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THE CLASSIFICATION OF MATRIX GI/M/1-TYPE MARKOV CHAINS WITH A TREE STRUCTURE AND ITS APPLICATIONS TO QUEUEING

QI-MING HE,* *Dalhousie University*

Abstract

In this paper, we study the classification of matrix GI/M/1-type Markov chains with a tree structure. We show that the Perron–Frobenius eigenvalue of a Jacobian matrix provides information for classifying these Markov chains. A fixed-point approach is utilized. A queueing application is presented to show the usefulness of the classification method developed in this paper.

Keywords: Markov chain; tree structure; matrix-analytic methods; fixed-point theory; GI/M/1-type Markov chain; queueing theory

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1. Introduction

The matrix GI/M/1-type Markov chain with a tree structure was introduced by Yeung and Sengupta [15]. Since then, such Markov chains have been used in the study of a number of queueing models (see [7] and [15]) as well as some telecommunications systems (see [14]). However, with the exception of [4] and [15], little work has been done on the classification of such Markov chains.

As was pointed out in [15], the classification problem of the GI/M/1-type Markov chains with a tree structure is a challenging problem. In [15], it was shown that the Perron–Frobenius eigenvalue of a matrix associated with the minimal fixed point of a nonlinear mapping provides information about the classification problem. In this paper, by using information related to a special fixed point of the nonlinear mapping, we identify a condition for the location of the minimal fixed point. The Markov chain is then classified according to where the minimal fixed point is located. We also show how to compute the special fixed point and, for a special case, where the special fixed point is located. Although the classification conditions obtained in this paper are not explicit, the results provide useful insight into the classification problem of such Markov chains and the method used in this paper is novel. Examples given in this paper show that our approach provides more information on the classification problem than that used in [15].

The methods used and the results obtained in this paper are closely related to the study of the classification of matrix M/G/1-type Markov chains with a tree structure. The matrix M/G/1-type Markov chain with a tree structure was introduced in [13]. The classification conditions of this class of Markov chains were found in [1], [4] and [5]. In a recent paper, [6], the scalar M/G/1-type, the scalar GI/M/1-type and the matrix M/G/1-type Markov chains with a tree structure were studied. Using fixed-point theory, degree theory, the mean-drift method and the invariant-measure method, it was shown that the Perron–Frobenius eigenvalue of a Jacobian matrix

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* Postal address: Department of Industrial Engineering, Dalhousie University, Halifax, Nova Scotia B3J 2X4, Canada. Email address: qi-ming.he@dal.ca

provides information for a complete classification of these Markov chains. The main results were obtained by exploring the relationship between the locations of fixed points of a nonlinear mapping and the classification problem. In this paper, we generalize the results obtained for the *scalar* GI/M/1 case to the *matrix* GI/M/1 case. Unlike the generalization from the scalar M/G/1 case to the matrix M/G/1 case (see [6]), the generalization from the scalar GI/M/1 case to the matrix GI/M/1 case is much more involved mathematically. The classification condition obtained in this paper gives an alternative to the classification condition given in [15] and an alternative to Neuts's condition for the classification of the classical GI/M/1-type Markov chains. We shall demonstrate the usefulness of the results by showing an application in queueing theory.

The rest of the paper is organized as follows. In Section 2, the Markov chain of interest is introduced and a mapping $\mathcal{A}$ is defined. Section 3 studies the relationships among the nonnegative fixed points of $\mathcal{A}$, especially the relationship between the minimal fixed point and other fixed points. In Section 4, a condition for positive recurrence of the Markov chain of interest is presented, along with some interesting numerical examples. In Section 5, the existence of a special fixed point (used in Section 4) in some convex set is proved for a special case. Finally, in Section 6, a queueing example is analyzed.

2. Markov chains of matrix GI/M/1-type with a tree structure

The following matrix GI/M/1-type Markov chain was introduced in [15].

Let $\aleph = \{J : J = j_1 j_2 \dots j_n,\ 1 \le j_i \le K,\ 1 \le i \le n,\ n \ge 1\} \cup \{0\}$, where K is a positive integer, i.e. each nonzero element J in $\aleph$ is a *string of integers* between 1 and K. The length of a string J in $\aleph$ is defined as the number of integers in the string and is denoted by $|J|$; if $J = 0$, then $|J| = 0$. The *addition operation* for strings in $\aleph$ is defined as $J + H = j_1 \dots j_n h_1 \dots h_i \in \aleph$ for $J = j_1 \dots j_n \in \aleph$ and $H = h_1 \dots h_i \in \aleph$. If we view each string $J \in \aleph$ as a node, then we obtain a *K-ary tree*. In the K-ary tree, each nonzero node J has a parent node and K children $\{J + 1, J + 2, \dots, J + K\}$; the root node $J = 0$ has no parent node. The node $J + k$ is called a type-k node.

We consider a Markov chain $\{(C_n, \eta_n),\ n \ge 0\}$, where C_n takes values in $\aleph$ and η_n takes integer values from 1 to m, where m is a positive integer. The random variable η_n is an auxiliary variable (also called the phase variable). The transition probabilities of the Markov chain are given as, for J and H in $\aleph$, $1 \le k \le K$,

$$\mathrm{P}\{C_{n+1} = J + k, \eta_{n+1} = j \mid C_n = J + H, \eta_n = i\} = A_{i,j}(H, k), \qquad 1 \le i, j \le m,$$
$$\mathrm{P}\{C_{n+1} = 0, \eta_{n+1} = j \mid C_n = H, \eta_n = i\} = B_{i,j}(H), \qquad 1 \le i, j \le m.$$

Let $\boldsymbol{A}(H, k)$ be the $m \times m$ matrix with (i, j)th element $A_{i,j}(H, k)$. Let $\boldsymbol{B}(H)$ be the $m \times m$ matrix with (i, j)th element $B_{i,j}(H)$. If $J = j_1 \dots j_n$, define $f(J, i) = j_{n-i+1} \dots j_n$, for $i = 1, \dots, n$, and $f(J, 0) = 0$. By the law of total probability, we have

$$\left(\boldsymbol{B}(J) + \sum_{i=0}^{|J|} \sum_{k=1}^{K} \boldsymbol{A}(f(J, i), k)\right)\boldsymbol{e} = \boldsymbol{e} \quad \text{for any } J \in \aleph, \tag{2.1}$$

where $\boldsymbol{e}$ is the column vector with all elements being 1.

Note 2.1. The K-ary tree defined above has a single root node 0. For some applications, the node 0 can have a parent node who has only one child (i.e. node 0). The results obtained in this paper hold for those Markov chains.

From the definition, it is clear that, in one transition, the Markov chain can move from the current node to one of its children, any node that is an immediate child of an ancestor of the current node, or the root node. The transition probabilities depend on the type of the targeted node. We assume that at least one of the matrices $\{\boldsymbol{A}(0, k),\ 1 \le k \le K\}$ is nonzero. Then $\{(C_n, \eta_n),\ n \ge 0\}$ is called a *Markov chain of matrix GI/M/1-type with a tree structure*. The Markov chain $\{(C_n, \eta_n),\ n \ge 0\}$ has the classical GI/M/1-type Markov chains ($K = 1$) and all GI/M/1-type Markov chains with a tree structure with $m = 1$ as special cases.

Next, we introduce *a mapping* $\mathcal{A}$, which plays a central role in this paper. Define

$$\mathcal{N}_m^K = \{\boldsymbol{X} : \boldsymbol{X} = (\boldsymbol{X}_1, \dots, \boldsymbol{X}_K),\ \text{where } \boldsymbol{X}_1, \dots, \boldsymbol{X}_K \text{ are } m \times m \text{ nonnegative matrices}\},$$

and

$$\mathcal{N}_m^K(\boldsymbol{Y}) = \{\boldsymbol{X} : \boldsymbol{X} \in \mathcal{N}_m^K \text{ and } \boldsymbol{X}_k \le \boldsymbol{Y}_k \text{ for } k = 1, \dots, K\} \quad \text{for } \boldsymbol{Y} \in \mathcal{N}_m^K.$$

It is clear that $\mathcal{N}_m^K$ and $\mathcal{N}_m^K(\boldsymbol{Y})$ are convex sets. The product of matrices in $\boldsymbol{X} \in \mathcal{N}_m^K$ is defined as follows; for $J = j_1 j_2 \dots j_{|J|-1} j_{|J|} \in \aleph$,

$$\boldsymbol{X}^{(J)} = \boldsymbol{X}_{j_1} \boldsymbol{X}_{j_2} \cdots \boldsymbol{X}_{j_{|J|-1}} \boldsymbol{X}_{j_{|J|}} \quad \text{for } \boldsymbol{X} \in \mathcal{N}_m^K.$$

The mapping $\mathcal{A}$: $\mathcal{N}_m^K \to \mathcal{N}_m^K$ is defined as follows: for $\boldsymbol{X} \in \mathcal{N}_m^K$,

$$\mathcal{A}(\boldsymbol{X}) = (\boldsymbol{A}_1^*(\boldsymbol{X}), \dots, \boldsymbol{A}_K^*(\boldsymbol{X})),$$

where

$$\boldsymbol{A}_k^*(\boldsymbol{X}) = \sum_{J \in \aleph} \boldsymbol{X}^{(J)} \boldsymbol{A}(J, k) \quad \text{for } k = 1, \dots, K.$$

For any $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_K)$ and $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots, \boldsymbol{Y}_K)$ in $\mathcal{N}_m^K$, if $\boldsymbol{X}_k \le \boldsymbol{Y}_k$ for $k = 1, \dots, K$, then we say that $\boldsymbol{X} \le \boldsymbol{Y}$. It is easy to see that the mappings $\boldsymbol{A}_k^*(\boldsymbol{X})$, for $k = 1, \dots, K$, are continuous and nondecreasing, i.e. for any $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathcal{N}_m^K$ with $\boldsymbol{X} \le \boldsymbol{Y}$, $\mathcal{A}(\boldsymbol{X}) \le \mathcal{A}(\boldsymbol{Y})$. An $\boldsymbol{X}$ in $\mathcal{N}_m^K$ is called a *fixed point* of $\mathcal{A}$ if $\boldsymbol{X} = \mathcal{A}(\boldsymbol{X})$. In the rest of the paper, 'a fixed point' means 'a nonnegative fixed point of the mapping $\mathcal{A}$'. A fixed point $\boldsymbol{X}_{\min}$ is called the *minimal fixed point* in $\mathcal{N}_m^K$ if $\boldsymbol{X}_{\min} \le \boldsymbol{Y}$ for any other fixed point $\boldsymbol{Y}$ in $\mathcal{N}_m^K$.

