# A FIXED POINT APPROACH TO THE CLASSIFICATION OF MARKOV CHAINS WITH A TREE STRUCTURE

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## ABSTRACT

In this paper, we study the classification problem of discrete time and continuous time Markov processes with a tree structure. We first show some useful properties associated with the fixed points of a nondecreasing mapping. Mainly we find the conditions for a fixed point to be the minimal fixed point by using fixed point theory and degree theory. We then use these results to identify conditions for Markov chains of *M/G/*1 type or *GI/M/*1 type with a tree structure to be positive recurrent, null recurrent, or transient. The results are generalized to Markov chains of matrix *M/G/*1 type with a tree structure. For all these cases, a relationship between a certain fixed point, the matrix of partial differentiation (Jacobian) associated with the fixed point, and the classification of the Markov chain with a tree structure is established. More specifically, we show that the Perron-Frobenius eigenvalue of the matrix of partial differentiation associated with a certain fixed point provides information for a complete classification of the Markov chains of interest.

*Key words*: Markov chains, ergodicity, tree structure, matrix analytic methods, fixed point theory, degree theory

## 1. INTRODUCTION

This paper identifies conditions for the classification of Markov chains of *M/G/*1 type or *GI/M/*1 type with a tree structure, i.e., conditions for the Markov chains to be positive recurrent, null recurrent, or transient. A fixed point approach is utilized to solve the problem. While some of the classification conditions of Markov chains of *M/G/*1 type with a tree structure have already been obtained in HE [11, 12], the results for Markov chains of *GI/M*/1 type with a tree structure are all new. Geometric insight into all the classification conditions is gained. In addition, this paper presents an application of the fixed point theory in the study of Markov chains with some special structure.

Markov chains of *M/G/*1 type and *GI/M*/1 type with a tree structure were introduced in Takine, Sengupta, and Yeung [22] and Yeung and Sengupta [24] respectively, as generalizations of the classical Markov chains of *M/G/*1 type or *GI/M*/1 type (Neuts [18, 19]). Since the queueing processes of some queueing systems with multiple types of customers can be formulated into such Markov chains, their study attracted considerable attention from researchers in recent years (Gajrat, et al. [6], HE [10, 11, 12], HE and Alfa [13], Latouche and Ramaswami [14], Takine, Sengupta, and Yeung [22], Van Houdt and Blondia [23], and Yeung and Sengupta [24]). Similar Markov chains were also studied by other researchers under the name string Markov chains (Gajrat, et al. [6] and Malyshev [16, 17]). An interesting problem in this area is the classification of these Markov chains. While classification conditions have been found for some of them (Gail, et al. [4, 5], Gajrat, et al. [6], HE [11, 12], and Neuts [18, 19]), others (e.g., the *GI/M*/1 case) are still unsolved. Thus, there is a need for more study on the classification problem.

According to Foster's criterion and the mean drift method (Cohen [2] and Fayolle, et al. [3]), the existence of some special solution to some nonlinear equations or inequalities determines whether an irreducible Markov chain is positive recurrent, null recurrent, or transient, where these equations or inequalities are usually determined by the transition probabilities of the Markov chain. Quite often, finding conditions for the existence of some special solution is used to solve the classification problem of Markov chains. In this paper, we focus on a nonlinear equation $\mathbf{x} = \mathcal{A}(\mathbf{x})$, where $\mathcal{A}$ is a nonlinear mapping, and try to identify conditions for the existence of fixed points to that equation in order to solve the classification problem.

For Markov chains of *M/G/*1 type with a tree structure, the mapping $\mathcal{A}$ and the equation $\mathbf{x} = \mathcal{A}(\mathbf{x})$ are related to the absorption probabilities of some boundary states of the Markov chain of interest ([22]). For Markov chains of *GI/M*/1 type with a tree structure, the mapping $\mathcal{A}$ and the equation $\mathbf{x} = \mathcal{A}(\mathbf{x})$ are related to the subinvariant measure of the Markov chain of interest (Seneta [21] and Yeung and Sengupta [24]). In this paper, we take a fixed point approach (Goebel and Kirk [9]) to study $\mathcal{A}$ and the equation $\mathbf{x} = \mathcal{A}(\mathbf{x})$. Using Brouwer's fixed point theorem and degree theory (Garcia and Zangwill [8] and Lloyd [15]), a relationship between the classification of the Markov chains with a tree structure, the fixed points of $\mathcal{A}$, and the Perron-Frobenius eigenvalues of the matrices of partial differentiation (*Jacobians*) associated with these fixed points is established. This approach not only produces new results, but also leads to new insight into the solutions obtained as well as geometric explanations to the classification conditions identified. In addition, we incorporate matrix analytic methods, the mean drift method, and the subinvariant measure approach in the solution process (Cohen [2], Fayolle, et al. [3], Latouche and Ramaswami [14], Neuts [18, 19], and Seneta [21]).

For Markov chains of *M/G/*1 type with a tree structure (both scalar and matrix cases), HE [11, 12] proved that the Perron-Frobenius eigenvalue of a nonnegative matrix provides complete information for their classification. By using a different approach (fixed point theory) in this paper, the same results are obtained with a much better understanding on how the classification conditions are formulated. New results and new proofs are obtained as well. For Markov chains of *GI/M/*1 type with a tree structure, conditions for a complete classification are obtained for the first time by utilizing fixed point theory and subinvariant (invariant) measures of Markov chains. The fixed points of the mapping $\mathcal{A}$ play an important role in the construction of the classification conditions. In fact, without understanding the role played by fixed points, it would be otherwise difficult to find these classification conditions.

While the main contribution of this paper is about the classification conditions of the Markov chains of interest, the results about the fixed points of the mapping $\mathcal{A}$ are interesting by their own rights. Usually, in fixed point theory and degree theory, the Jacobian of $\mathcal{A}$ is utilized to find conditions for the existence of a fixed point or to calculate the topological degree. The determinant or the sign of the determinant of the Jacobian plays a central role. Instead of using the determinant of the Jacobian, in this paper, we make use of the Perron-Frobenius eigenvalue and its corresponding eigenvector of the Jacobian at the fixed point to identify other possible fixed points. This approach is different from other existing methods.

This paper focuses primarily on discrete time Markov processes with a tree structure. Thus, we shall use "Markov chain" for "discrete time Markov process" throughout this paper. But all the results can be extended to continuous time Markov processes of *M/G/*1 type or *GI/M/*1 type with a tree structure.

The rest of the paper is organized as follows. In Section 2, the mapping $\mathcal{A}$ is introduced and some useful properties are proved by using fixed point theory and degree theory. Section 3 identifies the classification conditions for Markov chains of *M/G/*1 type with a tree structure. We choose first to present the results for the *M/G/*1 type Markov chains with a tree structure because they are easier to deal with. In Section 4, for Markov chains of matrix *M/G/*1 type with a tree structure, classification conditions are identified. Section 5 characterizes the classification conditions of Markov chains of *GI/M/*1 type with a tree structure. Finally, in Section 6, we summarize the results obtained in this paper.

## 2. PROPERTIES OF THE MAPPING $\mathcal{A}: \mathcal{R}_+^K \to \mathcal{R}_+^K$

In this section, we introduce a mapping $\mathcal{A}$ that is associated with Markov chains of *M/G/*1 type or *GI/M/*1 type with a tree structure. We shall focus on the fixed points of $\mathcal{A}$, especially the minimal fixed point. The mapping $\mathcal{A}$ is defined in a general way in this section and will be made explicit in Sections 3 and 5, where a particular type of Markov chain is under consideration. The results obtained in this section lay the basis for the analysis in Sections 3 and 5.

