# When will the Range of Prizes in Tournaments Increase in the Noise or in the Number of Players ?

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**Abstract**

The symmetric equilibrium resulting from the celebrated Tournament model of Lazear and Rosen has a range of compensation between winner and loser which is inversely proportional to $E[f(X)]$, the expectation of the additive noise's density. There seems to be a belief that this functional is always increasing in the noise's variability, which would agree with economic intuition - when output is noisier it should be less worthwhile to work hard. We show such is not the case for all distributions, and characterize classes where such is or is not the case. When the number of players $n$ grows, winning is more difficult so we would expect the required range of compensation to be larger. That would require that $E[f(Y)]$, where $Y = \max(X_1, \cdots, X_{n-1})$, will decrease in $n$. We examine the generality of this property. Finally we explore the same issues within a multiplicative model.

## 1. Introduction

One of the main goals of the basic Tournament Model (Lazear and Rosen 1981, Rosen 1986, Gibbons 1992, Section 2.2.D) is to explain the observed large range of compensation between a firm's top executive and those below him/her.

Since each worker's output is uncertain, the optimal range of compensation will depend on a functional of the noise distribution, as well as on the player's disutility of effort. Indeed, if the two players are similar, the symmetric Nash equilibrium for the prize range $w_H - w_L$ is given by (e.g. Gibbons)

$$(w_H - w_L)\int_{-\infty}^{\infty}[f(x)]^2dx = v'(e^*), \quad (1)$$

where $f(x)$ is the zero-mean noise's density, $v(e)$ is the disutility of effort function, assumed increasing and convex, and $e^*$ is the equilibrium effort. Thus, in particular, the prize range inversely depends on the integral of the square of the noise density. [1]

Gibbons (p. 81) argues that "... it is not worthwhile to work hard when output is noisy, because the outcome of the tournament is likely to be determined by luck rather than effort". He then shows that the function $\int_{-\infty}^{\infty}[f(x)]^2dx$ is indeed decreasing in the variance for a zero-mean normal distribution.

We wish to explore the generality of this claim. We show that $Ef(X) = \int_{-\infty}^{\infty}[f(x)]^2dx$, is *not always* decreasing in the variance of a distribution, and characterize classes of distributions for which the claim is or is not true.

We note that the quantity $\int[f(x)]^2dx \equiv Ef(x)$ bears resemblance to the *Tsallis entropy of order 2* (Tsallis 1988), which for a discrete probability mass function $(p_1, \cdots, p_n) \equiv \mathbf{p}$ is proportional to $\sum_{i=1}^{n} p_i^2 \equiv E(\mathbf{p})$. It is also somewhat related to the *Rényi entropy of order 2* (Rényi 1970, Cover and Thomas 1991, p. 499), which is proportional to $-\log_2 \mathrm{E}(\mathbf{p})$. It is known that these entropies achieve their maxima for the uniform distribution $p_i = 1/n, i = 1, \cdots, n$; that is, for the *most* variable distribution. But, as our random variable's range is $(-\infty, \infty)$, with zero mean, our functional's properties could be somewhat different.

How will/should the prize range depend on the number of players? Intuitively, since beating many rivals is exceedingly difficult, the incentive has to be sufficient to warrant the effort. That seems to suggest that $w_H - w_L$ should be increasing in the number of players (assuming that the winner receives $w_H$ and the rest $w_L$). We examine whether the Tournament model indeed displays that property. In turns out, again, that for some distributions the opposite behavior occurs. The desired property does, however, hold for symmetric unimodal distributions. We note that Nti (1997) has shown that in a rent-seeking game, increasing the number of players reduces the equilibrium effort per player, and may even result in decline in the total level of effort. But the rent-seeking model does not include explicit noise. For another instance of counter-intuitive behavior as the

[1] Lazear and Rosen, and some others, work with the difference between the two noise variables. Denoting the density of the difference by $h$, the condition of the symmetric Nash equilibrium can be written as $(w_H - w_L)h(0) = v'(e^*)$. While equivalent, that way of writing "masks" the dependence of $w_H - w_L$ on the noise's distribution.

number of players increase, see Hvide and Kristensen (1999).

The Tournament literature models noise as additive. But Lazear and Rosen and others comment that modeling noise as multiplicative will not change the qualitative conclusions. We thus wish to explore the veracity of that conjecture with respect to the two issues we investigate - the effect of increased noise and the number of players.

## 2. The impact of the variability of $X$ on $Ef(X)$

Assume that $X$ and $Y$ are two random variables. Let $f(x)$ ($g(x)$) be the density function of $X$ ($Y$) and $F(x)$ ($G(x)$) the distribution function. Throughout this paper, we assume that $f(x)$ is uniformly finite on the entire real line $(-\infty, \infty)$, i.e., $F(x)$ is continuous. The following example shows that $Ef(X)$ is not always decreasing in the variance.