3. Fixed points of the mapping $\mathcal{A}$

First, an elementary relationship between the minimal fixed point and other fixed points is given in the following lemma.

Lemma 3.1. *If the mapping $\mathcal{A}$ has a fixed point, then there exists a minimal fixed point $\boldsymbol{X}_{\min}$ in $\mathcal{N}_m^K$. In addition, if $\boldsymbol{Y}$ is a fixed point, then the subset $\mathcal{N}_m^K(\boldsymbol{Y})$ is invariant under the mapping $\mathcal{A}$.*

Proof. Suppose that there exists a fixed point $\boldsymbol{X}$ in $\mathcal{N}_m^K$. We now show that there exists a minimal nonnegative fixed point. For that purpose, we construct the following sequence: $\boldsymbol{X}[0] = 0$, $\boldsymbol{X}[n] = \mathcal{A}(\boldsymbol{X}[n-1])$ for $n \ge 1$. It is easy to verify that the sequence $\{\boldsymbol{X}[n],\ n \ge 0\}$ is nondecreasing. Since $\boldsymbol{X}$ is a fixed point and $\boldsymbol{X} \ge \boldsymbol{X}[0] = 0$, by induction, it is easy to prove that $\boldsymbol{X} \ge \boldsymbol{X}[n]$ for $n \ge 0$. Since $\{\boldsymbol{X}[n],\ n \ge 0\}$ is nondecreasing and is bounded by $\boldsymbol{X}$, the sequence $\{\boldsymbol{X}[n],\ n \ge 0\}$ converges and the limit is less than or equal to $\boldsymbol{X}$. It is readily seen that the limit is a fixed point of $\mathcal{A}$, which is less than or equal to any fixed point of $\mathcal{A}$. Therefore, that limit is the minimal fixed point $\boldsymbol{X}_{\min}$.

If $\boldsymbol{Y}$ is a fixed point, then, since the mapping $\mathcal{A}$ is monotone, we have $\mathcal{A}(\boldsymbol{X}) \le \mathcal{A}(\boldsymbol{Y}) = \boldsymbol{Y}$ for any $\boldsymbol{X}$ in $\mathcal{N}_m^K(\boldsymbol{Y})$. Therefore, the subset $\mathcal{N}_m^K(\boldsymbol{Y})$ is invariant under $\mathcal{A}$. This completes the proof of Lemma 3.1.

If $m = K = 1$, the Markov chain defined in Section 2 is the classical GI/M/1 Markov chain [9]. It is well known that $y = 1$ is a fixed point. It is also well known that the derivative of $\mathcal{A}$ at $y = 1$ determines where the minimal fixed point $\boldsymbol{X}_{\min}$ is located. Let $\mathcal{A}^{(1)}(1)$ be the derivative of $\mathcal{A}$ at $y = 1$. If $\mathcal{A}^{(1)}(1) > 1$, then $\boldsymbol{X}_{\min} < 1$; otherwise, $\boldsymbol{X}_{\min} = 1$. In this section, these results are generalized to cases with $m \geq 1$ and $K \geq 1$. If $\mathcal{A}$ has a fixed point $\boldsymbol{Y}$ in $\mathcal{N}_m^K(1)$ (defined in Section 4), $\boldsymbol{Y}$ can play a role similar to that of $y = 1$ for the $m = K = 1$ case. Instead of the derivative $\mathcal{A}^{(1)}(y)$ at $y = 1$, we consider the Jacobian matrix $\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})$ at $\boldsymbol{Y}$. Then we look at the Perron–Frobenius eigenvalue $\operatorname{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ to locate the minimal nonnegative fixed point of $\mathcal{A}$. Note that the Perron–Frobenius eigenvalue $\operatorname{sp}(\boldsymbol{X})$ of the nonnegative matrix $\boldsymbol{X}$ is the eigenvalue of $\boldsymbol{X}$ with the largest modulus. Next, we define the Jacobian matrix explicitly. For $1 \leq k,\ j \leq K$, let

$$\boldsymbol{A}_{j,k}^{*(1)}(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{J \in \aleph : J \neq 0} \sum_{n=1}^{|J|} \delta_{(j, j_n)} (\boldsymbol{X}_{j_1} \cdots \boldsymbol{X}_{j_{n-1}})^{\top} \otimes (\boldsymbol{Y}_{j_{n+1}} \cdots \boldsymbol{Y}_{j_{|J|}} \boldsymbol{A}(J, k)), \qquad \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{N}_m^K, \tag{3.1}$$

where '$\otimes$' denotes the Kronecker product of matrices [2], $\delta_{(\cdot,\cdot)}$ is the Kronecker delta function (i.e. $\delta_{(i,j)} = 1$ if $i = j$; $\delta_{(i,j)} = 0$ otherwise) and '$^{\top}$' denotes the matrix transpose. The matrix $\boldsymbol{A}_{j,k}^{*(1)}(\boldsymbol{X}, \boldsymbol{Y})$ is an $m^2 \times m^2$ matrix. Define $\mathcal{A}^{(1)}(\boldsymbol{X}, \boldsymbol{Y})$ to be the $(m^2 K) \times (m^2 K)$ matrix with (j, k)th block $\boldsymbol{A}_{j,k}^{*(1)}(\boldsymbol{X}, \boldsymbol{Y})$, $1 \leq j,\ k \leq K$. Clearly, every entry of $\mathcal{A}^{(1)}(\boldsymbol{X}, \boldsymbol{Y})$ is nondecreasing with respect to every entry of $\boldsymbol{X}$ and $\boldsymbol{Y}$. The matrix $\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})$ is the differentiation matrix of $\mathcal{A}$ at $\boldsymbol{Y}$ and is called the *Jacobian matrix*. In fact, if $m = 1$, $\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})$ is the matrix of partial differentiation of $\mathcal{A}$ at the point $\boldsymbol{Y}$. The relationship between $\boldsymbol{X}_{\min}$ and $\operatorname{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ is summarized in the following lemma.

Lemma 3.2. *Assume that $\boldsymbol{Y}$ is a fixed point in $\mathcal{N}_m^K$. Assume that $\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})$ is irreducible and that $\mathcal{A}^{(1)}(\boldsymbol{X}, \boldsymbol{Y})$, as a function of $\boldsymbol{X}$, is strictly increasing at $\boldsymbol{X} = \boldsymbol{Y}$ with respect to every element of $\boldsymbol{X}$. If* $\operatorname{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})) \leq 1$, *then $\boldsymbol{X}_{\min} = \boldsymbol{Y}$. In this case, $\boldsymbol{X}_{\min} = \boldsymbol{Y}$ is the only fixed point in $\mathcal{N}_m^K(\boldsymbol{Y})$. If* $\operatorname{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})) > 1$, *then $\boldsymbol{X}_{\min} < \boldsymbol{Y}$ (i.e. $\boldsymbol{X}_{\min} \leq \boldsymbol{Y}$ and $\boldsymbol{X}_{\min} \neq \boldsymbol{Y}$). In this case, the mapping $\mathcal{A}$ has at least two (different) fixed points in $\mathcal{N}_m^K(\boldsymbol{Y})$. On the other hand, if there exists a fixed point $\boldsymbol{Z}$ in $\mathcal{N}_m^K$ such that $\boldsymbol{Y} \leq \boldsymbol{Z}$ and $\boldsymbol{Y} \neq \boldsymbol{Z}$, then* $\operatorname{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})) < 1$.