We first introduce the domain of the mapping $\mathcal{A}$ and a set of strings of integers. Let $\mathcal{R}_+^K = \{\mathbf{x} = (x_1, x_2, \ldots, x_K)^T: x_k \geq 0, 1 \leq k \leq K\}$, where $K$ is a positive integer and "T" is for the transpose operation of matrix. It is easy to see that $\mathcal{R}_+^K$ is a convex cone and is usually called the nonnegative orthant of the vector space of dimension $K$ (see Rockafellar [20]). In this paper, we shall primarily work with the mapping $\mathcal{A}$ from $\mathcal{R}_+^K$ to $\mathcal{R}_+^K$ (except Section 4). For any two (column) vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{R}_+^K$, we say that $\mathbf{x}$ is less than $\mathbf{y}$ (denoted as $\mathbf{x} \leq \mathbf{y}$) if $x_k \leq y_k$ for $1 \leq k \leq K$. For any $\mathbf{y} \in \mathcal{R}_+^K$, define a polytope $\mathcal{R}_+^K(\mathbf{y})$ as:

$$\mathcal{R}_+^K(\mathbf{y}) = \{\mathbf{x}: \quad \mathbf{x} \leq \mathbf{y} \text{ and } \mathbf{x} \in \mathcal{R}_+^K\}. \tag{2.1}$$

The interior of $\mathcal{R}_+^K(\mathbf{y})$ is defined as int($\mathcal{R}_+^K(\mathbf{y})$) = $\{\mathbf{x}: \mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}) \text{ and } x_k < y_k, 1 \leq k \leq K\}$. For $K$=2, $\mathcal{R}_+^K(\mathbf{y})$ is the rectangle (polytope) determined by four points $\{(0, 0), (y_1, 0), (0, y_2), \mathbf{y} = (y_1, y_2)\}$. It is easy to see that $\mathcal{R}_+^K(\mathbf{y})$ is a convex set for any $\mathbf{y} \in \mathcal{R}_+^K$.

Next, we introduce a set of strings of integers. Let $\aleph = \{J: J = j_1 j_2 \cdots j_n, 1 \leq j_i \leq K, 1 \leq i \leq n, n > 0\} \cup \{0\}$. The length of a string $J$ in $\aleph$ is defined as the number of integers in the string and is denoted by $|J|$, except that $|J| = 0$ if $J = 0$. The following two operations related to strings in $\aleph$ are used in this paper.

*Addition operation*: for $J = j_1 \cdots j_n \in \aleph$ and $H = k_1 \cdots k_i \in \aleph$, then $J+H = j_1 \cdots j_n k_1 \cdots k_i \in \aleph$.
*Subtraction operation*: for $J = j_1 \cdots j_n \in \aleph$, $H = j_i \cdots j_n \in \aleph$, $i > 0$, then $J-H = j_1 \cdots j_{i-1} \in \aleph$, or
for $J = j_1 \cdots j_i \in \aleph$, $H = j_1 \cdots j_n \in \aleph$, $i > 0$, then $-J+H = j_{i+1} \cdots j_n \in \aleph$.

In order to define the mapping $\mathcal{A}$, we introduce a set of nonnegative real numbers associated with strings in $\aleph$. For every $J$ in $\aleph$ and $k$ ($1 \leq k \leq K$), a nonnegative real number $a(k, J)$ is defined with $0 \leq a(k, J) \leq 1$. We assume that at least one of $\{a(k, 0), 1 \leq k \leq K\}$ is positive. We also assume that for at least one pair $(k, J)$ with $|J| \geq 2$, $a(k, J) > 0$. The nonnegative real numbers $\{a(k, J), 1 \leq k \leq K \text{ and } J \in \aleph\}$ are related to transition probabilities of the Markov chains of interest and more explicit restrictions on $\{a(k, J), J \in \aleph \text{ and } 1 \leq k \leq K\}$ shall be imposed in Sections 3 and 5.

Let $N(J, k)$ be the number of times that the integer $k$ appears in the string $J$, for $1 \leq k \leq K$ and $J \in \aleph$. Then it is easy to see that $|J| = N(J, 1) + N(J, 2) + \cdots + N(J, K)$. Let $\mathbf{x}^{(J)} \equiv x_{j_1} x_{j_2} \cdots x_{j_{|J|}} = x_1^{N(J,1)} x_2^{N(J,2)} \cdots x_K^{N(J,K)}$ for any $J \in \aleph$ and $\mathbf{x} \in \mathcal{R}_+^K$. A function $a_k^*(\cdot): \mathcal{R}_+^K \to \mathcal{R}_+$, is defined as follows, for $1 \leq k \leq K$,

$$a_k^*(\mathbf{x}) \equiv \sum_{J \in \aleph} a(k, J) \mathbf{x}^{(J)} = \sum_{J \in \aleph} a(k, J) x_1^{N(J,1)} x_2^{N(J,2)} \cdots x_K^{N(J,K)}, \quad \text{for } \mathbf{x} \in \mathcal{R}_+^K. \tag{2.2}$$

The mapping $\mathcal{A}$: $\mathcal{R}_+^K \to \mathcal{R}_+^K$ is defined as $\mathcal{A}(\mathbf{x}) = (a_1^*(\mathbf{x}), a_2^*(\mathbf{x}), \cdots, a_K^*(\mathbf{x}))^{\mathrm{T}}$ for $\mathbf{x} \in \mathcal{R}_+^K$. It is easy to see that functions $\{a_k^*(\mathbf{x}), 1 \le k \le K\}$ are nondecreasing, nonnegative, and convex in their domain. Thus, for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{R}_+^K$ with $\mathbf{x} \le \mathbf{y}$, $\mathcal{A}(\mathbf{x}) \le \mathcal{A}(\mathbf{y})$, i.e., $\mathcal{A}$ is nondecreasing.

Let $\mathcal{F}_{ix}$ be the set of fixed points of the mapping $\mathcal{A}$, i.e.,

$$\mathcal{F}_{ix} = \{\mathbf{x}: \mathcal{A}(\mathbf{x}) = \mathbf{x} \text{ and } \mathbf{x} \in \mathcal{R}_+^K\}. \tag{2.3}$$

The mapping $\mathcal{A}$ and its fixed points play a central role in this paper. The fixed points of $\mathcal{A}$ and the Perron-Frobenius eigenvalues of Jacobians at these fixed points provide information for a classification of Markov chains with a tree structure. Thus, in the rest of this section, we study the mapping $\mathcal{A}$ and its fixed points (if exist), especially its minimal fixed point. The proofs in the following two sections are complicated. We suggest that readers use the cases with $K$=1 or $K$=2 to gain geometric intuition of the results and proofs.

## 2.1 The minimal fixed point $\mathbf{x}_{\min}$ of $\mathcal{A}$

We begin this subsection by showing the existence of a minimal fixed point in $\mathcal{F}_{ix}$ and finding its relationship with other fixed points of $\mathcal{A}$.

**Lemma 2.1** If $\mathcal{F}_{ix}$ is not empty, then $\mathcal{R}_+^K(\mathbf{y})$ is invariant under $\mathcal{A}$ for any $\mathbf{y} \in \mathcal{F}_{ix}$. If $\mathcal{F}_{ix}$ is not empty, then there exists a minimal fixed point $\mathbf{x}_{\min} \in \mathcal{F}_{ix}$ such that $\mathbf{x}_{\min} \le \mathbf{y}$ for any $\mathbf{y} \in \mathcal{F}_{ix}$.