**Example 1.** Let the random variable $X$ be uniform over the interval $[-1, 1]$. Random variable $Y$ has a uniform distribution on the union of intervals $(-2, -2+\delta)$ and $(2-\delta, 2)$, where $0 < \delta < 1$. Thus $E(X) = E(Y) = 0$. It is easy to verify that $Var(X) = 1/3 < (12 - 6\delta + \delta^2)/3 = Var(Y)$ (i.e., the variance of the random variable $Y$) for $0 < \delta < 1$. It is also easy to verify that $Y$ is larger than $X$ in convex order (Shaked and Shanthikumar 1994). Thus, it is reasonable to say that random variable $Y$ is more variable than $X$. However, $Ef(X)$ is less than $Eg(Y)$ as demonstrated in the following calculations.

$$
\begin{aligned}
Ef(X) &= \int_{-1}^{1} (\frac{1}{2})^2 dx = \frac{1}{2}. \\
Eg(Y) &= \int_{-2}^{-2+\delta} (\frac{1}{2\delta})^2 dx + \int_{2-\delta}^{2} (\frac{1}{2\delta})^2 dx = \frac{1}{2\delta}.
\end{aligned}
$$

When $0 < \delta < 1$, we have $Ef(X) < Eg(Y)$. Thus, the more variable random variable $Y$ actually has a larger value $Eg(Y)$. $\|$

A closer look at the structure of the function $g(x)$ leads to a more surprising conclusion: the variability of a random variable $Y$ has little to do with its corresponding $Eg(Y)$. To show this, we introduce the concept "support" of a density function,

The *support* of the density function $f(x)$ of the random variable $X$ is the set of points on which $f(x)$ is positive, i.e., $S_f = \{x : f(x) > 0\}$. Note that our definition of support is slightly different from the classical one (see Chow and Teicher 1988). We assume that $S_f$ can be composed from a countable number of *disjoint* intervals. We denote those disjoint intervals as $\{(a_k, b_k),\ b_k \leq a_{k+1} < b_{k+1}, -\infty < k < \infty\}$ for $f(x)$. Note that it is possible that $f(a_k) > 0$ or $f(b_k) > 0$ for some $k$. It is also possible that $\lim_{k\to-\infty} a_k = -\infty$ and $\lim_{k\to\infty} b_k = \infty$. So, $\cup_{k=-\infty}^{\infty}(a_k, b_k)$ includes all the points in $S_f$ with the possible exceptions of points $\{a_k, b_k, -\infty < k < \infty\}$. For later use, we introduce another random variable $Y$. The density function of $Y$, $g(x)$, has a support consisting of disjoint intervals $S_g = \{(a'_k, b'_k),\ b'_k \leq a'_{k+1} < b'_{k+1}, -\infty < k < \infty\}$. Now, we are ready to state and prove one of the main results of this paper.

**Proposition 1.** Consider the random variables $X$ and $Y$ defined above. Assume that there is a one-to-one relationship between the disjoint intervals of the supports of the density functions of $X$ and $Y$, i.e., for any $k$, there exists an $m$ such that $a'_m = a_k + c_k$ and $b'_m = b_k + c_k$ for some $c_k$ and vis versa. That is: the interval $(a'_m, b'_m)$ can be obtained by moving the interval $(a_k, b_k)$ to another location. If $f(x) = g(x + c_k)$ for all $x \in (a_k, b_k)$, $-\infty < k < \infty$, then $Ef(X) = Eg(Y)$.

**Proof.** By the assumption, the supports of $f(x)$ and $g(x)$ can be decomposed into the same number of disjoint intervals. By definition,

$$\begin{aligned} Ef(X) &= \int_{-\infty}^{\infty} (f(x))^2 dx = \sum_{k=-\infty}^{\infty} \int_{a_k}^{b_k} (f(x))^2 dx = \sum_{k=-\infty}^{\infty} \int_{a_k+c_k}^{b_k+c_k} (f(x-c_k))^2 dx \\ &= \sum_{m=-\infty}^{\infty} \int_{a'_m}^{b'_m} (g(x))^2 dx = Eg(Y). \end{aligned} \tag{2}$$

Also note that at the border points, $f(a_k) = g(a_k + c_k)$ or $f(b_k) = g(b_k + c_k)$ may or may not hold, but that has no impact on the equalities in equation (2). ||

Implications of Proposition 1 are interesting. First, Proposition 1 implies that the relationship between the variability (or the variance) of the random variable $X$ and its corresponding $Ef(X)$ is weak, especially when the support of its density function consists of more than one disjoint interval. Based on Proposition 1, one can construct numerous examples for which the variances of random variables are dramatically different but they all have the same value of $Ef(X)$. For instance, random variable $Y$ in Example 1 has the same value of $Ef(Y) = 1/(2\delta)$ as the uniform distribution in the interval $(-\delta, \delta)$. This example shows that the value of $Ef(X)$ can be arbitrarily large or small, which is independent of the mean and variance of the random variable $X$. Nonetheless, when the support of the density function of a random variable has a single interval, Proposition 1 provides little information about $Ef(X)$. Thus, the characteristics of $f(x)$ in each interval of its support should have larger impact on $Ef(X)$, which will be verified, at least partially, by Propositions 2 and 4.

According to Proposition 1, the study of $Ef(X)$ can focus on random variables with $b_k = a_{k+1}$ for all the intervals of the supports of their density functions. For this reason, we merge all these intervals together to get a single interval $(a_1, b_1)$ in which $f(x)$ is positive except at a countable number of points. We call the interval $(a_1, b_1)$ the *quasi-support* of $f(x)$ since $f(x)$ can be zero in this interval. It is possible that $(a_1, b_1) = (-\infty, \infty)$. In general, when $f(x)$ has a single quasi-support, the real line $(-\infty, \infty)$ is divided into three intervals $(-\infty, a_1]$, $(a_1, b_1)$, and $[b_1, \infty)$. The function $f(x)$ is zero in $(-\infty, a_1] \cup [b_1, \infty)$ if these intervals exist (with possible exceptions at $a_1$ and $b_1$). In the rest of this section, we assume that the density functions have a single quasi-support. According to Proposition 1, we understand that all the results hold when the quasi-support is divided in any way into a countable number of disjoint intervals that are relocated in any non-overlapping manner.