Proof. In the following proof, we shall use the *direct sum* of a matrix. For any matrix $\boldsymbol{X}$, its direct sum $\phi(\boldsymbol{X})$ is obtained by putting its rows into a single row vector, starting from the top [9]. A useful property of the direct sum is that, for three matrices $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$, $\phi(\boldsymbol{XYZ}) = \phi(\boldsymbol{Y})(\boldsymbol{X}^{\top} \otimes \boldsymbol{Z})$, given that the multiplications are valid. First, we have the following calculations:

$$\begin{aligned}
&\boldsymbol{A}_k^*(\boldsymbol{Y}) - \boldsymbol{A}_k^*(\boldsymbol{X}) \\
&\qquad = \sum_{J \in \aleph} (\boldsymbol{Y}^{(J)} - \boldsymbol{X}^{(J)}) \boldsymbol{A}(J, k) \\
&\qquad = \sum_{J \in \aleph : J \neq 0} \left\{ \sum_{n=1}^{|J|} \boldsymbol{X}_{j_1} \cdots \boldsymbol{X}_{j_{n-1}} (\boldsymbol{Y}_{j_n} - \boldsymbol{X}_{j_n}) \boldsymbol{Y}_{j_{n+1}} \cdots \boldsymbol{Y}_{j_{|J|}} \right\} \boldsymbol{A}(J, k) \\
&\qquad = \sum_{j=1}^{K} \left(\sum_{J \in \aleph : J \neq 0} \left\{ \sum_{n=1}^{|J|} \delta_{(j, j_n)} \boldsymbol{X}_{j_1} \cdots \boldsymbol{X}_{j_{n-1}} (\boldsymbol{Y}_{j_n} - \boldsymbol{X}_{j_n}) \boldsymbol{Y}_{j_{n+1}} \cdots \boldsymbol{Y}_{j_{|J|}} \right\} \boldsymbol{A}(J, k) \right).
\end{aligned} \tag{3.2}$$

Taking the direct sum on both sides of (3.2), we obtain that

$$
\begin{aligned}
&\phi(\boldsymbol{A}_k^*(\boldsymbol{Y})) - \phi(\boldsymbol{A}_k^*(\boldsymbol{X})) \\
&\quad = \sum_{j=1}^{K}\left(\phi(\boldsymbol{Y}_j - \boldsymbol{X}_j) \sum_{J\in\aleph:\ J\neq 0}\left\{\sum_{n=1}^{|J|}\delta_{(j,j_n)}(\boldsymbol{X}_{j_1}\cdots\boldsymbol{X}_{j_{n-1}})^{\top}\otimes(\boldsymbol{Y}_{j_{n+1}}\cdots\boldsymbol{Y}_{j_{|J|}}\boldsymbol{A}(J,k))\right\}\right) \\
&\quad = \sum_{j=1}^{K}(\phi(\boldsymbol{Y}_j - \boldsymbol{X}_j)\boldsymbol{A}_{j,k}^{*(1)}(\boldsymbol{X},\boldsymbol{Y})).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\phi(\mathcal{A}(\boldsymbol{Y}) - \mathcal{A}(\boldsymbol{X})) &\equiv (\phi(\boldsymbol{A}_1^*(\boldsymbol{Y})) - \phi(\boldsymbol{A}_1^*(\boldsymbol{X})), \dots, \phi(\boldsymbol{A}_K^*(\boldsymbol{Y})) - \phi(\boldsymbol{A}_K^*(\boldsymbol{X}))) \\
&= (\phi(\boldsymbol{Y}_1) - \phi(\boldsymbol{X}_1), \dots, \phi(\boldsymbol{Y}_K) - \phi(\boldsymbol{X}_K))\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y}) \\
&\equiv \phi(\boldsymbol{Y} - \boldsymbol{X})\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y}). \qquad (3.3)
\end{aligned}
$$

If $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) \leq 1$, we would like to show that $\boldsymbol{Y}$ is the minimal nonnegative fixed point. Suppose that $\boldsymbol{Y}$ is not the minimal fixed point. By Lemma 3.1, there exists another fixed point in $\mathcal{N}_m^K(\boldsymbol{Y})$. We denote that fixed point by $\boldsymbol{X}$. Then (3.3) becomes $\phi(\boldsymbol{Y} - \boldsymbol{X}) = \phi(\boldsymbol{Y} - \boldsymbol{X})\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})$. Since $\boldsymbol{X}$ is strictly smaller than $\boldsymbol{Y}$ (i.e. $\boldsymbol{X} < \boldsymbol{Y}$) and the matrix $\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})$ is irreducible, we must have $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})) < \mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}))$.

Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1^{\top}, \boldsymbol{\alpha}_2^{\top}, \dots, \boldsymbol{\alpha}_K^{\top})^{\top}$ be the right eigenvector of $\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})$ corresponding to the eigenvalue with the largest modulus. Since $\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})$ is irreducible, $\boldsymbol{\alpha}$ is positive. We normalize the vector $\boldsymbol{\alpha}$ by $\phi(\boldsymbol{Y})\boldsymbol{\alpha} = 1$. Postmultiplying by $\boldsymbol{\alpha}$ on both sides of (3.3), we obtain that

$$
\begin{aligned}
\phi(\boldsymbol{Y} - \boldsymbol{X})\boldsymbol{\alpha} &= \phi(\boldsymbol{Y} - \boldsymbol{X})\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{\alpha} \\
&< \phi(\boldsymbol{Y} - \boldsymbol{X})\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})\boldsymbol{\alpha} \\
&= (\phi(\boldsymbol{Y} - \boldsymbol{X})\boldsymbol{\alpha})\,\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})), \qquad (3.4)
\end{aligned}
$$

which implies that $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) > 1$, since $\phi(\boldsymbol{Y} - \boldsymbol{X})\boldsymbol{\alpha}$ is positive. That contradicts the assumption that $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) \leq 1$. Therefore, if $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) \leq 1$, there is no fixed point smaller than $\boldsymbol{Y}$. According to Lemma 3.1, $\boldsymbol{Y}$ is the minimal nonnegative fixed point.

If $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) > 1$, we would like to show that there is another fixed point that is smaller than $\boldsymbol{Y}$. Define $\mathcal{N}_m^K(\boldsymbol{Y}, \leq \varepsilon) = \{\boldsymbol{X} : \phi(\boldsymbol{X})\boldsymbol{\alpha} \leq 1 - \varepsilon,\ \boldsymbol{X} \in \mathcal{N}_m^K(\boldsymbol{Y})\}$ and $\mathcal{N}_m^K(\boldsymbol{Y}, = \varepsilon) = \{\boldsymbol{X} : \phi(\boldsymbol{X})\boldsymbol{\alpha} = 1 - \varepsilon,\ \boldsymbol{X} \in \mathcal{N}_m^K(\boldsymbol{Y})\}$. It is easy to see that both $\mathcal{N}_m^K(\boldsymbol{Y}, \leq \varepsilon)$ and $\mathcal{N}_m^K(\boldsymbol{Y}, = \varepsilon)$ are convex sets. We would like to show that, if ε is small enough, then the subset $\mathcal{N}_m^K(\boldsymbol{Y}, \leq \varepsilon)$ is invariant under the mapping $\mathcal{A}$. For any $\boldsymbol{X} \in \mathcal{N}_m^K(\boldsymbol{Y}, = \varepsilon)$, by (3.3),

$$
\begin{aligned}
&\phi(\mathcal{A}(\boldsymbol{X}))\boldsymbol{\alpha} - \phi(\mathcal{A}(\boldsymbol{Y}))\boldsymbol{\alpha} \\
&\quad = \phi(\boldsymbol{X} - \boldsymbol{Y})\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{\alpha} \\
&\quad = \phi(\boldsymbol{X} - \boldsymbol{Y})\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})\boldsymbol{\alpha} + \phi(\boldsymbol{X} - \boldsymbol{Y})[\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y}) - \mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})]\boldsymbol{\alpha} \\
&\quad = \mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}))\phi(\boldsymbol{X} - \boldsymbol{Y})\boldsymbol{\alpha} + \phi(\boldsymbol{X} - \boldsymbol{Y})[\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y}) - \mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})]\boldsymbol{\alpha}. \qquad (3.5)
\end{aligned}
$$

Note that $\phi(\boldsymbol{Y})\boldsymbol{\alpha} - \phi(\boldsymbol{X})\boldsymbol{\alpha} = 1 - \phi(\boldsymbol{X})\boldsymbol{\alpha} = \varepsilon$. The equality (3.5) implies that

$$\begin{aligned}\phi(\mathcal{A}(\boldsymbol{X}))\boldsymbol{\alpha} &= 1 - \mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}))\phi(\boldsymbol{Y}-\boldsymbol{X})\boldsymbol{\alpha} + \phi(\boldsymbol{X}-\boldsymbol{Y})[\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y}) - \mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})]\boldsymbol{\alpha}\\ &= 1 - \varepsilon - [\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) - 1]\varepsilon + \phi(\boldsymbol{Y}-\boldsymbol{X})[\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}) - \mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})]\boldsymbol{\alpha}.\end{aligned} \tag{3.6}$$