**Proof**. If $\mathcal{F}_{ix}$ is not empty, we choose any fixed point $\mathbf{y} \in \mathcal{F}_{ix}$. Since the mapping $\mathcal{A}$ is nondecreasing, we have $\mathcal{A}(\mathbf{x}) \le \mathcal{A}(\mathbf{y}) = \mathbf{y}$ for any $\mathbf{x} \in \mathcal{R}_+^K(\mathbf{y})$. Therefore, the set $\mathcal{R}_+^K(\mathbf{y})$ is invariant under the mapping $\mathcal{A}$. This simple fact is used repeatedly in this paper.

Now, we construct a sequence of points in $\mathcal{R}_+^K(\mathbf{y})$ that converges to the minimal fixed point in $\mathcal{F}_{ix}$ if $\mathcal{F}_{ix}$ is not empty. Let $\mathbf{x}[0] = (0, 0, \ldots, 0)^{\mathrm{T}}$,

$$\mathbf{x}[n+1] = \mathcal{A}(\mathbf{x}[n]), \quad n \ge 0. \tag{2.4}$$

Since the mapping $\mathcal{A}$ is nondecreasing, the sequence $\{\mathbf{x}[n], n \ge 0\}$ is nondecreasing. For any fixed point $\mathbf{y}$, it is easy to see that $\mathbf{y} \ge \mathbf{x}[0]$. By induction, it can be proved that $\mathbf{y} \ge \mathbf{x}[n]$ for $n \ge 0$. Since the polytope $\mathcal{R}_+^K(\mathbf{y})$ is bounded, $\{\mathbf{x}[n], n \ge 0\}$ must converge to a point in $\mathcal{R}_+^K(\mathbf{y})$. Denote the limit as $\mathbf{x}_{\min}$. Then $\mathbf{x}_{\min} \le \mathbf{y}$. Note that $(\mathbf{x}[1])_k = a(k, 0)$, $1 \le k \le K$, which implies that $\mathbf{x}_{\min} \ne 0$. This completes the proof of Lemma 2.1.

Next, we explore further the relationship between the minimal fixed point $\mathbf{x}_{\min}$ and other fixed points if $\mathcal{F}_{ix}$ is not empty. For that purpose, we introduce the following nonnegative matrix $\mathcal{A}^{(1)}(\mathbf{x})$ for any $\mathbf{x} \in \mathcal{R}_+^K$. For any $\mathbf{x} \in \mathcal{R}_+^K$ and $1\le k, j\le K$, let

$$a_{k,j}^{*(1)}(\mathbf{x}) \equiv \left.\frac{\partial a_k^*(\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{x}} = \sum_{J\in\aleph,\ J\neq 0} N(J,j)a(k,J)x_1^{N(J,1)}\cdots x_j^{N(J,j)-1}\cdots x_K^{N(J,K)}. \tag{2.5}$$

Let $\mathcal{A}^{(1)}(\mathbf{x})$ be a $K\times K$ matrix with $(k, j)$th element $a_{k,j}^{*(1)}(\mathbf{x})$. The matrix $\mathcal{A}^{(1)}(\mathbf{x})$ is called the *Jacobian* of $\mathcal{A}$ at $\mathbf{x}$ (i.e., the matrix of partial differentiation). Apparently, every element of $\mathcal{A}^{(1)}(\mathbf{x})$ is nondecreasing with respect to every element of $\mathbf{x}$. Let $sp(\mathcal{A}^{(1)}(\mathbf{x}))$ be the Perron-Frobenius eigenvalue of the nonnegative matrix $\mathcal{A}^{(1)}(\mathbf{x})$ (i.e., the eigenvalue with the largest modulus). For more details about nonnegative matrix, we refer to Gantmacher [7].

**Lemma 2.2** Assume that $\mathcal{F}_{ix}$ is not empty. Consider any fixed point $\mathbf{y} \in \mathcal{F}_{ix}$. Assume that $\mathcal{A}^{(1)}(\mathbf{y})$ is finite and irreducible. If $sp(\mathcal{A}^{(1)}(\mathbf{y})) \le 1$, then $\mathbf{x}_{\min} = \mathbf{y}$. For this case, $\mathbf{x}_{\min} = \mathbf{y}$ is the only fixed point of the mapping $\mathcal{A}$ in $\mathcal{R}_+^K(\mathbf{y})$. If $sp(\mathcal{A}^{(1)}(\mathbf{y})) > 1$, then $\mathbf{x}_{\min} \le \mathbf{y}$ and $\mathbf{x}_{\min} \neq \mathbf{y}$. For this case, there are at least two fixed points in $\mathcal{R}_+^K(\mathbf{y})$. In addition, $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) \le 1$ is always true.

**Proof.** For any $\mathbf{y} \in \mathcal{F}_{ix}$, we focus on the polytope $\mathcal{R}_+^K(\mathbf{y})$, which is invariant under the continuous mapping $\mathcal{A}$. According to the well-known Brouwer's fixed point theorem (see Goebel and Kirk [9]), there is at least one fixed point of $\mathcal{A}$ in $\mathcal{R}_+^K(\mathbf{y})$. Since $\mathbf{y}$ is a fixed point, we would like to know under what conditions there are fixed points in $\mathcal{R}_+^K(\mathbf{y})$ other than $\mathbf{y}$. It turns out that the Jacobian $\mathcal{A}^{(1)}(\mathbf{y})$ of $\mathcal{A}$ at $\mathbf{y}$ provides complete information to that question.

First, we consider the case with $sp(\mathcal{A}^{(1)}(\mathbf{y})) < 1$. For any $\mathbf{x} \in \mathcal{R}_+^K(\mathbf{y})$, we have $\mathcal{A}^{(1)}(\mathbf{x}) \le \mathcal{A}^{(1)}(\mathbf{y})$. If $sp(\mathcal{A}^{(1)}(\mathbf{y})) < 1$, then $sp(\mathcal{A}^{(1)}(\mathbf{x})) < 1$ for all $\mathbf{x} \in \mathcal{R}_+^K(\mathbf{y})$. Suppose that there exists a fixed point $\mathbf{x}_1 \in \mathcal{R}_+^K(\mathbf{y})$ and $\mathbf{x}_1 \neq \mathbf{y}$. Consider the closed line segment between $\mathbf{x}_1$ and $\mathbf{y}$: $\mathbf{x}(t) = (1-t)\mathbf{x}_1 + t\mathbf{y}$, $0\le t\le 1$. Denote by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_K)$ the left eigenvector of $\mathcal{A}^{(1)}(\mathbf{y})$ with $\alpha\mathbf{e}=1$ that corresponds to the eigenvalue $sp(\mathcal{A}^{(1)}(\mathbf{y}))$, where $\mathbf{e}$ is the column vector of ones. If $\mathcal{A}^{(1)}(\mathbf{y})$ is irreducible, then $sp(\mathcal{A}^{(1)}(\mathbf{y}))$ is positive and the vector $\alpha$ is positive (i.e., every element of $\alpha$ is positive). Let

$$\beta(t) = \alpha\mathcal{A}(\mathbf{x}(t)), \quad 0 \le t \le 1. \tag{2.6}$$

It is easy to verify that $\beta(t)$ is differentiable for $0\le t\le 1$, $\beta(0) = \alpha\mathbf{x}_1$, $\beta(1) = \alpha\mathbf{y}$, and $\beta'(t) = \alpha\mathcal{A}^{(1)}(\mathbf{x}(t))(\mathbf{y}-\mathbf{x}_1)$. By the *mean-value theorem* in calculus, we must have $\beta(1) - \beta(0) = \beta'(\xi)(1-0)$ for some $\xi$ between 0 and 1 ($0<\xi<1$). That leads to (note that $\mathbf{y}\ge\mathbf{x}_1$)

$$\alpha(\mathbf{y}-\mathbf{x}_1) = \alpha\mathcal{A}^{(1)}(\mathbf{x}(\xi))(\mathbf{y}-\mathbf{x}_1) \le \alpha\mathcal{A}^{(1)}(\mathbf{y})(\mathbf{y}-\mathbf{x}_1) = sp(\mathcal{A}^{(1)}(\mathbf{y}))\alpha(\mathbf{y}-\mathbf{x}_1). \quad (2.7)$$