In what follows, we identify some classes of random variables for which the variability does lead to a smaller or larger value of $Ef(X)$.

The function $f(x)$ is called unimodal (reverse-unimodal) if there exists a constant $c$ such that $f(x)$ is nondecreasing (nonincreasing) on $(a_1, b_1) \cap (-\infty, c)$ and nonincreasing (nondecreasing) on $(a_1, b_1) \cap (c, -\infty)$. Apparently, $f(x)$ attains its maximum (minimum) at $c$ when it is restricted to its quasi-support. We call $c$ the quasi-middle point of function $f(x)$. Let $X_c^- = \min\{c, X\}$ and $X_c^+ = \max\{c, X\}$. Note that these definitions include cases where $f(x)$ is monotone on its entire quasi-support. For such cases, we set $c = a_1$ or $c = b_1$. We define $Y_c^-$ and $Y_c^+$ similarly.

**Proposition 2.** Consider random variables $X$ and $Y$ defined in Proposition 1. Assume that $X_c^- \leq_{st} Y_c^-$ and $X_c^+ \geq_{st} Y_c^+$. If $f(x)$ is unimodal, then $Ef(X) \leq Eg(Y)$. If $f(x)$ is reverse-unimodal, then $Ef(X) \geq Eg(Y)$.

**Proof.** First, we note that $Ef(Y) = \int_{-\infty}^{\infty} f(y)g(y)dy = \int_{-\infty}^{\infty} g(x)f(x)dx = Eg(X)$.

Suppose that $f(x)$ and $g(x)$ are unimodal functions. Since $X_c^- \leq_{st} Y_c^-$ and $(f+g)(x)$ is nondecreasing in $(-\infty, c)$, we have $E(f+g)(X_c^-) \leq E(f+g)(Y_c^-)$ (Ross 1983 and Shaked and Shanthikumar 1994). Since $X_c^+ \geq_{st} Y_c^+$ and $(f+g)(x)$ is nonincreasing in $(-\infty, c)$, we have $E(f+g)(X_c^-) \leq E(f+g)(Y_c^-)$. When $f(x)$ and $g(x)$ are unimodal on their quasi-supports, they are also unimodal on $(-\infty, \infty)$. Then we have following calculations:

$$
\begin{aligned}
&E[f(X)+g(X)] = \int_{-\infty}^{\infty}[f(x)+g(x)]f(x)dx \\
&= \int_{-\infty}^{c}[f(x)+g(x)]f(x)dx + \int_{c}^{\infty}[f(x)+g(x)]f(x)dx \\
&= E[f(X_c^-)+g(X_c^-)] - [f(c)+g(c)](1-F(c)) \\
&\quad + E[f(X_c^+)+g(X_c^+)] - [f(c)+g(c)]F(c) \qquad (3) \\
&= E[(f+g)(X_c^-)] + E[(f+g)(X_c^+)] - [f(c)+g(c)] \\
&\le E[(f+g)(Y_c^-)] + E[(f+g)(Y_c^+)] - [f(c)+g(c)] \\
&= E[f(Y)+g(Y)].
\end{aligned}
$$

The inequality in the above equation is due to the monotonicity of the function $f(x)+g(x)$ in the intervals $(-\infty, c)$ and $(c, \infty)$. Since $Ef(Y) = Eg(X)$, the above inequality leads to $Ef(X) \le Eg(Y)$.

When $f(x)$ is reverse-unimodal on its quasi-support, we introduce a function $\hat{f}(x)$ as: $\hat{f}(x) = f(a_1+)$ when $-\infty < x \le a_1$ and $\hat{f}(x) = f(b_1-)$ when $b_1 \le x < \infty$. The function $\hat{f}(x)$ is reverse-unimodal on $(-\infty, \infty)$. We introduce $\hat{g}(x)$ in a similar manner. Since $F(x) = 0$ for $x \in (-\infty, a_1)$ and $F(x) = 1$ for $x \in (b_1, \infty)$, $F'(x) = 0$ in the two intervals (if the two intervals exist). Thus, $E\hat{f}(X) = Ef(X)$ and $E\hat{g}(Y) = Eg(Y)$. The rest of the proof is similar to the unimodal case. This completes the proof. ||

Many interesting cases can be solved using Proposition 2. For instance, results related to symmetric distribution functions are given in the following Corollary.

**Corollary 3.** Assume that $f(x)$ and $g(x)$ are symmetric at a point $c$, i.e., $f(x) = f(2c-x)$ and $g(x) = g(2c-x)$. Assume that $X_c^+ \ge_{st} Y_c^+$. If $f(x)$ and $g(x)$ are nonincreasing in their quasi-support on $(c, \infty)$. We have $Ef(X) \le Eg(Y)$. If $f(x)$ and $g(x)$ are nondecreasing on their quasi-support on $(c, \infty)$. We have $Ef(X) \ge Eg(Y)$.