Since $\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})$ and $\phi(\boldsymbol{X})$ are continuous in $\boldsymbol{X}$ and every element of $\boldsymbol{\alpha}$ is positive, we can choose a small enough ε such that

$$[\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}) - \mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})]\boldsymbol{\alpha} \le [\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) - 1]\boldsymbol{\alpha}$$

for any $\boldsymbol{X} \in \mathcal{N}_m^K(\boldsymbol{Y}, = \varepsilon)$. This inequality along with (3.6) implies that $\phi(\mathcal{A}(\boldsymbol{X}))\boldsymbol{\alpha} \le 1-\varepsilon$ for any $\boldsymbol{X} \in \mathcal{N}_m^K(\boldsymbol{Y}, = \varepsilon)$, that is, $\mathcal{A}(\boldsymbol{X}) \in \mathcal{N}_m^K(\boldsymbol{Y}, \le \varepsilon)$ for any $\boldsymbol{X} \in \mathcal{N}_m^K(\boldsymbol{Y}, = \varepsilon)$. For any $\boldsymbol{X} \in \mathcal{N}_m^K(\boldsymbol{Y}, \le \varepsilon)$, consider the closed line segment $\boldsymbol{X}(t) = (1-t)\boldsymbol{X} + t\boldsymbol{Y}, 0 \le t \le 1$. It can be shown that there exists a t_1 such that $\boldsymbol{X}(t_1) \in \mathcal{N}_m^K(\boldsymbol{Y}, = \varepsilon)$ and $\boldsymbol{X} \le \boldsymbol{X}(t_1)$. This implies that $\mathcal{A}(\boldsymbol{X}) \le \mathcal{A}(\boldsymbol{X}(t_1)) \in \mathcal{N}_m^K(\boldsymbol{Y}, \le \varepsilon)$. Thus, we have shown that the convex subset $\mathcal{N}_m^K(\boldsymbol{Y}, \le \varepsilon)$ is invariant under the mapping $\mathcal{A}$. Since $\mathcal{A}$ is continuous, by the well-known Brouwer theorem [3], there is a fixed point in $\mathcal{N}_m^K(\boldsymbol{Y}, \le \varepsilon)$, which is smaller than $\boldsymbol{Y}$. Thus, the minimal fixed point is smaller than $\boldsymbol{Y}$.

Suppose that there exists a fixed point $\boldsymbol{Z}$ of $\mathcal{A}$ in $\mathcal{N}_m^K$ such that $\boldsymbol{Y} < \boldsymbol{Z}$. By replacing $\boldsymbol{X}$ by $\boldsymbol{Z}$ in (3.4) and changing the direction of the inequality, we obtain that $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) < 1$. This completes the proof of Lemma 3.2.

For cases with $m,\ K \ge 2$, in general, there is no explicit expression for any fixed point of $\mathcal{A}$ in $\mathcal{N}_m^K$. Thus, the condition given in Lemma 3.2 cannot be explicit. The implication of this is that the classification condition (presented in Section 4) may not be explicit.

4. Classification of the Markov chain of interest

According to Neuts's conditions in matrix-analytic methods (see [9] and [10]) and classification conditions for matrix M/G/1-type Markov chains with a tree structure (see [4] and [5]), the classification of the Markov chains of interest has much to do with the fixed points of the mapping $\mathcal{A}$. According to Yeung and Sengupta [15], the Markov chain defined in Section 2 is positive recurrent if and only if the minimal fixed point $\boldsymbol{X}_{\min}$ satisfies $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K}) < 1$. Note that, in the literature of matrix-analytic methods, the matrices $\boldsymbol{X}_{\min,1}, \ldots, \boldsymbol{X}_{\min,K}$ are usually denoted by $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_K$. In this section, we show that the locations of the fixed points of $\mathcal{A}$ classify the Markov chain of interest. Let

$$\begin{aligned}\mathcal{N}_m^K(1) &= \{\boldsymbol{X} : \boldsymbol{X} \in \mathcal{N}_m^K \text{ and } \mathrm{sp}(\boldsymbol{X}_1 + \cdots + \boldsymbol{X}_K) = 1\},\\ \mathcal{N}_m^K(\le 1) &= \{\boldsymbol{X} : \boldsymbol{X} \in \mathcal{N}_m^K \text{ and } \mathrm{sp}(\boldsymbol{X}_1 + \cdots + \boldsymbol{X}_K) \le 1\}.\end{aligned}$$

By assuming that $\mathcal{A}$ has a fixed point in $\mathcal{N}_m^K(1)$, the results given in Section 3 can be utilized to allocate the minimal fixed point in $\mathcal{N}_m^K(\le 1)$. We now translate these results into a classification condition for the Markov chain of interest.

Theorem 4.1. *Assume that the Markov chain $\{(C_n, \eta_n),\ n \ge 0\}$ defined in Section 2 is irreducible and aperiodic. Also assume that the mapping $\mathcal{A}$ has a fixed point $\boldsymbol{Y}$ in $\mathcal{N}_m^K(1)$ and that $\boldsymbol{Y}_1 + \cdots + \boldsymbol{Y}_K$ is irreducible. If the Jacobian matrix $\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})$ is irreducible and if $\mathcal{A}^{(1)}(\boldsymbol{X},\boldsymbol{Y})$ is strictly increasing in $\boldsymbol{X}$, then the Markov chain $\{(C_n, \eta_n),\ n \ge 0\}$ is positive recurrent if and only if* $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) > 1$.

Proof. If $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) > 1$, then, according to Lemma 3.2, there exists a fixed point in $\mathcal{N}_m^K(\boldsymbol{Y})$ that is smaller than $\boldsymbol{Y}$. That implies that the minimal nonnegative solution $\boldsymbol{X}_{\min}$ is smaller than $\boldsymbol{Y}$. Since $\mathrm{sp}(\boldsymbol{Y}_1+\cdots+\boldsymbol{Y}_K)=1$ and $\boldsymbol{Y}_1+\cdots+\boldsymbol{Y}_K$ is irreducible, we must have $\mathrm{sp}(\boldsymbol{X}_{\min,1}+\cdots+\boldsymbol{X}_{\min,K}) < 1$. Further, the matrix $\sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}B(J)$ is stochastic because, by (2.1),

$$
\begin{aligned}
\sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{B}(J)\boldsymbol{e} &= \sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}\bigg(\boldsymbol{e}-\sum_{k=1}^{K}\sum_{i=0}^{|J|}\boldsymbol{A}(f(J,i),k)\boldsymbol{e}\bigg)\\
&= \sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{e}-\sum_{k=1}^{K}\bigg(\sum_{J\in\aleph}\sum_{i=0}^{|J|}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{A}(f(J,i),k)\bigg)\boldsymbol{e}\\
&= \sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{e}-\sum_{k=1}^{K}\bigg(\sum_{i=0}^{\infty}\sum_{J\in\aleph:|J|\ge i}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{A}(f(J,|J|-i),k)\bigg)\boldsymbol{e}\\
&= \sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{e}-\sum_{k=1}^{K}\bigg(\sum_{i=0}^{\infty}\sum_{J,H\in\aleph:|J|=i}\boldsymbol{X}_{\min}^{(J+H)}\boldsymbol{A}(f(J+H,|H|),k)\bigg)\boldsymbol{e}\\
&= \sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{e}-\sum_{k=1}^{K}\bigg(\sum_{i=0}^{\infty}\sum_{J\in\aleph:|J|=i}\boldsymbol{X}_{\min}^{(J)}\sum_{H\in\aleph}\boldsymbol{X}_{\min}^{(H)}\boldsymbol{A}(H,k)\bigg)\boldsymbol{e}\\
&= \sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}\boldsymbol{e}-\bigg(\sum_{i=0}^{\infty}\sum_{J\in\aleph:|J|=i}\boldsymbol{X}_{\min}^{(J)}\bigg)\bigg(\sum_{k=1}^{K}\boldsymbol{X}_{\min,k}\bigg)\boldsymbol{e}\\
&= \bigg(\sum_{i=0}^{\infty}\sum_{J\in\aleph:|J|=i}\boldsymbol{X}_{\min}^{(J)}\bigg)\bigg(\boldsymbol{I}-\sum_{k=1}^{K}\boldsymbol{X}_{\min,k}\bigg)\boldsymbol{e}\\
&= \bigg(\sum_{i=0}^{\infty}\bigg(\sum_{k=1}^{K}\boldsymbol{X}_{\min,k}\bigg)^{i}\bigg)\bigg(\boldsymbol{I}-\sum_{k=1}^{K}\boldsymbol{X}_{\min,k}\bigg)\boldsymbol{e}\\
&= \bigg(\boldsymbol{I}-\sum_{k=1}^{K}\boldsymbol{X}_{\min,k}\bigg)^{-1}\bigg(\boldsymbol{I}-\sum_{k=1}^{K}\boldsymbol{X}_{\min,k}\bigg)\boldsymbol{e}=\boldsymbol{e},
\end{aligned}
$$

where $\boldsymbol{I}$ denotes the identity matrix.

Let $\boldsymbol{\pi}(J)=\pi(0)\boldsymbol{X}_{\min}^{(J)}$ for $J\in\aleph$, where $\boldsymbol{\pi}(0)$ is a nonnegative and nonzero vector satisfying $\boldsymbol{\pi}(0)=\boldsymbol{\pi}(0)\sum_{J\in\aleph}\boldsymbol{X}_{\min}^{(J)}B(J)$. Then it is straightforward to verify that $\{\boldsymbol{\pi}(J),\ J\in\aleph\}$ is an invariant measure that is finite. Since the Markov chain is irreducible and aperiodic, $\{\boldsymbol{\pi}(J),\ J\in\aleph\}$ is unique up to a positive constant. According to Theorem 5.5 of [11], the Markov chain is positive recurrent.

