*(Y)$. While this can be proved formally (and we will, shortly, for completeness), it is rather obviously true since, if one stocked in the pooled case exactly the sum of the optimal unpooled stocks, one could do at least as well, and thus, by acting optimally in the pooled case, one can do still better. The optimal stock, however, does not necessarily go down with pooling ($Q_{X+Y}^* > Q_X^* + Q_Y^*$ is possible), and neither do the expected sales.

This note focuses on the benefits of risk-pooling, i.e., the benefit function $\Phi^*(X) + \Phi^*(Y) - \Phi^*(X + Y)$, which is always nonnegative. As discussed in the Introduction, we shall investigate the relationship between the demand variability and the benefits of risk-pooling. We first show an example where the benefits of risk-pooling are *decreasing* in demand variability.

3 Counterexample

We say that X is smaller than Y in convex ordering, denoted by $X \leq_c Y$, if $E[f(X)] \leq E[f(Y)]$ for all convex functions f (e.g., Shaked and Shanthikumar 1994). Let random de-

mand $X_n = 0$, w.p. (with probability) $1 - 1/n$; and n , w.p. $1/n$, for any positive integer n . It is easy to verify that $E(X_n) = 1 \forall n$ and $Var(X_n) = n - 1$. It is also easy to verify that $X_n \leq_c X_{n+1}$, i.e., X_{n+1} is more variable than X_n with respect to the convex ordering.

Consider another random demand $Y = 0$, w.p. 0.5; and 1, w.p. 0.5. We assume that X_n and Y are independent. Let c be the unit ordering cost ($0 < c < 1$). We are interested in the benefits of risk-pooling $\Phi^*(X_n) + \Phi^*(Y) - \Phi^*(X_n + Y)$ as a function of n .

We assume that $0 < c < 0.5$ and that n satisfies $cn > 1 + c$. Then we have

$$\begin{aligned}\Phi^*(X_n) &= \min_{0 \leq Q < \infty} \{cQ + E(X_n - Q)^+\} \\ &= \min\left\{\min_{0 \leq Q \leq n} \{cQ + (n - Q)/n\}, \min_{n < Q} \{cQ\}\right\} = \min\{1, cn\} = 1.\end{aligned}$$

It is also easy to show that the random variable $X_n + Y = 0$, w.p. $(1 - 1/n)/2$; 1, w.p. $(1 - 1/n)/2$; n , w.p. $1/(2n)$; and $n + 1$, w.p. $1/(2n)$. By routine calculations, we have

$$\Phi^*(X_n + Y) = c + 1 - \frac{1}{2n}.$$

Then we obtain the benefits of risk-pooling as

$$\Phi^*(X_n) + \Phi^*(Y) - \Phi^*(X_n + Y) = 1 + \Phi^*(Y) - c - 1 + \frac{1}{2n} = \Phi^*(Y) - c + \frac{1}{2n}.$$

The above expression implies that the benefits of risk-pooling decrease when the demand variability (of X_n) increases (even though the mean demand remains unchanged). Note that in this example $Q_{X_n}^* = Q_Y^* = 0$, implying that $\Phi^*(X_n) = E(X_n)$ and $\Phi^*(Y) = E(Y)$, while $Q_{X_n+Y}^* = 1$, and thus $\Phi^*(X_n + Y) < E(X_n + Y)$.

While the example may not be quite realistic economically, the possibility of such behavior means that the concept of increased variability needs to be refined so that the intuitive expectations will be validated.

4 Costs and Improvements as Function of Mean-Preserving Variability Increases

Let

$$X_\alpha = \alpha X + (1 - \alpha)\mu_X, \quad 0 \leq \alpha \leq 1, \quad (2)$$

where $\mu_X \equiv E(X)$. Denote by $Var(X)$ the variance of X . Then, $E(X_\alpha) = E(X)$ and $Var(X_\alpha) = \alpha^2 Var(X)$, which is nondecreasing in α . Thus, the random demand X_α becomes more variable when α increases. Throughout this paper we assume that α takes values between 0 and 1. This mean-preserving transformation (spread) is a common device for exploring the implications of changing variability in probabilistic micro-economics (e.g., Baron 1970, Sandmo 1971, Hey 1979). While X_α usually does not belong to the same parametric family as X (though it does so in normal case), X can have any distribution. It was previously used in operations management by Gerchak and Mossman 1992, Gupta and Gerchak 1995, and Gerchak 2000. We shall first show this transformation is consistent with convex ordering.