**Proof.** Since $f(x)$ $(g(x))$ is nonincreasing in its quasi-support on $(c, \infty)$, it is nondecreasing in its quasi-support on $(-\infty, c)$. Since $X_c^+ \ge_{st} Y_c^+$, we have $X_c^- \le_{st} Y_c^-$. All conclusions are obtained by Proposition 2. ||

By Corollary 3, it is easy to see that among all the uniform distributions on $(-a, a)$ for all real $a$ $(> 0)$, $Ef(X)$ is decreasing in the parameter $a$. This implies that for this class of distributions, larger variability means a smaller $Ef(X)$. By Proposition 1 and Corollary 3, it is clear why $Eg(Y) \leq Ef(X)$ in Example 1. Note that Example 1 shows that symmetry of $f(x)$ together with $X_0^+ \geq_{st} Y_0^+$ does not ensure that $Ef(X) \geq Eg(Y)$. Monotonicity of the density function $f(x)$ does play a positive role.

By Corollary 3, $Ef(X)$ is decreasing in the parameter $\sigma$ for all Normal distributions with (fixed) mean $\mu$ (as observed in the Tournaments literature).

We now briefly discuss a simple transform defined as: $X_{c,d} = cX + d$ for $c > 0$ and $d$. Denote by $f_{c,d}(x)$ the density function of random variable $X_{c,d}$. It can be verified that $f_{c,d}(x) = f((x-d)/c)/c$. Thus, $Ef_{c,d}(X_{c,d}) = Ef(X)/c$ for $c > 0$. Thus, $Ef_{c,d}(cX + d)$ is decreasing in $c$. Since $Var(X_{c,d})$ is increasing in $c$, it is clear that $Ef(X)$ is nonincreasing with respect to variability under the transform. By setting $d = (1 - c)E(X)$, we obtain a mean-preserving transformation (Sandmo 1972, Baron 1972). We conclude that under such transformation $Ef(X_{c,(1-c)E(X)})$ *does* decrease in the variability of $X_c$. So the conjecture voiced by Gibbons is correct within this family of distributions, but not in general.

## 3. Effect of Number of Players

If the output of player $i$, $i = 1, \cdots, n$, depends on his/her effort $e_i$ in the additive form $y_i = e_i + x_i$, then the probability that player $i$ will win is

$$P[y_i(e_i) > y_j(e_j^*),\ j \neq i] = P(e_i + x_i > e_j^* + x_j,\ j \neq i)$$

$$= P(x_j < e_i - e_j^* + x_i,\ j \neq i)$$

which for the symmetrical case equals

$$\int_{-\infty}^{\infty} [F(e_i - e^* + x)]^{n-1} f(x) dx$$

so

$$\frac{d}{de_i} P = \int_{-\infty}^{\infty} (n-1)[F(x)]^{n-2}[f(x)]^2 dx$$

at $e_i = e^*$.

Thus the symmetric equilibrium is

$$(w_H - w_L) \int_{-\infty}^{\infty} (n-1)[F(x)]^{n-2}[f(x)]^2 dx = v'(e^*).$$

Let

$$A(n) \equiv \int_{-\infty}^{\infty} (n-1)[F(x)]^{n-2}[f(x)]^2 dx. \tag{4}$$

We wish to explore when $A(n)$ is decreasing in $n$.

Let $X$ be a random variable with density function $f(x)$ and distribution function $F(x)$. Let $\{X_n,\ n \geq 0\}$ be independent random variables with a common distribution $F(x)$. By definition, it is easy to see that

$$A(n) = \int_{-\infty}^{\infty} f(x) d(F(x))^{n-1} = Ef(\max\{X_1, X_2, ..., X_{n-1}\}), \text{ for } \ n \geq 2. \tag{5}$$

For later use, we denote $(F(x))^n$ by $F_n(x)$, which is a proper distribution function. Obviously, $F_n(x) \geq F_{n+1}(x)$, which corresponds to $\max\{X_1, ..., X_n\} \leq_{st} \max\{X_1, ..., X_n, X_{n+1}\}$. We shall write $X^{(n)} = \max\{X_1, ..., X_n\}$ and $X^{(n)} \leq_{st} X^{(n+1)}$. We also write $A(n) = \int_{-\infty}^{\infty} f(x) d(F(x))^{n-1} = Ef(X^{(n-1)})$. The limit of $A(n)$ is obtained explicitly from equation (5) as follows:

$$\lim_{n\to\infty} A(n) = \limsup_{x\to\sup(S_f)} f(x),$$

where $\sup(S_f) = \lim_{k\to\infty} b_k$. The above equality holds since $f(x)$ is uniformly finite on the entire real line and $\int_{-\infty}^{t} f(x)d(F(x))^n \to 0$ when $F(t) < 1$ for any $t$. Thus, if the support of $f(x)$ does not stretch to $+\infty$, i.e., $\sup(S_f) < \infty$, the limit of $A(n)$ is given by the above formula. Otherwise, the limit of $A(n)$ is zero.

Assume that the support $S_f$ of $f(x)$ consists of disjoint intervals $\{(a_k, b_k),\ b_k \leq a_{k+1} < b_{k+1}, -\infty < k < \infty\}$. It is clear that $F(x)$ changes its value only in these intervals. Consequently, $F_n(x) = (F(x))^n$ changes its value only in these intervals as well. Intuitively, this means that the derivative $((F(x))^n)' = 0$ when $x$ is outside of the support $S_f$. We are now ready to give a series of characterizations of $A(n)$. We begin with an extension of Proposition 1.

**Proposition 4.** Consider the random variables $X$ and $Y$ defined in Proposition 1. The function $A(n)$ is the same for $X$ and $Y$.