On the other hand, if the Markov chain is positive recurrent, according to [15], $\{\boldsymbol{\pi}(J),\ J\in\aleph\}$ must be the stationary distribution of $\{(C_n,\eta_n),\ n\ge 0\}$ and $\mathrm{sp}(\boldsymbol{X}_{\min,1}+\cdots+\boldsymbol{X}_{\min,K})<1$. Thus, we have a fixed point in $\mathcal{N}_m^K(\boldsymbol{Y})$ that is smaller than $\boldsymbol{Y}$. According to Lemma 3.2, we must have $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}))>1$. This completes the proof of Theorem 4.1.

Theorem 4.1 implies that the location of the minimal nonnegative fixed point of $\mathcal{A}$ determines whether or not the Markov chain is positive recurrent. But the corresponding condition depends on the fixed point $\boldsymbol{Y}$ in $\mathcal{N}_m^K(1)$, which is not given explicitly. We shall discuss the existence of a fixed point of $\mathcal{A}$ in $\mathcal{N}_m^K(1)$ in Section 5.

Corollary 4.1. *Assume that the Markov chain* $\{(C_n, \eta_n),\ n \geq 0\}$ *defined in Section 2 is irreducible and aperiodic. Assume also that the mapping* $\mathcal{A}$ *has a fixed point* $\boldsymbol{Y}$ *in* $\mathcal{N}_m^K(1)$ *and* $\boldsymbol{Y}_1 + \cdots + \boldsymbol{Y}_K$ *is irreducible. Then* $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{X}_{\min}, \boldsymbol{X}_{\min})) \leq 1$. *If the Markov chain is positive recurrent, then* $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K}) < 1$ *and* $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{X}_{\min}, \boldsymbol{X}_{\min})) \leq 1 < \mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$*; otherwise,* $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K}) = 1$ *and* $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{X}_{\min}, \boldsymbol{X}_{\min})) = \mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})) \leq 1$.

Proof. The first conclusion is from Lemma 3.2. The rest of the results are from Theorem 4.1.

Note 4.1. When $m = 1$ and $K \geq 1$, Theorem 4.1 is consistent with Theorem 5.4 of [6]. When $m \geq 1$ and $K = 1$, the condition given in Theorem 4.1 is different from Neuts's condition. But the two conditions are equivalent to each other; this is shown in Appendix A.

Note 4.2. Corollary 4.1 shows that, with respect to the classification of the Markov chains, $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ (if the fixed point $\boldsymbol{Y}$ exists) provides more accurate information than $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K})$, especially when the Markov chain is transient. Although it has not been proved mathematically, numerical examples and the results in [6] indicate that $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})) < 1$ may hold if the Markov chain is transient. Thus, $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ may provide information about transience of the Markov chain that is not available from $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K})$, since $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K}) = 1$ for both null recurrent and transient cases.

Based on Theorem 4.1 and the above discussion, we propose the following procedure for the classification of the Markov chain defined in Section 2:

1. Compute all the transition blocks $\{\boldsymbol{A}(J, k),\ J \in \aleph,\ 1 \leq k \leq K\}$.
2. Check irreducibility and periodicity of the Markov chain.
3. Compute a fixed point $\boldsymbol{Y}$ in $\mathcal{N}_m^K(1)$ (Section 5) and construct the matrix $\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})$.
4. Check whether or not $\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})$ is irreducible and calculate $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$.

Next, we present a numerical example in order to analyse $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K})$, $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{X}_{\min}, \boldsymbol{X}_{\min}))$ and $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ numerically.

Example 4.1. Consider a matrix GI/M/1-type Markov chain with $m = K = 2$,

$$\begin{aligned}
\boldsymbol{A}(0,1) &= \begin{pmatrix} 0 & 0.1 \\ \mu & \mu \end{pmatrix}, \\
\boldsymbol{A}(0,2) &= \begin{pmatrix} 0.1 & 0 \\ \mu & \mu \end{pmatrix}, \\
\boldsymbol{A}(11,1) = \boldsymbol{A}(21,1) &= \begin{pmatrix} 0.1 & 0.4 \\ 0.25-\mu & 0.25-\mu \end{pmatrix}, \\
\boldsymbol{A}(12,1) = \boldsymbol{A}(22,1) &= \begin{pmatrix} 0.2 & 0.1 \\ 0.5-2\mu & 0 \end{pmatrix}, \\
\boldsymbol{A}(11,2) = \boldsymbol{A}(21,2) &= \begin{pmatrix} 0.1 & 0.2 \\ 0.25-\mu & 0.25-\mu \end{pmatrix}
\end{aligned}$$

TABLE 1: Perron–Frobenius eigenvalues for Example 4.1.

μ	0.1	0.15	0.17	0.1785	0.18	0.20	0.24
$\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K})$	0.4454	0.7416	0.9120	0.9992	1	1	1
$\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{X}_{\min}, \boldsymbol{X}_{\min}))$	0.6254	0.8597	0.9572	0.9996	0.9928	0.8898	0.6640
$\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$	1.3745	1.1403	1.0427	1.0003	0.9928	0.8898	0.6640

and

$$\boldsymbol{A}(12, 2) = \boldsymbol{A}(22, 2) = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.5 - 2\mu \end{pmatrix},$$

with all other matrices $\boldsymbol{A}(J, k)$ equal to zero and $0 < \mu < 0.25$. Since the transition from node 11 to node 2 in one step is possible, this Markov chain is not an M/G/1-type Markov chain. Therefore, the results obtained in [5] and [6] cannot apply. Thus, we apply Theorem 4.1 to classify the Markov chain. Information for classification is given in Table 1.

Table 1 shows that the Markov chain is positive recurrent if $\mu < 0.1785$ and null recurrent or transient if $\mu > 1.8$. The Perron–Frobenius eigenvalue $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K})$ is nondecreasing with respect to μ. The Perron–Frobenius eigenvalue $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ is nonincreasing with respect to μ. This is quite intuitive since, when μ is increasing, the probabilities that the Markov chain moves away from the root node 0 are increasing.

Table 1 shows that, if the Markov chain is null recurrent (for some μ between 1.7 and 1.8), then $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{X}_{\min}, \boldsymbol{X}_{\min}))$ and $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ are close to 1. If $\mu \geq 0.18$, then the Markov chain is null recurrent or transient and $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K}) = 1$. On the other hand, if $\mu \geq 0.18$, then $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ is smaller than 1, which shows clearly that the Markov chain is transient.

5. Fixed points in $\mathcal{N}_m^K(1)$

To use Theorem 4.1, we need to compute a fixed point $\boldsymbol{Y}$ in $\mathcal{N}_m^K(1)$ and $\boldsymbol{X}_{\min}$ in $\mathcal{N}_m^K(\leq 1)$. A computation procedure for $\boldsymbol{X}_{\min}$ has been given in the proof of Lemma 3.1. A fixed point $\boldsymbol{Y}$ in $\mathcal{N}_m^K(1)$ can be computed as follows. Let $Y_k[0] = (1/K)\boldsymbol{I}$ for $k = 1, \ldots, K$. For $n \geq 1$, let $\boldsymbol{Y}[n] = \mathcal{A}(\boldsymbol{Y}[n-1])$. Then the limit of the sequence $\{\boldsymbol{Y}[n],\ n \geq 0\}$, if it exists and is in $\mathcal{N}_m^K(1)$, can be chosen as $\boldsymbol{Y}$. Unfortunately, the convergence of $\{\boldsymbol{Y}[n],\ n \geq 0\}$ has not yet been proved. There is no guarantee that (i) $\{\boldsymbol{Y}[n],\ n \geq 0\}$ converges and (ii) the limit is in $\mathcal{N}_m^K(1)$ if the sequence does converge. Nonetheless, numerical experimentation shows that, in general, $\{\boldsymbol{Y}[n],\ n \geq 0\}$ does converge to a fixed point of $\mathcal{A}$ in $\mathcal{N}_m^K(1)$. Thus, the procedure given in Section 4 can be applied to classify the Markov chain defined in Section 2.

To prove that there is a fixed point in $\mathcal{N}_m^K(1)$, the idea used in [6] is to show that the convex set $\mathcal{N}_m^K(1)$ is invariant under the mapping $\mathcal{A}$. However, numerical experimentation shows that $\mathcal{N}_m^K(1)$ may not be invariant under $\mathcal{A}$ for the matrix GI/M/1 case, though a fixed point exists. In the rest of this section, we identify some conditions on the transition probabilities $\{\boldsymbol{A}(J, k),\ J \in \aleph, 1 \leq k \leq K\}$ such that $\mathcal{N}_m^K(1)$ is invariant under $\mathcal{A}$. Consequently, the existence of a fixed point in $\mathcal{N}_m^K(1)$ is ensured.