Proposition 1 *For any nonnegative random demand X , we have $X_\alpha \leq_c X$ for $0 \leq \alpha \leq 1$.*

Proof. To show the result, we note that $\mu_X \leq_c X$, so $E(\mu_X - x)^+ \leq E(X - x)^+$ holds for any x . Then we have, for any x ,

$$\begin{aligned} E(X_\alpha - x)^+ &= E(\alpha(X - x) + (1 - \alpha)(\mu_X - x))^+ \\ &\leq \alpha E(X - x)^+ + (1 - \alpha)(\mu_X - x)^+ \\ &\leq \alpha E(X - x)^+ + (1 - \alpha)E(X - x)^+. \end{aligned}$$

Thus, $E(X_\alpha - x)^+ \leq E(X - x)^+$ holds for any x . Therefore, (e.g., Shaked and Shanthikumar, 1994, Thm. 2.A.1) since the means are equal, $X_\alpha \leq_c X$. ||

Gerchak and Mossman showed that $Q_\alpha^* = \alpha Q_X^* + (1 - \alpha)\mu_X$. That implies that the stock is increasing in demand variability if and only if $F^{-1}(1 - c) > \mu_X$. We shall show that the benefits of risk pooling always increase in the variabilities of individual random demands under such mean-preserving transformation.

Let $\{X_1, X_2, \dots, X_n\}$ be n random demands (nonnegative random variables), where n is a positive integer. Let $X_{i,\alpha_i} = \alpha_i X_i + (1 - \alpha_i)\mu_{X_i}$, for $0 \leq \alpha_i \leq 1$ and $1 \leq i \leq n$. The benefits of risk-pooling of the n demands are defined as, for $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\phi(\alpha) \equiv \sum_{i=1}^n \Phi^*(X_{i,\alpha_i}) - \Phi^*\left(\sum_{i=1}^n X_{i,\alpha_i}\right). \quad (3)$$

The following theorem presents the main result of this paper.

Theorem 2 For $0 \leq t \leq 1$ and $0 \leq \alpha_i \leq 1$, $1 \leq i \leq n$,

$$\phi(t\alpha) = t\phi(\alpha); \quad \phi(\alpha) = \sum_{i=1}^n \alpha_i \Phi^*(X_i) - \Phi^*\left(\sum_{i=1}^n \alpha_i X_i\right). \quad (4)$$

The function $\phi(t\alpha)$ is nonnegative. In addition, the function $\phi(\alpha)$ is non-decreasing in each α_i for $1 \leq i \leq n$. The benefits of risk-pooling are upper-bounded:

$$\phi(t\alpha) \leq t(1-c) \sum_{i=1}^n \alpha_i \mu_{X_i}.$$

The optimal risk-pooled order size has the following property:

$$Q_{X_1, t\alpha_1 + \dots + X_n, t\alpha_n}^* = tQ_{\alpha_1 X_1 + \dots + \alpha_n X_n}^* + \sum_{i=1}^n (1 - t\alpha_i) \mu_{X_i}. \quad \parallel$$

The linear relationship in equation (4) is due to the mean-preserving transformation used. Since $\phi(\alpha)$ is nonnegative, Theorem 2 shows that the benefits of risk-pooling increase in the variabilities of individual random demands. But that property is not always shared by the optimal order size.

We divide the proof of Theorem 2 into a series of lemmas and properties. Some of these are of independent managerial/economic interest. We shall comment on the implications of various results when they are presented. The first result establishes that variability increases the expected costs as well as bounds the expected costs for any random demand.