**Proof.** The proof is similar to that of Proposition 1. $\|$

Proposition 4 implies that $A(n)$ is invariant under the transform $\{X_n+c, n \geq 1\}$ for any constant $c$. Proposition 4 implies that the study of $A(n)$ can focus on density functions with a single interval of support (quasi-support). Proposition 4 shows that the variability of $X$ has a weak connection to the monotonicity of $A(n)$.

It it easy to prove that for a uniform distribution on $(a, b)$, $A(n) = 1/(b-a)$, which is independent of $n$. This result can be easily generalized as follows.

**Proposition 5.** Suppose that $X$ has a uniform distribution over its support. Then $A(n) = 1/[\sum_{-\infty}^{\infty}(b_k - a_k)]$, which is independent of $n$.

**Proof.** When $X$ has a uniform distribution over its support, the total length of $S_f$ must be positive and finite, i.e., $0 < \sum_{k=-\infty}^{\infty}(b_k - a_k) < \infty$ and $f(x) = 1/[\sum_{k=-\infty}^{\infty}(b_k - a_k)]$. The conclusion is obtained by equation (5). ||

Basically, Proposition 5 considered a class of density functions that are constant over its support. Next, we consider density functions that are monotone over its entire support. Apparently, $A(n)$ is no longer constant but the following Proposition shows that $A(n)$ is monotone with respect to $n$ when $f(x)$ is monotone in $x$ over its support.

**Proposition 6.** If $f(x)$ is nondecreasing on its support $S_f$, then $A(n)$ is nondecreasing. If $f(x)$ is nonincreasing on its support $S_f$ and is uniformly bounded, then $A(n)$ is nonincreasing.

**Proof**. First, we introduce the following $\hat{f}(x)$:

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in S_f; \\ f(b_k), & \text{if } b_k < x < a_{k+1}. \end{cases}$$

The value of $\hat{f}(x)$ at border points $\{a_k, b_k, -\infty < k < \infty\}$ can be arbitrary as long as they are uniformly finite. When $f(x)$ is nondecreasing on its support, $\hat{f}(x)$ is a nondecreasing function. Since $X^{(n-1)} \leq_{st} X^{(n)}$ and $\hat{f}(x)$ is nondecreasing, we have $E\hat{f}(X^{(n-1)}) \leq E\hat{f}(X^{(n)})$. Since the derivative $F_n'(x) = 0$ when $x$ is outside of the support $S_f$, we have

$$E\hat{f}(X^{(n)}) = \sum_{k=-\infty}^{\infty}\left(\int_{a_k}^{b_k} + \int_{b_k}^{a_{k+1}}\right)\hat{f}(x)dF_n(x) = \sum_{k=-\infty}^{\infty}\int_{a_k}^{b_k} f(x)dF_n(x) = Ef(X^{(n)}). \tag{6}$$

Thus, $Ef(X^{(n-1)}) \leq Ef(X^{(n)})$, which is equivalent to $A(n) \leq A(n+1)$.

Similarly, it can be proved that $A(n)$ is nonincreasing when $f(x)$ is nonincreasing on its entire support. ||

Based on Proposition 6, numerous examples can be easily constructed for increasing $A(n)$ or decreasing $A(n)$. The exponential and hyperexponential distributions are examples of the latter.

The conclusions in Proposition 6 hold for larger class of density functions. We use the following example to demonstrate this point.

**Example 2.** Consider random variable $X$ with the following density function:

$$f(x) = \begin{cases} 0, & x < 0; \\ 1 - x, & 0 \le x \le 1; \\ 0, & 1 < x < 2; \\ 3 - x, & 2 \le x \le 3. \end{cases}$$

By direct calculation, it can be found that

$$\begin{aligned} A(n) &= (0.5)^{n-2}(n-1)\left[\int_0^1 x^2(1-x^2)^{n-2}dx + \int_0^1 x^2(2-x^2)^{n-2}dx\right] \\ &\approx (n-1)\int_0^1 x^2(1-\frac{x^2}{2})^{n-2}dx \to 0, \end{aligned}$$

(decreasingly) when $n$ goes to infinity by the mean value theorem in calculus.

Consider another random variable $Y$ with with the following density function:

$$g(x) = \begin{cases} 0, & x < 0; \\ x, & 0 \le x \le 1; \\ 0, & 1 < x < 2; \\ x - 2, & 2 \le x \le 3. \end{cases}$$

It is clear that $X$ and $Y$ have the same means and variances. By direct calculation, it can be found that

$$\begin{aligned} A(n) &= \frac{n-1}{2^{n-2}}\left[\frac{1}{2n-1} + \sum_{i=0}^{n-2}\begin{pmatrix} n-2 \\ i \end{pmatrix}\frac{1}{2i+3}\right] \\ &\approx (n-1)\sum_{i=0}^{n-2}\begin{pmatrix} n-2 \\ i \end{pmatrix}(\frac{1}{2})^{n-2}\frac{1}{2i+3} \\ &= E\left(\frac{n-1}{2U_{n-2}+3}\right) = E\left(\frac{1}{2\frac{U_{n-2}}{n-1} + \frac{3}{n-1}}\right), \end{aligned}$$

where $U_{n-2}$ is the Binomial distribution with parameters $n-2$ and 0.5. It is well-known that $U_{n-2}/(n-2)$ converges to 0.5 with probability one when $n$ goes to infinity (Chow and Teicher, 1988). Then $A(n)$ converges to 1 (increasingly) when $n$ goes to infinity. $\|$

Example 2 suggests that conclusions in Proposition 6 may be true for cases where $f(x)$ is monotone when restricted to every individual interval in its support. Example 2 shows that the shape of $f(x)$ has a major impact on the monotonicity of the sequence $A(n)$, $n \geq 2$.