Assume that there is a stochastic matrix $\boldsymbol{A}$ satisfying the following condition: for any $\varepsilon > 0$, there exists an $N > 0$ such that

$$\max_{J \in \aleph:\ |J|=n} \left\{ \left\| \boldsymbol{A} - \sum_{k=1}^{K} \sum_{i=0}^{N} \boldsymbol{A}(f(J, i), k) \right\|_{\max} \right\} \leq \varepsilon, \tag{5.1}$$

where the norm $\|.\|_{\max}$ is defined as $\|\boldsymbol{X}\|_{\max} = \max_{1\le i,j\le m}\{|(\boldsymbol{X})_{i,j}|\}$. We assume that the matrix $\boldsymbol{A}$ is irreducible. Then the Perron–Frobenius eigenvalue $\mathrm{sp}(\boldsymbol{A})$ of $\boldsymbol{A}$ is 1. Let $\boldsymbol{\theta}$ be the probability invariant vector of $\boldsymbol{A}$, that is $\boldsymbol{\theta A} = \boldsymbol{\theta}$ and $\boldsymbol{\theta e} = 1$. Since $\boldsymbol{A}$ is irreducible, every element of $\boldsymbol{\theta}$ is positive. Define

$$\mathcal{N}^K_{m,\boldsymbol{\theta}}(1) = \{\boldsymbol{X} : \boldsymbol{X} \in \mathcal{N}^K_m \text{ and } \boldsymbol{\theta}(\boldsymbol{X}_1 + \cdots + \boldsymbol{X}_K) = \boldsymbol{\theta}\}.$$

It is easy to see that $\mathcal{N}^K_{m,\boldsymbol{\theta}}(1) \subset \mathcal{N}^K_m(1)$ and $\mathcal{N}^K_{m,\boldsymbol{\theta}}(1)$ is a convex set. The next lemma shows that, under the condition (5.1), the mapping $\mathcal{A}$ has a fixed point in $\mathcal{N}^K_{m,\boldsymbol{\theta}}(1)$. Consequently, $\mathcal{A}$ has a fixed point in $\mathcal{N}^K_m(1)$.

Lemma 5.1. *Under the condition (5.1), the set $\mathcal{N}^K_{m,\boldsymbol{\theta}}(1)$ is invariant under the mapping $\mathcal{A}$. Thus, the mapping $\mathcal{A}$ has a fixed point in $\mathcal{N}^K_{m,\boldsymbol{\theta}}(1)$.*

Proof. For $\boldsymbol{X} \in N^K_{m,\boldsymbol{\theta}}(1)$, by definition, we have $\boldsymbol{\theta}(\boldsymbol{X}_1 + \cdots + \boldsymbol{X}_K) = \boldsymbol{\theta}$ and, for $N > 0$,

$$\begin{aligned}
\boldsymbol{\theta}\sum_{k=1}^{K}\sum_{i=0}^{N}\sum_{J\in\aleph:\,|J|=i}\boldsymbol{X}^{(J)}\boldsymbol{A}(J,k) &= \boldsymbol{\theta}\sum_{k=1}^{K}\sum_{i=0}^{N}\sum_{J\in\aleph:\,|J|=i}(\boldsymbol{X}_1+\cdots+\boldsymbol{X}_K)^{N-i}\boldsymbol{X}^{(J)}\boldsymbol{A}(J,k)\\
&= \boldsymbol{\theta}\sum_{J\in\aleph:\,|J|=N}\boldsymbol{X}^{(J)}\sum_{k=1}^{K}\sum_{i=0}^{N}\boldsymbol{A}(f(J,i),k)\\
&= \boldsymbol{\theta}\sum_{J\in\aleph:\,|J|=N}\boldsymbol{X}^{(J)}\boldsymbol{A} + \boldsymbol{\theta}\sum_{J\in\aleph:\,|J|=N}\boldsymbol{X}^{(J)}\left(\sum_{k=1}^{K}\sum_{i=0}^{N}\boldsymbol{A}(f(J,i),k) - \boldsymbol{A}\right)\\
&= \boldsymbol{\theta}(\boldsymbol{X}_1+\cdots+\boldsymbol{X}_K)^N\boldsymbol{A} + \boldsymbol{\theta}\sum_{J\in\aleph:\,|J|=N}\boldsymbol{X}^{(J)}\left(\sum_{k=1}^{K}\sum_{i=0}^{N}\boldsymbol{A}(f(J,i),k) - \boldsymbol{A}\right)\\
&= \boldsymbol{\theta} + \boldsymbol{\theta}\sum_{J\in\aleph:\,|J|=N}\boldsymbol{X}^{(J)}\left(\sum_{k=1}^{K}\sum_{i=0}^{N}\boldsymbol{A}(f(J,i),k) - \boldsymbol{A}\right).
\end{aligned}\tag{5.2}$$

By the condition (5.1), (5.2) and the monotone convergence theorem, we obtain that, for any $\boldsymbol{X} \in \mathcal{N}^K_{m,\boldsymbol{\theta}}(1)$,

$$\begin{aligned}
\boldsymbol{\theta}\sum_{k=1}^{K}\boldsymbol{A}^*_k(\boldsymbol{X}) &= \boldsymbol{\theta}\sum_{k=1}^{K}\sum_{J\in\aleph}\boldsymbol{X}^{(J)}\boldsymbol{A}(J,k)\\
&= \lim_{N\to\infty}\boldsymbol{\theta}\sum_{k=1}^{K}\sum_{i=0}^{N}\sum_{J\in\aleph:\,|J|=i}\boldsymbol{X}^{(J)}\boldsymbol{A}(J,k)\\
&= \boldsymbol{\theta} + \lim_{N\to\infty}\boldsymbol{\theta}\sum_{J\in\aleph:\,|J|=N}\boldsymbol{X}^{(J)}\left(\sum_{k=1}^{K}\sum_{i=0}^{N}\boldsymbol{A}(f(J,i),k) - \boldsymbol{A}\right)\\
&= \boldsymbol{\theta} + \lim_{N\to\infty}\boldsymbol{\theta}\sum_{J\in\aleph:\,|J|=N}\boldsymbol{X}^{(J)}\boldsymbol{\varepsilon}(N) \quad \left(\text{where } \boldsymbol{\varepsilon}(N) = \sum_{k=1}^{K}\sum_{i=0}^{N}\boldsymbol{A}(f(J,i),t) - \boldsymbol{A}\right)\\
&= \boldsymbol{\theta} + \lim_{N\to\infty}\boldsymbol{\theta\varepsilon}(N) = \boldsymbol{\theta},
\end{aligned}$$

where the matrix $\boldsymbol{\varepsilon}(N)$ tends to 0 as $N \to \infty$ (by the condition (5.2)). Thus, the convex set $\mathcal{N}^K_{m,\boldsymbol{\theta}}(1)$ is invariant under the mapping $\mathcal{A}$. By Brouwer's fixed-point theorem, there is a fixed point of $\mathcal{A}$ in $\mathcal{N}^K_{m,\boldsymbol{\theta}}(1)$. This completes the proof of Lemma 5.1.

The condition (5.1) is satisfied for a number of easy-to-check cases. Intuitive explanations of (5.1) can be seen in these special cases as well. We give two such examples below. The first case is as follows:

$$\lim_{n\to\infty} \max_{|J|=n} \{\boldsymbol{B}(J)\} = 0, \qquad J \in \aleph, \tag{5.3}$$

$$\boldsymbol{B}(J) = \boldsymbol{B}(j+J) + \sum_{k=1}^{K} \boldsymbol{A}(j+J,k) \quad \text{for } j = 1,\dots,K,\ J \in \aleph. \tag{5.4}$$

The condition (5.3) implies that the Markov chain does not jump from a 'remote' node directly to the root node 0. The condition (5.4) ensures that there is a certain level of homogeneity in the transition probabilities associated with the phase variable $\{\eta_n, n \geq 0\}$. That condition can be satisfied by stochastic systems for which the phase variable (environmental factor) is independent of system actions. One such example is shown in Section 6. When the conditions (5.3) and (5.4) are satisfied, the matrix $\boldsymbol{A}$ is given by $\boldsymbol{A} = \boldsymbol{B}(0) + \boldsymbol{A}(0,1) + \cdots + \boldsymbol{A}(0,K)$. By the law of total probability, the matrix $\boldsymbol{A}$ is a stochastic matrix.

The second case is as follows: for a given positive integer N,

$$\boldsymbol{A}(J,k) = 0, \qquad J \in \aleph \text{ and } |J| > N, \tag{5.5}$$

$$\boldsymbol{A} = \sum_{k=1}^{K} \sum_{i=0}^{|J|} \boldsymbol{A}(f(J,i),k), \qquad J \in \aleph \text{ and } |J| = N, \tag{5.6}$$

where $\boldsymbol{A}$ is a stochastic matrix. The condition (5.5) implies that, in one transition, the Markov chain can only 'jump' at most N levels towards the root node. Similar to (5.4), the condition (5.6) ensures that there is some kind of homogeneity in the transition process. Given the conditions (5.5) and (5.6), it is easy to verify that (5.1) is satisfied.

To end this section, we point out that finding fixed points of the mapping $\mathcal{A}$ is a key to the classification of GI/M/1-type Markov chains with a tree structure. But the computation of such fixed points can be expensive. Alternatively, it might be worthwhile to find sufficient conditions for positive recurrence where the fixed points ($\boldsymbol{X}_{\min}$ and $\boldsymbol{Y}$) are not involved (see Lemmas 6.1 and 6.2 in [5]). In fact, in [8], some linear programming problems are constructed such that their optimal solutions provide information for classifying M/G/1-type Markov chains with a tree structure. It is interesting to extend that approach to GI/M/1-type Markov chains with a tree structure.

6. Stability of an SM[K]/G[K]/1/LCFS queue

In this section, we consider an SM[K]/G[K]/1/LCFS preemptive repeat queue. The issue is system stability. For more on the study of queues with multiple types of customers, see [7] and [12].