Since $\mathbf{y} - \mathbf{x}_1 \ge 0$ and $\mathbf{y} - \mathbf{x}_1 \ne 0$, we have $\alpha(\mathbf{y} - \mathbf{x}_1) > 0$. Then equation (2.7) leads to $sp(\mathcal{A}^{(1)}(\mathbf{y})) \ge 1$, which contradicts $sp(\mathcal{A}^{(1)}(\mathbf{y})) < 1$. Therefore, if $sp(\mathcal{A}^{(1)}(\mathbf{y})) < 1$, then $\mathbf{x}_{\min} = \mathbf{y}$, which implies that $\mathbf{x}_{\min} = \mathbf{y}$ is the only fixed point in $\mathcal{R}_+^K(\mathbf{y})$.

Second, we consider the case with $sp(\mathcal{A}^{(1)}(\mathbf{y})) = 1$. Since at least one string $J$ with $|J| \ge 2$ has $a(k, J) > 0$ for some $k$, we must have $\mathcal{A}^{(1)}(\mathbf{x}) \le \mathcal{A}^{(1)}(\mathbf{y})$ and $\mathcal{A}^{(1)}(\mathbf{x}) \ne \mathcal{A}^{(1)}(\mathbf{y})$ for any interior point $\mathbf{x}$ of $\mathcal{R}_+^K(\mathbf{y})$ (i.e., $\mathbf{x}$ with $x_k < y_k$ for $1 \le k \le K$). Since $\mathcal{A}^{(1)}(\mathbf{y})$ is irreducible, we must have $sp(\mathcal{A}^{(1)}(\mathbf{x})) < sp(\mathcal{A}^{(1)}(\mathbf{y})) = 1$ for $\mathbf{x}$ such that $x_k < y_k$ for all $1 \le k \le K$. Similar to the above case with $sp(\mathcal{A}^{(1)}(\mathbf{y})) < 1$, it can be proved that there is no fixed point in the interior of $\mathcal{R}_+^K(\mathbf{y})$. Now, suppose that there exists another fixed point $\mathbf{z} = (y_1, \ldots, y_i, z_{i+1}, \ldots, z_K)^T$ with $1 \le i < K$ and $z_k < y_k$ for $i+1 \le k \le K$. Then we consider a mapping $\underline{\mathcal{A}} = (a^*_{i+1}(\mathbf{x}), ..., a^*_K(\mathbf{x}))^T$ with $x_k = y_k$ for $1 \le k \le i$ and $z_k \le x_k \le y_k$ for $i+1 \le k \le K$, i.e., $\underline{\mathcal{A}}$ is a mapping from $\mathcal{R}_+^{K-i}$ to $\mathcal{R}_+^{K-i}$ with variables $\{x_k, i+1 \le k \le K\}$. The vector $(y_{i+1}, \ldots, y_K)^T$ is a fixed point of $\underline{\mathcal{A}}$. The vector $(z_{i+1}, \ldots, z_K)^T$ is a fixed point of $\underline{\mathcal{A}}$ in the interior of $\mathcal{R}_+^{K-i}((y_{i+1}, ..., y_K)^T)$. If we denote

$$\mathcal{A}^{(1)}(\mathbf{y}) = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad (2.8)$$

then the corresponding Jacobian of the mapping $\underline{\mathcal{A}}$ is the matrix $A_4$ at the point $\mathbf{y}$. Since $\mathcal{A}^{(1)}(\mathbf{y})$ is irreducible, we must have $sp(A_4) < sp(\mathcal{A}^{(1)}(\mathbf{y})) = 1$. Then by equation (2.7), we know that the mapping $\underline{\mathcal{A}}$ cannot have any fixed point in $\mathrm{int}(\mathcal{R}_+^{K-i}((y_{i+1}, ..., y_K)^T))$. This contradicts the assumption that $(z_{i+1}, \ldots, z_K)^T$ is an interior fixed point of $\underline{\mathcal{A}}$. Therefore, $\mathcal{A}$ does not have any fixed point that is less than $\mathbf{y}$.

Lastly, we consider the case with $sp(\mathcal{A}^{(1)}(\mathbf{y})) > 1$. We want to show that there is another fixed point in $\mathcal{R}_+^K(\mathbf{y})$. The idea is to cut an area around $\mathbf{y}$ in $\mathcal{R}_+^K(\mathbf{y})$ and show that there is a fixed point in the remaining area of $\mathcal{R}_+^K(\mathbf{y})$. The area to be cut from $\mathcal{R}_+^K(\mathbf{y})$ is defined as $\{\mathbf{x}: \mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}), \alpha\mathbf{x} > \alpha\mathbf{y} - \varepsilon\}$ for some positive $\varepsilon$. Note that the selection of the direction $\alpha$ is critical for the following proof. Then the area still under consideration is

$$\mathcal{R}_+^K(\mathbf{y}, \varepsilon) = \{\mathbf{x}: \mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}), \alpha\mathbf{x} \le \alpha\mathbf{y} - \varepsilon\}. \quad (2.9)$$

It is easy to see that if $\varepsilon$ is positive and small enough, both $\mathcal{R}_+^K(\mathbf{y}, \varepsilon)$ and $\mathcal{R}_+^K(\mathbf{y}) - \mathcal{R}_+^K(\mathbf{y}, \varepsilon)$ are not empty and convex. Figure 2.1 shows the relationship between all the subsets involved and explains geometrically why this approach works for $K$=2.

For any $\mathbf{x}_1$ in the subset $\{\mathbf{x}: \mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}), \alpha\mathbf{x} = \alpha\mathbf{y} - \varepsilon\}$, which is the common boundary of the removed area and the remaining area, we introduce the closed line segment $\mathbf{x}(t) = (1-t)\mathbf{x}_1 + t\mathbf{y}$, $0 \le t \le 1$, to link $\mathbf{x}_1$ and $\mathbf{y}$. Considering the function $\beta(t)$ defined in equation (2.6) and using the mean-value theorem, we obtain

$$\begin{aligned}\alpha\mathbf{y} - \alpha\mathcal{A}(\mathbf{x}_1) &= \alpha\mathcal{A}^{(1)}(\mathbf{x}(\xi))(\mathbf{y} - \mathbf{x}_1) \\ &= sp(\mathcal{A}^{(1)}(\mathbf{y}))\alpha(\mathbf{y} - \mathbf{x}_1) + \alpha\left[\mathcal{A}^{(1)}(\mathbf{x}(\xi)) - \mathcal{A}^{(1)}(\mathbf{y})\right](\mathbf{y} - \mathbf{x}_1).\end{aligned} \tag{2.10}$$

Equation (2.10) and $\alpha\mathbf{x}_1 = \alpha\mathbf{y} - \varepsilon$ lead to

$$\alpha\mathcal{A}(\mathbf{x}_1) = \alpha\mathbf{y} - \varepsilon - (sp(\mathcal{A}^{(1)}(\mathbf{y})) - 1)\varepsilon + \alpha\left[\mathcal{A}^{(1)}(\mathbf{y}) - \mathcal{A}^{(1)}(\mathbf{x}(\xi))\right](\mathbf{y} - \mathbf{x}_1). \tag{2.11}$$