Now, as in Section 2, we explore the impact of symmetry of the random variable $X$ on $A(n)$. First, lets have a look at one more example.

**Example 3.** Consider random variables $X$ and $Y$ with the following density functions.

$$f(x) = \begin{cases} 0, & x < -2; \\ -(x+1), & -2 \leq x \leq -1; \\ 0, & -1 < x < 1; \\ x-1, & 1 \leq x \leq 2. \end{cases} \qquad g(x) = \begin{cases} 0, & x < -2; \\ x+2, & -2 \leq x \leq -1; \\ 0, & -1 < x < 1; \\ 3-x, & 1 \leq x \leq 2. \end{cases}$$

The two functions are not monotone on their corresponding supports, but they are monotone in each interval of their supports. In fact, they are increasing in one interval and decreasing in another. Therefore, Proposition 6 does not apply. But $f(x)$ and $g(x)$ possess a symmetry property that leads to the monotonicity of $A(n)$, which is shown next.

First, we introduce the function $h_n(t) = t^{n-1} - (1-t)^{n-1}$, for $0.5 \leq t \leq 1$ and $n \geq 2$.

**Lemma 7**. The function $h_n(x)$ has $h_n(0.5) = 0$, $h_n(1) = 1$, and $0 \leq h_n(t) \leq 1$ for $0.5 \leq t \leq 1$. $h_n(t)$ is increasing in $[0.5, 1]$ and $h_n(t) \geq h_{n+1}(t)$.

**Proof.** It is easy to see that $h_n(t)$ is increasing in $[0.5, 1]$. We also have the following:

$$\begin{aligned} h_n(t) - h_{n+1}(t) &= t^{n-1} - (1-t)^{n-1} - t^n + (1-t)^n \\ &= t^{n-1}(1-t) - (1-t)^{n-1}t \\ &= t(1-t)[t^{n-2} - (1-t)^{n-2}] \geq 0, \end{aligned}$$

when $0.5 \leq t \leq 1$. ||

**Proposition 8**. Assume that the random variable $X$ is symmetric at point $c$. Then $A(2) = A(3)$. If $f(x)$ is nonincreasing on its support in $[c, \infty)$, then $A(n)$ is nonincreasing. If $f(x)$ is nondecreasing on its support in $[c, \infty)$, then $A(n)$ is nondecreasing.

**Proof.** For $n \geq 2$, we have

$$\begin{aligned} A(n) &= \int_{-\infty}^{\infty} f(x)d(F(x))^{n-1} = \int_{-\infty}^{c} f(x)d(F(x))^{n-1} + \int_{c}^{\infty} f(x)d(F(x))^{n-1} \\ &= \int_{\infty}^{c} f(2c-x)d(F(2c-x))^{n-1} + \int_{c}^{\infty} f(x)d(F(x))^{n-1} \\ &= -\int_{c}^{\infty} f(x)d(1-F(x))^{n-1} + \int_{c}^{\infty} f(x)d(F(x))^{n-1} \\ &= \int_{c}^{\infty} f(x)d[(F(x))^{n-1} - (1-F(x))^{n-1}] = \int_{c}^{\infty} f(x)dh_n(F(x)). \end{aligned}$$

The fourth equality is due to symmetry, i.e., $F(2c-x) = 1 - F(x)$. According to Lemma 7, the function $h_n(F(x))$ is a proper probability distribution function (define $h_n(F(x)) = 0$ for $x < c$). Since $h_n(F(x)) \geq h_{n+1}(F(x))$, we say that $h_n(F(x)) \leq_{st} h_{n+1}(F(x))$ (with minor abuse of notation).

When $n = 2$, $A(2) = \int_c^{\infty} f(x)dh_2(F(x)) = 2\int_c^{\infty} f(x)^2 dx$. When $n = 3$, $A(3) = \int_c^{\infty} f(x)dh_3(F(x))$ $= 2\int_c^{\infty} f(x)^2 dx$. Therefore, $A(2) = A(3)$.

When $f(x)$ is nonincreasing in its support in $[c, \infty)$, we can introduce a nonincreasing function $\hat{f}(x)$ in $[c, \infty)$ as we have done in the proof of Proposition 6. Then it is easy to obtain that $A(n)$ is nonincreasing.

Similarly, we can show that when $f(x)$ is nondecreasing in $[c, \infty)$, $A(n)$ is nondecreasing. $\|$

The monotonicity of $A(n)$ associated with Example 3 is obtained immediately from Proposition 8. Furthermore, Proposition 8 implies that normal distributions $A(n)$ is decreasing in $n$.

## 4. Multiplicative Model

For the multiplicative model $y(e) = eZ$, where $Z$ is a non-negative random variable with mean 1, to be equivalent to the additive model $y(e) = e + X$, one needs to have

$$P(eZ \leq z) = P(e + X \leq z) \qquad \forall z.$$

Denoting the distribution of $Z$ by $H$ and that of $X$ by $F$, that translates to

$$H(x) = F(e(x-1)) \qquad \forall x.$$

or

$$h(x) = ef(e(x-1)) \qquad \forall x.$$

It is thus clear that the distribution $H(x)$ which achieves equivalence depends on the effort level $e$. Thus by selecting a *single* (fixed) distribution for the multiplier $Z$, which is what one would normally do in adopting a multiplicative model, the resulting model will *not* have a single (fixed) additive counterpart for every effort level. Also, although the multiplicative model can be written in the additive form $\log y - \log e + \log z$, two $z$ multipliers with equal means (one) do not generally have equal $E(\log z)$. Thus it is not apriori obvious that the properties of the prize range in both models will be similar.