The queueing system of interest is defined as follows. Customers arrive in the queueing system according to a continuous-time semi-Markov chain $\{(\xi_n, \tau_n),\ n \geq 0\}$ with m phases. The variable ξ_n is the phase of the semi-Markov chain immediately after the nth transition. The variable τ_n is the time of the nth transition with $\tau_0 = 0$. The arrivals of customers occur at the transition epochs of the semi-Markov process. Let k_n be the type of the customer associated with the nth transition. Define, for $k = 1, \dots, K$,

$$P\{\xi_{n+1} = j, \tau_{n+1} \le t, k_{n+1} = k \mid \xi_n = i\} = d_{k,i,j}(t), \qquad 1 \le i, j \le m,\ n \ge 0,$$

where t is a nonnegative real number. The function $d_{k,i,j}(t)$ is the probability that the next customer arrives within time t of the arrival of the preceding customer, the phase of the underlying semi-Markov process becomes j after the arrival and the next customer is of type k, given that the phase was initially i. We assume that $d_{k,i,j}(0) = 0$. Let $\boldsymbol{D}_k(t)$ be an $m \times m$ matrix with (i, j)th entry $d_{k,i,j}(t)$. The matrices $\{\boldsymbol{D}_k(t),\ 1 \le k \le K\}$ provide all the information about the semi-Markov arrival process. Let

$$\begin{aligned}\boldsymbol{D}(t) &= \boldsymbol{D}_1(t) + \cdots + \boldsymbol{D}_K(t),\\ \boldsymbol{D}_k &= \lim_{t\to\infty} \boldsymbol{D}_k(t),\\ \boldsymbol{D} &= \boldsymbol{D}_1 + \cdots + \boldsymbol{D}_K.\end{aligned}$$

The matrix $\boldsymbol{D}$ is the probability transition matrix of the embedded Markov chain of the semi-Markov process $\{(\xi_n, \tau_n),\ n \ge 0\}$ at transition epochs. We assume that $\boldsymbol{D}$ is irreducible. Let $\boldsymbol{\theta}_a$ be the probability invariant vector of $\boldsymbol{D}$, that is, $\boldsymbol{\theta}_a \boldsymbol{D} = \boldsymbol{\theta}_a$ and $\boldsymbol{\theta}_a \boldsymbol{e} = 1$. In steady state, the interarrival time of the semi-Markov process can be calculated as $E_{\boldsymbol{\theta}_a}(\tau) = \boldsymbol{\theta}_a \int_0^\infty t\boldsymbol{D}(\mathrm{d}t)\boldsymbol{e}$. The arrival rate of customers is given as $\lambda = (E_{\boldsymbol{\theta}_a}(\tau))^{-1}$, i.e. the average number of customers arriving per unit time. The arrival rate of type-k customers is given by $\lambda_k = \lambda \boldsymbol{\theta}_a \boldsymbol{D}_k \boldsymbol{e}$ for $k = 1, \dots, K$.

For a type-k customer, its service time s_k has a general distribution with mean $\mathrm{E}\, s_k = 1/\mu_k$ for $k = 1, \dots, K$. The service process and arrival process are independent. All customers, regardless of their types, join a single queue. The service discipline for all customers is last-come-first-served (LCFS) preemptive repeat. When a customer enters the system, it pushes the customer in service (if any) out of the server and begins its own service. When the server becomes available, the youngest customer in queue reenters the server and begins its service like a new customer.

Let $q(t)$ be the queue string at time t, that is, $q(t)$ records the types of customers in queue at time t and their order in queue. For instance, $q(t) = 2313$ means that there are 4 customers in the system at time t. The *youngest* customer, who is in the server, is of type 3. The second youngest is of type 1. The *oldest* customer is of type 2 and the second oldest is of type 3. Let $q_n = q(\tau_n-)$, i.e. the queue seen by the nth arriving customer. Let k_n be the type of the nth customer. Then $\{(q_n, k_n, \xi_n),\ n \ge 0\}$ is a Markov chain of matrix GI/M/1-type with a tree structure. The transition matrices of that Markov chain are given as follows: for $k = 1, \dots, K$,

$$\boldsymbol{A}(0, k) = \begin{pmatrix} \boldsymbol{0} \\ \displaystyle\int_0^\infty P\{s_k > t\}(\boldsymbol{D}_1(\mathrm{d}t), \dots, \boldsymbol{D}_K(\mathrm{d}t)) \\ \boldsymbol{0} \end{pmatrix},$$

$$\boldsymbol{A}(k + J, k) = \begin{pmatrix} \displaystyle\int_0^\infty P\{s_J + s_1 \le t < s_{k+J} + s_1\}(\boldsymbol{D}_1(\mathrm{d}t), \dots, \boldsymbol{D}_K(\mathrm{d}t)) \\ \vdots \\ \displaystyle\int_0^\infty P\{s_J + s_K \le t < s_{k+J} + s_K\}(\boldsymbol{D}_1(\mathrm{d}t), \dots, \boldsymbol{D}_K(\mathrm{d}t)) \end{pmatrix}, \qquad J \in \aleph,$$

$$\boldsymbol{A}(k + J, j) = \boldsymbol{0}, \qquad k \ne j,\ J \in \aleph,$$

and

$$\boldsymbol{B}(J) = \begin{pmatrix} \displaystyle\int_0^\infty \mathrm{P}\{s_J + s_1 \le t\}(\boldsymbol{D}_1(\mathrm{d}t), \dots, \boldsymbol{D}_K(\mathrm{d}t)) \\ \vdots \\ \displaystyle\int_0^\infty \mathrm{P}\{s_J + s_K \le t\}(\boldsymbol{D}_1(\mathrm{d}t), \dots, \boldsymbol{D}_K(\mathrm{d}t)) \end{pmatrix}, \qquad J \in \aleph,$$

where $s_J = \sum_{n=1}^{|J|} s_{j_n}$, the sum of the service times of customers in the queue $J = j_1 \dots j_{|J|}$. Note that in the above, for convenience, we write $\int_0^\infty f(t)(\boldsymbol{D}_1(\mathrm{d}t), \dots, \boldsymbol{D}_K(\mathrm{d}t))$ for the matrix $(\int_0^\infty f(t)\boldsymbol{D}_1(\mathrm{d}t), \dots, \int_0^\infty f(t)\boldsymbol{D}_K(\mathrm{d}t))$. It is easy to see that the condition (5.3) is satisfied. The condition (5.4) is satisfied because

$$\mathrm{P}\{s_{J+h} \le t\} = \mathrm{P}\{s_{k+J+h} \le t\} + \mathrm{P}\{s_{J+h} \le t < s_{k+J+h}\}, \qquad t > 0.$$

The results obtained in Section 4 can be applied to find whether or not the queueing system is stable. For some special cases, explicit stability conditions can be found. For instance, for the M[K]/M[K]/1 queue, where customers arrive according to K independent Poisson processes and all service times are exponentially distributed, it can be shown that the queueing system is stable if and only if $\rho := \lambda_1/\mu_1 + \cdots + \lambda_K/\mu_K < 1$. For general cases, we follow the procedure presented in Section 4 to do the stability check.

Example 6.1. We consider an SM[2]/G[2]/1/LCFS queue. Parameters of this queueing system are given as follows: $K = m = 2$, $\boldsymbol{D}_1(t) = 0.7\boldsymbol{D}(t)$, $\boldsymbol{D}_2(t) = 0.3\boldsymbol{D}(t)$ and

$$\boldsymbol{D}(t) = \begin{pmatrix} d_{1,1}(t) & d_{1,2}(t) \\ d_{2,1}(t) & d_{2,2}(t) \end{pmatrix},$$

where

$$d_{1,1}(t) = \begin{cases} 0, & t < 1, \\ 0.3, & t \ge 1, \end{cases} \qquad d_{1,2}(t) = \begin{cases} 0, & t < 2, \\ 0.7, & t \ge 2, \end{cases}$$

$$d_{2,1}(t) = \begin{cases} 0, & t < 5, \\ 1, & t \ge 5, \end{cases} \qquad d_{2,2}(t) = 0;$$

and

$$\mathrm{P}\{s_1 \le t\} = \begin{cases} 0, & t < 1.6, \\ 0.1, & 1.6 \le t < 2, \\ 0.9, & t \ge 2, \end{cases} \qquad \mathrm{P}\{s_2 \le t\} = \begin{cases} 0, & t < 2, \\ \mu, & 2 \le t < 5, \\ 1-\mu, & t \ge 5, \end{cases}$$

where μ is a parameter between 0 and 1. By routine calculations, we have $\boldsymbol{\theta}_a = (0.59, 0.41)$, $\lambda = 0.36$, $\lambda_1 = 0.25$, $\lambda_2 = 0.11$, $\mu_1 = 1/1.96$ and $\mu_2 = 1/(5-3\mu)$. Then the classical traffic intensity is given as $\rho = \lambda_1/\mu_1 + \lambda_2/\mu_2 = 0.25 \times 1.96 + 0.11(5 - 3\mu) = 1.04 - 0.33\mu$. For $\mu \in (0, 1)$, we have $1.04 > \rho > 0.71$.

For $\mu \in (0, 1)$, similar to Example 4.1, we compute ρ, $\mathrm{sp}(\boldsymbol{X}_{\min,1} + \cdots + \boldsymbol{X}_{\min,K})$ and $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}))$ to check the stability of the queueing system. The results are given in Table 2.