We can choose small enough $\varepsilon$ so that $\alpha\left[\mathcal{A}^{(1)}(\mathbf{y}) - \mathcal{A}^{(1)}(\mathbf{x})\right] \le (sp(\mathcal{A}^{(1)}(\mathbf{y})) - 1)\alpha$ for all $\{\mathbf{x}: \mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}), \alpha\mathbf{x} \ge \alpha\mathbf{y} - \varepsilon\}$, since $sp(\mathcal{A}^{(1)}(\mathbf{y}))-1>0$, $\alpha$ is positive, and $\mathcal{A}^{(1)}(.)$ is continuous at $\mathbf{y}$. Then equation (2.11) leads to $\alpha\mathcal{A}(\mathbf{x}_1) \le \alpha\mathbf{y}-\varepsilon$, i.e., $\mathcal{A}(\mathbf{x}_1) \in \mathcal{R}_+^K(\mathbf{y}, \varepsilon)$ for any $\mathbf{x}_1$ in the border set $\{\mathbf{x}: \mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}), \alpha\mathbf{x} = \alpha\mathbf{y} - \varepsilon\}$.

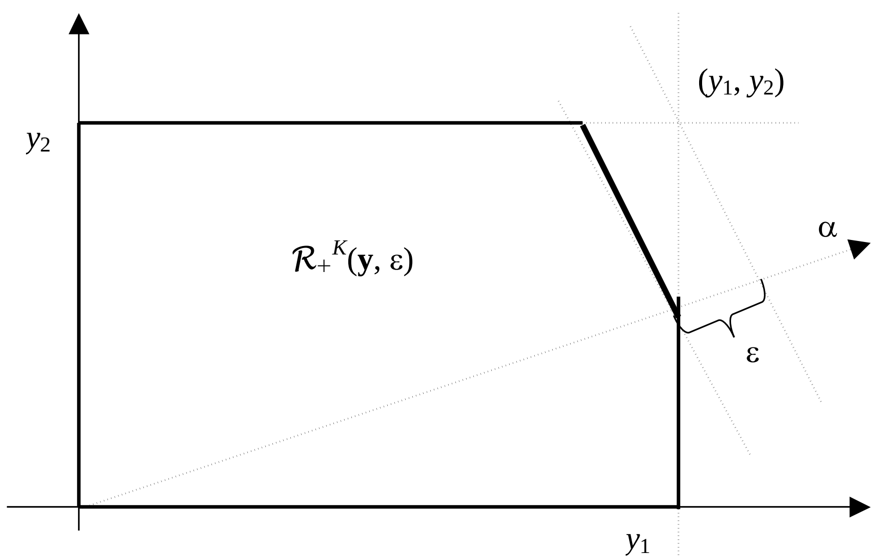


**Figure 2.1** The convex set $\mathcal{R}_+^K(\mathbf{y}, \varepsilon)$ when $K$=2

For any $\mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}, \varepsilon)$, consider the closed line segment between $\mathbf{x}$ and $\mathbf{y}$: $\mathbf{x}(t) = (1-t)\mathbf{x} + t\mathbf{y}$, $0 \le t \le 1$. The projection of $\mathbf{x}(t)$ on the direction $\alpha$ is given by $\alpha\mathbf{x}(t) = (1-t)\alpha\mathbf{x} + t\alpha\mathbf{y}$. Then we have $\alpha\mathbf{x}(0) = \alpha\mathbf{x} \le \alpha\mathbf{y}-\varepsilon$ and $\alpha\mathbf{x}(1) = \alpha\mathbf{y}$. Since $\alpha\mathbf{x}(t)$ is continuous in $t$, it is easy to see that there exists $t_1$ such that $0 \le t_1 \le 1$ and $\alpha\mathbf{x}(t_1) = \alpha\mathbf{y}-\varepsilon$, i.e., $\mathbf{x}(t_1) = (1-t_1)\mathbf{x} + t_1\mathbf{y}$ is in the border set $\{\mathbf{x}: \mathbf{x} \in \mathcal{R}_+^K(\mathbf{y}), \alpha\mathbf{x} = \alpha\mathbf{y} - \varepsilon\}$. Since $\mathbf{x} \le \mathbf{y}$, we have $\mathbf{x} \le \mathbf{x}(t_1)$, which implies that $\alpha\mathcal{A}(\mathbf{x}) \le \alpha\mathcal{A}(\mathbf{x}(t_1))$ and $\mathcal{A}(\mathbf{x}) \in \mathcal{R}_+^K(\mathbf{y}, \varepsilon)$. Therefore, $\mathcal{R}_+^K(\mathbf{y}, \varepsilon)$ is invariant under $\mathcal{A}$. By Brouwer's fixed point theorem (see [9]), there must be a fixed point in $\mathcal{R}_+^K(\mathbf{y},\varepsilon)$. Thus, the minimal fixed point must be in $\mathcal{R}_+^K(\mathbf{y},\varepsilon) \subset \mathcal{R}_+^K(\mathbf{y})$.

Since $sp(\mathcal{A}^{(1)}(\mathbf{y})) > 1$ for any fixed point $\mathbf{y}$ implies that there is another fixed point smaller than $\mathbf{y}$, we must have $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) \le 1$. This completes the proof of Lemma 2.2.

Lemma 2.2 shows that the Perron-Frobenius eigenvalue of $\mathcal{A}^{(1)}(\mathbf{x})$ is always equal to or less than one at the minimal fixed point (if exists), while it must be larger than 1 at all other fixed points (if exist). Intuitively, $sp(\mathcal{A}^{(1)}(\mathbf{y})) \le 1$ at a fixed point $\mathbf{y}$ implies that $\mathcal{A}(\mathbf{x})$ is not increasing faster than $\mathbf{x}$ around the fixed point $\mathbf{y}$ at least in the direction $\alpha$. Since $\mathcal{A}^{(1)}(\mathbf{x})$ is nondecreasing with respect to each variable $x_k$, it is clear that $\mathcal{A}(\mathbf{x})$ is not increasing faster than $\mathbf{x}$ in $\mathcal{R}_+^K(\mathbf{y})$. Since $\mathcal{A}(0) \ge 0$, then $\mathbf{y}$ is the first point that $\mathbf{x}$ catches up with $\mathcal{A}(\mathbf{x})$. Therefore, there is no fixed point within $\mathcal{R}_+^K(\mathbf{y})$. If $sp(\mathcal{A}^{(1)}(\mathbf{y})) > 1$ at the fixed point $\mathbf{y}$, $\mathcal{A}(\mathbf{x})$ is increasing faster than $\mathbf{x}$ around the fixed point $\mathbf{y}$ in at least one direction (i.e., $\alpha$). Thus, $\mathcal{A}(\mathbf{x})$ is catching up with $\mathbf{x}$ around $\mathbf{y}$ in $\mathcal{R}_+^K(\mathbf{y})$. Since $\mathcal{A}(0) \ge 0$, there must be a point within $\mathcal{R}_+^K(\mathbf{y})$ where $\mathcal{A}(\mathbf{x})$ falls behind $\mathbf{x}$, i.e., there is another fixed point within $\mathcal{R}_+^K(\mathbf{y})$.

**Note**: The irreducibility of $\mathcal{A}^{(1)}(\mathbf{y})$ has much to do with the irreducibility of the Markov chains considered in this paper, though they are not equivalent. The irreducibility of $\mathcal{A}^{(1)}(\mathbf{x})$ implies that the corresponding Markov chain can go from one type of node to any other types of nodes.