Now, for the multiplicative model

$$P[y_i(e_i) > y_j(e_j^*)] = P\left(Z_i > \frac{e_j^* Z_j}{e_i}\right) = \int_0^\infty \bar{H}_i\left(\frac{e_j^* z_j}{e_i}\right) h_j(z_j) dz_j.$$

Thus

$$\frac{d}{de_i} P[y_i(e_i) > y_j(e_j^*)] = \frac{e_j^*}{e_i^2} \int_0^\infty z_j h_i\left(\frac{e_j^* z_j}{e_i}\right) h_j(z_j) dz_j \ ,$$

So the Nash equilibrium is

$$(w_H - w_L)\frac{e_j^*}{e_i^2} \int_0^\infty z_j h_i\left(\frac{e_j^* z_j}{e_i}\right) h_j(z_j) dz_j = v'(e_i),$$

and in the symmetric case it becomes

$$(w_H - w_L) \int_0^\infty z[h(z)]^2 dz = ev'(e). \tag{7}$$

Thus the functional of interest is $E[Zh(Z)] = \int_0^\infty z[h(z)]^2 dz$, where $E(Z) = 1$. Is the effect of larger variability of $Z$ on that functional similar to what we obtain in the additive model?

Now, if $Z$ has support $[0, \infty]$ and $E(Z) = 1$, then for the mean-preserving transformation $Z_c = cZ + (1-c)$,

$$h_c(z) = \frac{1}{c} h\left(\frac{z-(1-c)}{c}\right),$$

So

$$\begin{aligned} \textstyle\int_0^\infty z[h_c(z)]^2 dz &= \int_0^\infty z\{\frac{1}{c} h\left(\frac{z-(1-c)}{c}\right)\}^2 dz \\ &= \int_{-(1-c)/c}^\infty y\,[h(y)]^2\,dy + \frac{1-c}{c}\int_{-(1-c)/c}^\infty [h(y)]^2 dy. \end{aligned}$$

Since $(1-c)/c$ is decreasing in $c$, each of the above terms is decreasing in $c$. Thus $\int_0^\infty z[h_c(z)]^2 dz$ is decreasing in $c$ for $E(Z_c) = 1$.

Is the functional $E[Zh(Z)]$ always decreasing with respect to the variability of $Z$? The answer is no. To prove this conclusion, we introduce a transform to translate the functional $E[Zh(Z)]$ into a functional of the form $E[g(Y)]$. Then some of the results obtained in Section 2 can be applied. Let

$$g(x) = \frac{1}{2E[Z]} h(\sqrt{x}), \quad \text{for} \quad x \geq 0. \tag{8}$$

**Lemma 9** The function $g(x)$ is a density function in $(0, \infty)$. Define a random variable $Y$ having the density function $g(x)$. When $E[Z] = 1$, we have $E[Zh(Z)] = 2E[g(Y)]$ and $E[\frac{1}{\sqrt{Y}}] = 1$. On the other hand, if $E[Zh(Z)] = 2E[g(Y)]$ and $E[\frac{1}{\sqrt{Y}}] = 1$, $h(x) = 2g(x^2)$ is a proper density function with mean 1.

**Proof.** Since

$$\int_0^\infty g(y)dy = \int_0^\infty \frac{1}{2E[Z]} h(\sqrt{y})dy = \frac{1}{2E[Z]} \int_0^\infty h(x)dx^2 = \frac{1}{E[Z]} \int_0^\infty xh(x)dx = 1,$$

$g(x)$ is a density function. When $E[Z] = 1$, we have

$$E[g(Y)] = \int_0^\infty (g(y))^2 dy = \int_0^\infty \frac{1}{4}(h(\sqrt{y}))^2 dy = \frac{1}{4} \int_0^\infty (h(x))^2 dx^2 = \frac{1}{2} \int_0^\infty x(h(x))^2 dx.$$

Finally, by $E[Z] = 1$, we have

$$E[\frac{1}{\sqrt{Y}}] = \int_0^\infty \frac{1}{\sqrt{y}} g(y)dy = \int_0^\infty \frac{1}{2x} h(x)dx^2 = \int_0^\infty h(x)dx = 1.$$

Similarly, we can prove that $h(x) = 2g(x^2)$ is a proper density function under the given conditions. If the random variable $Z$ has the density function $h(x)$, then $E(Z) = 1$. This completes the proof. ||

An immediate consequence of Lemma 9 is that some of the results obtained in Section 2 can be applied to the multiplicative case. For instance, Propositions 1, 2, and 3 hold provided that $E[\frac{1}{\sqrt{Y}}] = 1$ holds for all the random variables involved. Nonetheless, the constraint $E[\frac{1}{\sqrt{Y}}] = 1$ restricts the use of some of the results obtained.

Another immediate result from Lemma 9 is that $E[Zh(Z)]$ may not be decreasing with respect to the variability of $Z$. According to Lemma 9, counterexamples can be constructed similar to those given in Sections 2 and 3 (plus that constraint $E[\frac{1}{\sqrt{Y}}] = 1$ must be satisfied). We present the following example.