Lemma 3 For any nonnegative random demands X and Y , if $Y \leq_c X$, i.e., Y is less than X in the convex ordering, then $\Phi^*(Y) \leq \Phi^*(X)$. Consequently, we have

$$c\mu_X \leq \Phi^*(X) \leq \mu_X. \quad (5)$$

Proof. By definition, if $Y \leq_c X$, $E(Y - x)^+ \leq E(X - x)^+$ holds for any x . Then we have

$$\Phi^*(X) = cQ_X^* + E(X - Q_X^*)^+ \geq cQ_X^* + E(Y - Q_X^*)^+ \geq \Phi^*(Y).$$

It is well-known that $\mu_X \leq_c X$ (Shaked and Shanthikumar, 1994). Choose $Y \equiv \mu_X$. We obtain $\Phi^*(\mu_X) \leq \Phi^*(X)$. Since $\Phi^*(\mu_X) = c\mu_X$ (note that $0 < c < 1$), we have $c\mu_X \leq \Phi^*(X)$. By choosing $Q = 0$ in equation (1), we obtain $\Phi^*(X) \leq \mu_X$. ||

Next, we present some results concerning the optimization problem in equation (1) when the demand undergoes a mean-preserving transformation. The proofs of the following results are straightforward.

Lemma 4 *For any nonnegative random demand X and a nonnegative constant y , we have*

$$\Phi^*(X + y) = \Phi^*(X) + cy. \quad (6)$$

The corresponding optimal order size is given by $Q_X^ + y$. Also, we have*

$$\Phi^*(\alpha X) = \alpha\Phi^*(X). \quad (7)$$

The optimal order size is αQ_X^ . Consequently, we have*

$$\Phi^*(X_\alpha) = \alpha\Phi^*(X) + (1 - \alpha)c\mu_X. \quad (8)$$

The optimal order size is given by $\alpha Q_X^ + (1 - \alpha)\mu_X$. ||*

Note that Gerchak and Mossman (1992) claimed that both order size and costs undergo the same transformation as the demand, which is true for the order size but not the costs (due to the presence of c in the last term of (8)).

The results in Lemma 4 are quite intuitive. When the random demand increases by a constant, the optimal decision is to add that constant to the original optimal solution, and the extra cost is then cy . When the random demand increases proportionally, the optimal order size and costs increase by the same multiplier.

Note that since the right-hand side of equation (8) can be written as $\alpha[\Phi^*(X) - c\mu_X] + c\mu_X$, and, by Lemma 3, $\Phi^*(X) \geq c\mu_X$, it follows that $\Phi^*(X_\alpha)$ is linearly increasing in α .

Now, we consider models with two random demands and explore the benefits of risk-pooling. It is well-known that

$$\Phi^*(X + Y) \leq \Phi^*(X) + \Phi^*(Y). \quad (9)$$

The pooled costs are bounded as follows:

$$c(\mu_X + \mu_Y) \leq \Phi^*(X + Y) \leq \Phi^*(X) + \Phi^*(Y) \leq \mu_X + \mu_Y. \quad (10)$$

The inequalities in (10) imply that the benefits of risk-pooling are always smaller than or equal to $(1 - c)(\mu_X + \mu_Y)$. If c is close to 1, the benefits of risk-pooling is immaterial. That

is not surprising, since when c is close to 1 the optimal Q approaches zero (no production) and the objective function thus approaches the expected demand, which is additive.

As for the benefits of risk-pooling under mean-preserving transformation, we consider two scenarios: a) $X_\alpha + Y_\alpha$ and b) $X_\alpha + Y$, where $0 \leq \alpha \leq 1$. In the first scenario, the variabilities of both demands are changed in a “coordinated” fashion. In the second scenario, only the variability of one of the demands is changed.