## 2.2 A larger fixed point if $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < 1$

Lemma 2.2 shows that $sp(\mathcal{A}^{(1)}(\mathbf{y})) > 1$ at any fixed point $\mathbf{y}$ implies that there is a smaller fixed point. In this subsection, we show that if $sp(\mathcal{A}^{(1)}(\mathbf{y})) < 1$ at any fixed point $\mathbf{y}$, then there is a larger fixed point. Since the minimal fixed point $\mathbf{x}_{\min}$ is nonzero and $\mathbf{x}_{\min}$ is the only possible fixed point with $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < 1$ (Lemma 2.2), we shall focus on $\mathbf{x}_{\min}$. We assume that the Jacobian $\mathcal{A}^{(1)}(\mathbf{x}_{\min})$ is irreducible.

Denote by $\mathbf{x}_{\min} + \mathcal{R}_+^K = \{\mathbf{y}: \mathbf{y} = \mathbf{x}_{\min} + \mathbf{x} \text{ for } \mathbf{x} \in \mathcal{R}_+^K\}$. We shall prove that there is another fixed point (other than $\mathbf{x}_{\min}$) in $\mathbf{x}_{\min} + \mathcal{R}_+^K$ if $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < 1$ and $\mathcal{A}^{(1)}(\mathbf{x}_{\min})$ is irreducible. The idea is to identify a subset in $\mathbf{x}_{\min} + \mathcal{R}_+^K$ that contains a fixed point and then to find the fixed point in the subset. For that purpose, let us consider the function

$$\beta_{\mathbf{d}}(t) = \alpha\mathcal{A}(\mathbf{x}_{\min} + t\mathbf{d}), \quad t \ge 0, \tag{2.12}$$

for any direction $\mathbf{d} \in \mathcal{R}_+^K$, which is normalized by $\alpha\mathbf{d} = 1$, where $\alpha$ is the left eigenvector of $\mathcal{A}^{(1)}(\mathbf{x}_{\min})$ corresponding to the eigenvalue $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min}))$. The vector $\alpha$ is normalized by $\alpha\mathbf{e} = 1$.

The function $\beta_{\mathbf{d}}(t)$ is the projection of the vector $\mathcal{A}(\mathbf{x}_{\min}+t\mathbf{d})$ on the direction $\alpha$. Since $\alpha$ is positive, the function $\beta_{\mathbf{d}}(t)$ is well-defined for any direction $\mathbf{d}$ in $\mathcal{R}_+^K$. It is easy to verify that

$$\beta_{\mathbf{d}}(0) = \alpha\mathcal{A}(\mathbf{x}_{\min}) = \alpha\mathbf{x}_{\min}; \tag{2.13}$$
$$\beta'_{\mathbf{d}}(t) = \alpha\mathcal{A}^{(1)}(\mathbf{x}_{\min}+t\mathbf{d})\mathbf{d}, \text{ and } \beta'_{\mathbf{d}}(0) = \alpha\mathcal{A}^{(1)}(\mathbf{x}_{\min})\mathbf{d} = sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < 1.$$

Since every element of $\mathcal{A}^{(1)}(\mathbf{x}_{\min}+t\mathbf{d})$ is a nondecreasing function of $t$, $\beta'_{\mathbf{d}}(t)$ is a nondecreasing function of $t$. Thus, $\beta_{\mathbf{d}}(t)$ is nondecreasing and convex. We also consider a linear function $l_{\mathbf{d}}(t) = \alpha(\mathbf{x}_{\min} + t\mathbf{d}) = \alpha\mathbf{x}_{\min} + t$, for $t\geq 0$. It is clear that

$$l_{\mathbf{d}}(0) = \alpha\mathbf{x}_{\min}, \quad l'_{\mathbf{d}}(t) = 1, \quad \text{and} \quad l'_{\mathbf{d}}(0) = 1. \tag{2.14}$$

The function $l_{\mathbf{d}}(t)$ is the projection of the vector $\mathbf{x}_{\min} + t\mathbf{d}$ on the direction $\alpha$. Comparing the functions $\beta_{\mathbf{d}}(t)$ and $l_{\mathbf{d}}(t)$, we know that they have the same value at $t = 0$. But $\beta_{\mathbf{d}}(t)$ is smaller than $l_{\mathbf{d}}(t)$ for small and positive $t$. If $\beta'_{\mathbf{d}}(t)$ is increasing to infinity as $t$ goes to infinity, then $\beta_{\mathbf{d}}(t)$ will eventually pass $l_{\mathbf{d}}(t)$ when $t$ goes to infinity. That implies that there is a finite and unique point $t_{\mathbf{d}}>0$ such that $\beta_{\mathbf{d}}(t_{\mathbf{d}}) = l_{\mathbf{d}}(t_{\mathbf{d}})$. Note that $t_{\mathbf{d}}$ can be infinite for some direction $\mathbf{d}$. Now, we define

$$\begin{aligned}\Omega_\alpha &= \{\mathbf{x}_{\min} + t_{\mathbf{d}}\mathbf{d}\colon\ \mathbf{d}\in\mathcal{R}_+^K,\ \alpha\mathbf{d} = 1,\ \beta_{\mathbf{d}}(t_{\mathbf{d}}) = l_{\mathbf{d}}(t_{\mathbf{d}})\}\\ &= \{\mathbf{x}_{\min} + \mathbf{x}\colon\ \alpha\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) = \alpha(\mathbf{x}_{\min}+\mathbf{x}) \text{ and } \mathbf{x}\in\mathcal{R}_+^K\}.\end{aligned} \tag{2.15}$$

Apparently, any fixed point in the set $\mathbf{x}_{\min} + \mathcal{R}_+^K$ must be in the set $\Omega_\alpha$. Thus, we shall focus on $\Omega_\alpha$ and identify possible fixed point(s) of $\mathcal{A}$ in the set $\Omega_\alpha$. Next, we introduce a subset of $\mathcal{R}_+^K$ that is associated with $\Omega_\alpha$. Let

$$\Gamma_\alpha = \{\mathbf{x}_{\min} + \mathbf{x}\colon\ \exists\ \mathbf{y} \text{ such that } \mathbf{x}_{\min} + \mathbf{y} \in\Omega_\alpha \text{ and } \mathbf{x} = t\mathbf{y} \text{ for some } 0\leq t\leq 1\}. \tag{2.16}$$

Intuitively, $\Gamma_\alpha$ is a convex hull generated by all the points in $\Omega_\alpha$ and $\mathbf{x}_{\min}$. In other words, $\Gamma_\alpha$ is obtained by cutting the convex cone $\mathbf{x}_{\min} + \mathcal{R}_+^K$ by $\Omega_\alpha$. We have the following result about $\Gamma_\alpha$.

**Lemma 2.3** For any point $\mathbf{x}_{\min} + \mathbf{x} \in\Gamma_\alpha$, $\alpha\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) \leq \alpha(\mathbf{x}_{\min}+\mathbf{x})$. The set $\Gamma_\alpha$ is convex.

**Proof**. For any $\mathbf{x}_{\min} + \mathbf{x} \in\Gamma_\alpha$, we must have $\mathbf{x} = t\mathbf{d}$ for some $\mathbf{d}\in\mathcal{R}_+^K$ with $\alpha\mathbf{d} = 1$ and some $0\leq t\leq t_{\mathbf{d}}$. By definition, $\alpha\mathcal{A}(\mathbf{x}_{\min}+t\mathbf{d}) = \beta_{\mathbf{d}}(t) \leq l_{\mathbf{d}}(t) = \alpha\mathbf{x}_{\min} + t = \alpha(\mathbf{x}_{\min}+ t\mathbf{d})$.