From Table 2, we learn that, if $\mu = 0.5$, then we have $\rho = 0.88$ but $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})) = 0.998$. Consequently, the queueing system is unstable. Thus, ρ does not provide accurate information

TABLE 2: Perron–Frobenius eigenvalues and ρ for Example 6.1.

μ	0.1	0.3	0.5	0.52	0.55	0.6	0.7	0.9
ρ	1.007	0.94	0.88	0.87	0.86	0.84	0.81	0.74
$\mathrm{sp}(\boldsymbol{X}_{\min,1}+\cdots+\boldsymbol{X}_{\min,K})$	1	1	1	0.956	0.885	0.786	0.640	0.475
$\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}))$	0.833	0.892	0.998	1.009	1.025	1.053	1.109	1.213

for system stability, which is a well-known fact. For $\mu < 0.5$, the queueing system is unstable and $\mathrm{sp}(\boldsymbol{X}_{\min,1}+\cdots+\boldsymbol{X}_{\min,K}) = 1$. For such cases, Table 2 shows clearly that $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) < 1$. Therefore, $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y}))$ is a better indicator than $\mathrm{sp}(\boldsymbol{X}_{\min,1}+\cdots+\boldsymbol{X}_{\min,K})$ for system (in)stability.

Appendix A. Equivalence of Neuts's condition and Theorem 4.1 when $K = 1$

If $K = 1$, then all strings have the form $J = 1\ldots1$. The Markov chain defined in Section 2 becomes a classical GI/M/1-type Markov chain. Let $\boldsymbol{A}(|J|) = \boldsymbol{A}(J, 1)$. Let $\boldsymbol{Y} = \{Y\}$ be a fixed point of $\mathcal{A}$ in $\mathcal{N}_m^1(1)$. By (3.1), $\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})$ becomes

$$\begin{aligned}
&\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})\\
&\quad=\sum_{n=1}^{\infty}[\boldsymbol{I}^{\top}\otimes(\boldsymbol{Y}^{n-1}\boldsymbol{A}(n))+(\boldsymbol{Y}^{1})^{\top}\otimes(\boldsymbol{Y}^{n-2}\boldsymbol{A}(n))+\cdots+(\boldsymbol{Y}^{n-1})^{\top}\otimes\boldsymbol{A}(n)]\\
&\quad=\sum_{n=1}^{\infty}[\boldsymbol{I}^{\top}\otimes\boldsymbol{Y}^{n-1}+(\boldsymbol{Y}^{1})^{\top}\otimes\boldsymbol{Y}^{n-2}+\cdots+(\boldsymbol{Y}^{n-1})^{\top}\otimes\boldsymbol{I}][\boldsymbol{I}\otimes\boldsymbol{A}(n)]. \qquad (\mathrm{A}.1)
\end{aligned}$$

Theorem 4.1 states that $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})) > 1$ (plus some other conditions) if and only if the Markov chain is positive recurrent. It is readily seen that $\boldsymbol{\theta}\boldsymbol{Y} = \boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is defined in Section 4. Let $\boldsymbol{u}$ be the right eigenvector of $\boldsymbol{Y}$ corresponding to the eigenvalue 1, i.e. $\boldsymbol{Y}\boldsymbol{u} = \boldsymbol{u}$. The vector $\boldsymbol{u}$ is normalized by $\boldsymbol{\theta}\boldsymbol{u} = 1$. *Neuts's condition* states that the Markov chain is positive recurrent if and only if $\boldsymbol{\theta}\sum_{n=0}^{\infty} n\boldsymbol{A}(n)\boldsymbol{e} > 1$. We now prove the equivalence of the two conditions for positive recurrence of the Markov chain of interest.

Premultiplying by $\boldsymbol{I}\otimes(\boldsymbol{I}-\boldsymbol{Y}+\boldsymbol{u}\boldsymbol{\theta})$ on both sides of (A.1) yields that

$$\begin{aligned}
&(\boldsymbol{I}\otimes(\boldsymbol{I}-\boldsymbol{Y}+\boldsymbol{u}\boldsymbol{\theta}))\mathcal{A}^{(1)}(\boldsymbol{Y},\boldsymbol{Y})\\
&\quad=(\boldsymbol{I}\otimes\boldsymbol{u})\bigg(\sum_{n=1}^{\infty}\bigg(\sum_{j=0}^{n-1}(\boldsymbol{Y}^{j})^{\top}\bigg)\otimes\boldsymbol{\theta}\boldsymbol{A}(n)\bigg)\\
&\qquad+\sum_{n=1}^{\infty}\bigg[\sum_{j=0}^{n-1}(\boldsymbol{Y}^{j})^{\top}\otimes\boldsymbol{Y}^{n-1-j}-\sum_{j=0}^{n-1}(\boldsymbol{Y}^{j})^{\top}\otimes\boldsymbol{Y}^{n-j}\bigg][\boldsymbol{I}\otimes\boldsymbol{A}(n)]. \qquad (\mathrm{A}.2)
\end{aligned}$$

Since $(\boldsymbol{I}-\boldsymbol{Y}+\boldsymbol{u}\boldsymbol{\theta})\boldsymbol{u} = \boldsymbol{u}$, we have

$$(\boldsymbol{I}-\boldsymbol{Y}+\boldsymbol{u}\boldsymbol{\theta})^{-1}\boldsymbol{u} = \boldsymbol{u}.$$

The invertibility of the matrix $\boldsymbol{I} - \boldsymbol{Y} + \boldsymbol{u\theta}$ can be proved routinely (see [9]). Premultiplying by $\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{Y} + \boldsymbol{u\theta})^{-1}$ on both sides of (A.2) yields that

$$\begin{aligned}\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y}) = (\boldsymbol{I} \otimes \boldsymbol{u})\bigg(\sum_{n=1}^{\infty}\bigg(\sum_{j=0}^{n-1}(\boldsymbol{Y}^j)^\top\bigg) \otimes (\boldsymbol{\theta A}(n))\bigg) \\ + (\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{Y} + \boldsymbol{u\theta})^{-1}) \\ \times \sum_{n=1}^{\infty}\bigg[\sum_{j=0}^{n-1}(\boldsymbol{Y}^j)^\top \otimes \boldsymbol{Y}^{n-1-j} - \sum_{j=0}^{n-1}(\boldsymbol{Y}^j)^\top \otimes \boldsymbol{Y}^{n-j}\bigg][\boldsymbol{I} \otimes \boldsymbol{A}(n)]. \qquad \text{(A.3)}\end{aligned}$$

Note that $\boldsymbol{Y}^\top\boldsymbol{\theta}^\top = \boldsymbol{\theta}^\top$ since $\boldsymbol{\theta} = \boldsymbol{\theta Y}$. Also note that $\boldsymbol{Y} = \sum_{n=0}^{\infty} \boldsymbol{Y}^n \boldsymbol{A}(n)$. Postmultiplying $\boldsymbol{\theta}^\top \otimes \boldsymbol{e}$ on both sides of (A.3) yields that

$$\begin{aligned}&\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})(\boldsymbol{\theta}^\top \otimes \boldsymbol{e}) \\ &\quad = \boldsymbol{\theta}^\top \otimes \boldsymbol{u}\bigg(\boldsymbol{\theta}\sum_{n=1}^{\infty} n\boldsymbol{A}(n)\boldsymbol{e}\bigg) \\ &\qquad + (\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{Y} + \boldsymbol{u\theta})^{-1})\sum_{n=1}^{\infty}[\boldsymbol{I} \otimes \boldsymbol{I} - \boldsymbol{I} \otimes \boldsymbol{Y}^n][\boldsymbol{I} \otimes \boldsymbol{A}(n)](\boldsymbol{\theta}^\top \otimes \boldsymbol{e}) \\ &\quad = \boldsymbol{\theta}^\top \otimes \boldsymbol{u}\bigg(\boldsymbol{\theta}\sum_{n=1}^{\infty} n\boldsymbol{A}(n)\boldsymbol{e}\bigg) + (\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{Y} + \boldsymbol{u\theta})^{-1})(\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{Y}))(\boldsymbol{\theta}^\top \otimes \boldsymbol{e}) \\ &\quad = \boldsymbol{\theta}^\top \otimes \boldsymbol{u}\bigg(\boldsymbol{\theta}\sum_{n=1}^{\infty} n\boldsymbol{A}(n)\boldsymbol{e}\bigg) + (\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{Y} + \boldsymbol{u\theta})^{-1})(\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{Y} + \boldsymbol{u\theta} - \boldsymbol{u\theta}))(\boldsymbol{\theta}^\top \otimes \boldsymbol{e}) \\ &\quad = \boldsymbol{\theta}^\top \otimes \boldsymbol{e} + (\boldsymbol{\theta}^\top \otimes \boldsymbol{u})\bigg\{\bigg(\boldsymbol{\theta}\sum_{n=1}^{\infty} n\boldsymbol{A}(n)\boldsymbol{e}\bigg) - 1\bigg\}. \qquad \text{(A.4)}\end{aligned}$$

If $\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})$ is irreducible, then (A.4) indicates that $\mathrm{sp}(\mathcal{A}^{(1)}(\boldsymbol{Y}, \boldsymbol{Y})) > 1$ if and only if $\boldsymbol{\theta}\sum_{n=0}^{\infty} n\boldsymbol{A}(n)\boldsymbol{e} > 1$. This completes the proof.

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