Proposition 5 *For any two nonnegative random demands X and Y , we have,*

$$\Phi^*(X_\alpha) + \Phi^*(Y_\alpha) - \Phi^*(X_\alpha + Y_\alpha) = \alpha[\Phi^*(X) + \Phi^*(Y) - \Phi^*(X + Y)]. \quad (11)$$

Thus, the benefits of risk-pooling are non-decreasing in α . Also, we have

$$\Phi^*(X_\alpha) + \Phi^*(Y_\alpha) - \Phi^*(X_\alpha + Y_\alpha) \leq \alpha(1 - c)(\mu_X + \mu_Y).$$

Proof. By Lemma 3, we have

$$\begin{aligned} & \Phi^*(X_\alpha + Y_\alpha) - \Phi^*(X_\alpha) - \Phi^*(Y_\alpha) \\ = & \alpha\Phi^*(X + Y) + (1 - \alpha)c(\mu_X + \mu_Y) - \alpha\Phi^*(X) - (1 - \alpha)c\mu_X - \alpha\Phi^*(Y) - (1 - \alpha)c\mu_Y \\ = & \alpha[\Phi^*(X + Y) - \Phi^*(X) - \Phi^*(Y)] \end{aligned}$$

By equation (9), the benefits of risk-pooling are non-decreasing in α . The upper-bound on the benefits of risk-pooling is obtained by using the inequalities given in equation (10). ||

Proposition 6 *For any two nonnegative random demands X and Y , the benefits of risk-pooling $\Phi^*(X_\alpha) + \Phi^*(Y) - \Phi^*(X_\alpha + Y)$, are non-decreasing in α .*

Proof. For any $\delta > 0$, by equation (9),

$$\Phi^*(\alpha X + Y) - \Phi^*((\alpha + \delta)X + Y) \geq -\Phi^*(\delta X) = -\delta\Phi^*(X).$$

By Lemma 4,

$$\begin{aligned} & \Phi^*(X_{\alpha+\delta}) - \Phi^*(X_{\alpha+\delta} + Y) - [\Phi^*(X_\alpha) - \Phi^*(X_\alpha + Y)] \\ = & (\alpha + \delta)\Phi^*(X) - \Phi^*((\alpha + \delta)X + Y) - [\alpha\Phi^*(X) - \Phi^*(\alpha X + Y)] \\ = & \delta\Phi^*(X) + \Phi^*(\alpha X + Y) - \Phi^*((\alpha + \delta)X + Y) \\ \geq & \delta\Phi^*(X) - \delta\Phi^*(X) = 0. \end{aligned}$$

Therefore, the benefits of risk-pooling are non-decreasing in α . ||

By equations (9) and (10) and Proposition 6, it is easy to show that, for any two nonnegative random demands X and Y and $0 \leq \alpha, \beta \leq 1$, $\phi(\alpha, \beta) = \Phi^*(X_\alpha) + \Phi^*(Y_\beta) - \Phi^*(X_\alpha + Y_\beta)$ is non-decreasing in both α and β . Therefore, the benefits of risk-pooling are non-decreasing with respect to the variabilities of (correlated) individual random demands. In addition, we have $\phi(\alpha, \beta) \leq (1 - c)(\alpha\mu_X + \beta\mu_Y)$, which implies that the benefits of risk-pooling is insignificant if c is close to 1.

In summary, we have shown that Theorem 2 holds for a special case where two demands are involved. Using Lemma 4, equations (9) and (10), and Propositions 5 and 6, Theorem 2 can be proved directly. Details are omitted.

5 Concluding Remarks

This Note provided an answer to an interesting question pertaining to implications of risk pooling, raised by several authors: Do the benefits of risk-pooling increase in the variabilities of the original demands? While this behavior is not guaranteed for an arbitrary setting, it does hold for a structured model of increased variability. Our positive results do not assume any particular distributional forms or independence of demands.

Our analysis made heavy use of a mean-preserving transformation - a cardinal model for changing variability. In this case, a mean-preserving transformation provides information about the cardinal magnitude of change in benefits, not just the sign of change. The mean-preserving transformation defines an ordering of random variables that is stronger than the convex ordering. An interesting topic for future research is to explore the weakest stochastic variability ordering for which the benefits of risk-pooling are still non-decreasing.

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