Note that the function $\alpha\mathcal{A}(\mathbf{x})$ is a convex function of $\mathbf{x}$. For any two points $\mathbf{x}_{\min}+\mathbf{x}$ and $\mathbf{x}_{\min}+\mathbf{z}$ in $\Gamma_\alpha$ and $0<\lambda<1$,

$$\begin{aligned}\alpha\mathcal{A}(\lambda(\mathbf{x}_{\min}+\mathbf{x})+(1-\lambda)(\mathbf{x}_{\min}+\mathbf{z})) &\leq \lambda\alpha\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x})+(1-\lambda)\alpha\mathcal{A}(\mathbf{x}_{\min}+\mathbf{z})\\ &\leq \lambda\alpha(\mathbf{x}_{\min}+\mathbf{x})+(1-\lambda)\alpha(\mathbf{x}_{\min}+\mathbf{z})\\ &= \alpha\left[\lambda(\mathbf{x}_{\min}+\mathbf{x})+(1-\lambda)(\mathbf{x}_{\min}+\mathbf{z})\right].\end{aligned} \tag{2.17}$$

Since $\mathbf{x}_{min}+\mathbf{x}$ and $\mathbf{x}_{min}+\mathbf{z}$ are in $\Omega_\alpha$, it can be shown that, in the direction $\lambda(\mathbf{x}_{min}+\mathbf{x})+(1-\lambda)(\mathbf{x}_{min}+\mathbf{z})$, there is a finite point in $\Omega_\alpha$. Thus, $\lambda(\mathbf{x}_{min}+\mathbf{x})+(1-\lambda)(\mathbf{x}_{min}+\mathbf{z})$ is in $\Gamma_\alpha$. Therefore, $\Gamma_\alpha$ is a convex set. This completes the proof of Lemma 2.3.

**Lemma 2.4** For any point $\mathbf{x}_{min} + \mathbf{x} \in \Omega_\alpha$, either $\mathcal{A}(\mathbf{x}_{min}+\mathbf{x}) - (\mathbf{x}_{min}+ \mathbf{x}) = 0$ holds or the vector $\mathcal{A}(\mathbf{x}_{min} +\mathbf{x}) - (\mathbf{x}_{min} + \mathbf{x})$ is perpendicular to $\alpha$.

**Proof**. Suppose that $\mathbf{x}_{min} + \mathbf{x} \in\Omega_\alpha$ and $\mathcal{A}(\mathbf{x}_{min} + \mathbf{x}) \neq \mathbf{x}_{min}+ \mathbf{x}$. The vector $\mathcal{A}(\mathbf{x}_{min} + \mathbf{x}) - (\mathbf{x}_{min}+ \mathbf{x})$ is not zero. By definition, $\alpha\mathcal{A}(\mathbf{x}_{min} + \mathbf{x}) = \alpha(\mathbf{x}_{min}+ \mathbf{x})$ if $\mathbf{x}_{min}+ \mathbf{x} \in\Omega_\alpha$. Then $\alpha[\mathcal{A}(\mathbf{x}_{min} + \mathbf{x}) - (\mathbf{x}_{min} + \mathbf{x})] = 0$, i.e., $\mathcal{A}(\mathbf{x}_{min} + \mathbf{x}) - (\mathbf{x}_{min} + \mathbf{x})$ is perpendicular to $\alpha$. This completes the proof of Lemma 2.4.

Let $S_\alpha = \{\mathbf{x}: \alpha\mathbf{x} = 0\}$, i.e., the hyperplane that is perpendicular to $\alpha$. Note that $S_\alpha$ is a subspace of the dimension $K$–1. By Lemma 2.4, $\mathcal{A}(\mathbf{x}_{min} + \mathbf{x})$ is in the affine set $\mathbf{x}_{min} + \mathbf{x} + S_\alpha$ for any $\mathbf{x}_{min} + \mathbf{x} \in\Omega_\alpha$.

Let $\underline{t} = \inf\{t_{\mathbf{d}}: \mathbf{d}\in \mathcal{R}_+^K\}$. Next, we show that $\underline{t}>0$, i.e., $\mathbf{x}_{min}$ is not in $\Omega_\alpha$ and $\mathbf{x}_{min}$ is not the limit of any convergent sequence in $\Omega_\alpha$.

**Lemma 2.5** The constant $\underline{t}$ is positive, i.e., $\underline{t} > 0$.

**Proof**. If $\underline{t} = 0$, then there exists a sequence $\{\mathbf{d}(n), n\geq 0\}$ such that $t_{\mathbf{d}(n)} \to 0$ when $n \to \infty$. Since $\mathbf{d}(n) \in \mathcal{R}_+^K$ with $\alpha\mathbf{d}(n) = 1$, the sequence $\{\mathbf{d}(n), n\geq 0\}$ must have a convergent subsequence. Denote the limit as $\mathbf{d}(\infty)$. Since $\beta'_{\mathbf{d}(\infty)}(0) = sp(\mathcal{A}^{(1)}(\mathbf{x}_{min})) < 1$, we must have that $t_{\mathbf{d}(\infty)} > 0$, which contradicts the fact that $t_{\mathbf{d}(n)} \to 0$ when $n \to \infty$. This completes the proof of Lemma 2.5.

To show that there is a fixed point of $\mathcal{A}$ in $\Omega_\alpha$, we consider the boundary of $\Omega_\alpha$. This is a typical method used in fixed point theory. Let $\partial\Omega_\alpha = \{\mathbf{x}_{min} + \mathbf{x}: \mathbf{x}_{min} + \mathbf{x} \in\Omega_\alpha$ with at least one $x_k = 0\}$, i.e., the boundary set of $\Omega_\alpha$. For convenience, we shall use $\Omega_\alpha - \mathbf{x}_{min}$ for $\{\mathbf{x}: \mathbf{x}_{min} + \mathbf{x} \in\Omega_\alpha\}$ and $\partial\Omega_\alpha - \mathbf{x}_{min}$ for $\{\mathbf{x}: \mathbf{x}_{min} + \mathbf{x} \in\Omega_\alpha$ with at least one $x_k = 0\}$.

In order to identify a fixed point of the mapping $\mathcal{A}$ on $\Omega_\alpha$, we need to know the topological degree of the mapping $\mathcal{A}(\mathbf{x}_{min}+\mathbf{x})-(\mathbf{x}_{min}+\mathbf{x})$ at the vector zero on the set $\Omega_\alpha - \mathbf{x}_{min}$. Consider a mapping $U$: $\Omega_\alpha - \mathbf{x}_{min} \to S_\alpha$ (satisfying certain conditions). We shall denote the topological degree of the mapping $U$ at the vector zero on the set $\Omega_\alpha - \mathbf{x}_{min}$ as $r(U, \Omega_\alpha - \mathbf{x}_{min}, 0)$. Let $U^{-1}(0) = \{\mathbf{x} \in \Omega_\alpha - \mathbf{x}_{min}: U(\mathbf{x}) = 0\}$. Then the topological degree of $U$ at the vector zero is defined as the sum of the signs of the determinants of the Jacobians at the points in $U^{-1}(0)$, i.e.,

$$r(U, \Omega_\alpha - \mathbf{x}_{min}, 0) = \sum_{\mathbf{x}\in U^{-1}(0)} sign(\det(U^{(1)}(\mathbf{x}))), \qquad (2.18)$$

where det(·) is the determinant of a matrix and $sign(x) = 1$ if $x>0$, 0 if $x=0$; –1 if $x<0$. We refer to Garcia and Zangwill [8] and Lloyd [15] for more details about the topological degree of a mapping. In general, if the topological degree is nonzero, $U$ has at least one zero point in its domain. The following lemma is the key in identifying the fixed point of the mapping $\mathcal{A}$ in $\Omega_\alpha$.