**Example 4.** Consider a random variable $Y$ with density function $g(y) = 1$ for $y \in (a, a + 0.5) \cup (b, b + 0.5)$ and $g(y) = 0$ for $y \notin (a, a + 0.5) \cup (b, b + 0.5)$ , where $0 \leq a \leq 0.5$ and $1 \leq b$. Apparently, $g(y)$ is a density function. The condition $E[\frac{1}{\sqrt{Y}}] = 1$ is equivalent to

$$1 = \frac{1}{\sqrt{a} + \sqrt{a + 0.5}} + \frac{1}{\sqrt{b} + \sqrt{b + 0.5}}.$$

For $0.0625 < a \leq 0.5$, we can find $b$ $(\geq 1)$ so that the above equation is satisfied, i.e., $E[\frac{1}{\sqrt{Y}}] = 1$.

It is easy to verify that $E[g(Y)] = 1$, which is independent of the values of $a$ and $b$. Let $Z$ be a random variable with a density function $h(x) = 2g(x^2)$. The support of the function $h(x)$ is $(\sqrt{a}, \sqrt{a+0.5}) \cup (\sqrt{b}, \sqrt{b+0.5})$. By Lemma 9, $E[Z] = 1$ and $E[Zh(Z)] = 2$. It can be shown that the variance of $Z$ is approximately $0.25\sqrt{b}$ when $b$ is large. Thus, the variance of $Z$ goes to infinity when $b$ goes to infinity. This implies that $E[g(Y)]$ with $E[\frac{1}{\sqrt{Y}}] = 1$ has a weak relationship with the variability of $Y$.

Several special cases can be dealt with easily. For instance, let $Z$ be a random variable with a uniform distribution on $(1-a, 1+a)$ for $0 < a < 1$. Then it is easy to verify that $E[Zh(Z)] = 1/(2a)$, which is decreasing with respect to the variance of $Z$. For a Gamma distribution, $h(x) = \alpha^\alpha x^{\alpha-1}e^{-\alpha x}/\Gamma(\alpha)$ for $\alpha > 0$, it is easy to find that when $E[Z] = 1$, $E[Zh(Z)] = \Gamma(2\alpha)/(2^\alpha\Gamma(\alpha))^2$, which is decreasing with respect to $\alpha = E[(Z - E[Z])^2]$, i.e., decreasing with respect to the variance of $Z$.

## Multiplicative Model with $n$ players

For $n$ *players*, the multiplicative model becomes

$$P[y_i(e_i) > y_j(e_j^*)\ \forall j \neq i] = P\left(Z_j < \frac{e_i}{e_j^*} Z_i\ \forall j \neq i\right)$$

which for the symmetric model equals

$$\int_0^\infty [H\left(\frac{e_i}{e^*} z_i\right)]^{n-1} h(z_i) dz_i,$$

and the derivative w.r.t. $e_i$ is

$$\frac{1}{e^*}\int_0^\infty (n-1)z[H(z)]^{n-2}[h(z)]^2 dz$$

Thus the symmetric equilibrium is

$$(w_H - w_L)\int_0^\infty (n-1)z[H(z)]^{n-2}[h(z)]^2 dz = ev'(e).$$

Thus our interest is in the functional

$$B(n) = \int_0^\infty (n-1)z[H(z)]^{n-2}[h(z)]^2 dz \tag{9}$$

for a random variable $Z$ with $E(Z) = 1$. We wish to find when $B(n)$ is indeed decreasing in $n$. First, we rewrite $B(n)$ in the following way.

$$B(n) = \int_0^\infty zh(z)d[H(z)]^{n-1}. \tag{10}$$

Now, similarly to the limit of $A(n)$,

$$\lim_{n\to\infty} B(n) = \limsup_{z\to\sup(S_f)} zh(z),$$

Let $w(z) = zh(z)$ and $Z^{(n)} = \max\{Z_1, ..., Z_n\}$. Then $B(n) = E(w(Z^{(n)}))$. It is then clear that, based on the results obtained in Section 3, the monotonicity of $B(n)$ has a close relationship with the monotonicity of the function $w(x)$. Proposition 6 in Section 3 holds after replacing $f(x)$ with $w(x)$, given that $E(Z) = 1$. We use the following examples to illustrate that point.

**Example 5.** Let $Z$ be a random variable with a uniform distribution on $(1-a, 1+a)$ for $0 < a < 1$. Then $xh(x) = x/(2a)$, which is increasing on the support of $g(x)$: $(1-a, 1+a)$. Then $B(n)$ is increasing in $n$. In fact, it is easy to derive $B(n) = (1-a)/(2a) + (n-1)/n$.

Let $Z$ be a random variable with a density function $h(x) = x/2$ for $x \in (0, 2)$ and otherwise $h(x) = 0$. Then $E(Z) = 1$. Since $xh(x) = x^2/2$ is increasing on its support $(0, 2)$, by Proposition 6, $B(n)$ is increasing in $n$. By direct calculation, we found that $B(n) = 2(n-1)/n$.

Let $Z$ be a random variable with a density function $h(x) = (2-x)/[x(1.5-b)]$ on the interval $(b, b+1)$ (the support of $h(x)$), where $b$ is between 0.5 and 0.75 that satisfies the equation $2.5 = b + 2\ln(1 + 1/b)$. It can be proved that $h(x)$ is a proper density function on $(b, b+1)$ and $E(Z) = 1$. Clearly, $xh(x) = (2-x)/(1.5-b)$ is decreasing on the support $(b, b+1)$ of $h(x)$. By Proposition 6, $B(n)$ is decreasing in $n$.

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