Let $t^* = \sup\{t_{\mathbf{d}}:\ \mathbf{d}\in \mathcal{R}_+^K\}$, which is attained at direction $\mathbf{d}^*$ and can be infinite. By definition, we have

$$\mathbf{x}_{\min} + t^*\mathbf{d}^* = \arg\sup\{\alpha(\mathbf{x}_{\min} + \mathbf{x}):\quad \mathbf{x}_{\min} + \mathbf{x} \in \Omega_\alpha\}. \tag{2.19}$$

**Lemma 2.6** Assume that $t^*$ is finite. Then $\mathcal{A}(\mathbf{x})$ has at least one fixed point in $\Omega_\alpha$. If $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x})-(\mathbf{x}_{\min}+\mathbf{x})$ is nonzero for $\mathbf{x}$ in the boundary set $\partial\Omega_\alpha - \mathbf{x}_{\min}$, the topological degree of the mapping $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x})-(\mathbf{x}_{\min}+\mathbf{x})$ at the vector zero on the set $\Omega_\alpha - \mathbf{x}_{\min}$ is nonzero.

**Proof**. First, we consider a special case $K$=2 (see Figure 2.2). According to Lemma 2.3, $\Gamma_\alpha$ is a convex set. Then $\Omega_\alpha$ is a convex set (surface) as well. If $K$=2, the set $\partial\Omega_\alpha$ has two points $\{\mathbf{x}_{\min}+\mathbf{x}_{(1,0)}, \mathbf{x}_{\min}+\mathbf{x}_{(0,1)}\}$, where $\mathbf{x}_{\min}+\mathbf{x}_{(1,0)} = (x_{\min,1}+y_1, x_{\min,2})$ and $\mathbf{x}_{\min}+\mathbf{x}_{(0,1)} = (x_{\min,1}, x_{\min,2}+y_2)$ with $y_1>0$ and $y_2>0$. The hyperplane $S_\alpha$ has one dimension. Denote by $\mathbf{u} = (u_1, u_2)$ the only nonzero direction in $S_\alpha$. Since $\alpha\mathbf{u} = 0$, we must have $u_1u_2 < 0$. Assume that $u_1 < 0$ and $u_2 > 0$. By the definition of $\Omega_\alpha$, for any $\mathbf{x}_{\min}+\mathbf{x}$ in $\Omega_\alpha$, we must have $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) = (\mathbf{x}_{\min}+\mathbf{x}) + s_{\mathbf{x}}\mathbf{u}$ for some real number $s_{\mathbf{x}}$. Suppose that $\mathcal{A}(x_{\min,1}+y_1, x_{\min,2}) = (\mathbf{x}_{\min}+\mathbf{x}_{(1,0)}) + s_1\mathbf{u}$ and $\mathcal{A}(x_{\min,1}, x_{\min,2}+y_2) = (\mathbf{x}_{\min}+\mathbf{x}_{(0,1)}) + s_2\mathbf{u}$. If $s_1$ or $s_2$ is zero, we already find a new fixed point of the mapping $\mathcal{A}$. Otherwise, since the set $\mathbf{x}_{\min} + \mathcal{R}_+^K$ is invariant under $\mathcal{A}$, it is clear that $s_1 > 0$ and $s_2 < 0$. Since $\mathcal{A}$ is a continuous mapping on $\Omega_\alpha$, $s_{\mathbf{x}}$ must be a continuous function of $\mathbf{x}$. Therefore, there must be a point $\mathbf{x}_{\min}+\mathbf{x}$ on $\Omega_\alpha$ such that $s_{\mathbf{x}} = 0$, i.e., $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) = (\mathbf{x}_{\min}+\mathbf{x})$. This completes the proof of the case $K$=2.

The key in the above proof is that the mapping $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x})-(\mathbf{x}_{\min}+\mathbf{x}) \equiv U_1(\mathbf{x})$, which is well-defined on $\Omega_\alpha - \mathbf{x}_{\min}$ with its image in the subspace $S_\alpha$, has a topological degree of 1 or –1 at the vector zero on the set $\Omega_\alpha - \mathbf{x}_{\min}$, i.e. $r(U_1, \Omega_\alpha-\mathbf{x}_{\min}, 0) = 1$ or –1. Thus, according to degree theory, $U_1$ must have a zero point on $\Omega_\alpha - \mathbf{x}_{\min}$ ($U_1(\mathbf{x}) = 0$), i.e., $\mathcal{A}$ has a fixed point on $\Omega_\alpha$. To extend the results to the general case $K>2$, the main task is to show that the topological degree of $U_1$ is nonzero (possibly 1 or –1).

For any point $\mathbf{x}_{\min}+\mathbf{x}$ in $\partial\Omega_\alpha$, we denote its projection on the linear line $\mathbf{x}_{\min}+t\alpha^T$ as $\mathbf{x}_{\min}+t_{\mathbf{x},\alpha}\alpha^T$, where $t_{\mathbf{x},\alpha}= \alpha\mathbf{x}/(\alpha\alpha^T)$. Consider the vector $\mathbf{x}_{\min}+t_{\mathbf{x},\alpha}\alpha^T - (\mathbf{x}_{\min}+\mathbf{x}) = t_{\mathbf{x},\alpha}\alpha^T - \mathbf{x}$. Apparently, the vector $t_{\mathbf{x},\alpha}\alpha^T - \mathbf{x}$ is in $S_\alpha$. We introduce a mapping $U_2(\mathbf{x}) = t_{\mathbf{x},\alpha}\alpha^T - \mathbf{x}$: $\Omega_\alpha-\mathbf{x}_{\min} \to S_\alpha$. Choose a special direction $\mathbf{h} = \alpha^T/(\alpha\alpha^T)$ and $\mathbf{z} = t_{\mathbf{h}}\mathbf{h}$. Then it is easy to see that $t_{\mathbf{z},\alpha} = t_{\mathbf{h}}/(\alpha\alpha^T)$ and $U_2(\mathbf{z}) = t_{\mathbf{z},\alpha}\alpha^T - \mathbf{z} = 0$. So, $\mathbf{z}$ is the point where a vector penetrates the set $\Omega_\alpha-\mathbf{x}_{\min}$ in the direction $\alpha$. Therefore, the mapping $U_2$ has one and only one zero point on $\Omega_\alpha-\mathbf{x}_{\min}$ since $\Gamma_\alpha$ is convex. Since $\Omega_\alpha-\mathbf{x}_{\min}$ is a connected area, then the topological degree of $U_2$ at the vector zero on $\Omega_\alpha-\mathbf{x}_{\min}$ must be either 1 or –1, i.e., $r(U_2, \Omega_\alpha-\mathbf{x}_{\min}, 0) = 1$ or –1. Intuitively, the vector $\alpha$ is positive so that $\partial\Omega_\alpha - \mathbf{x}_{\min}$ touches every hyperplane of $\mathcal{R}_+^K$. Since the set $\Gamma_\alpha$ is convex, if $\mathbf{x}$ goes around $\partial\Omega_\alpha - \mathbf{x}_{\min}$, $t_{\mathbf{x},\alpha}\alpha^T - \mathbf{x}$ will circle the direction $\alpha^T$ exactly once. Thus, the topological degree of $U_2$ is either 1 or –1.

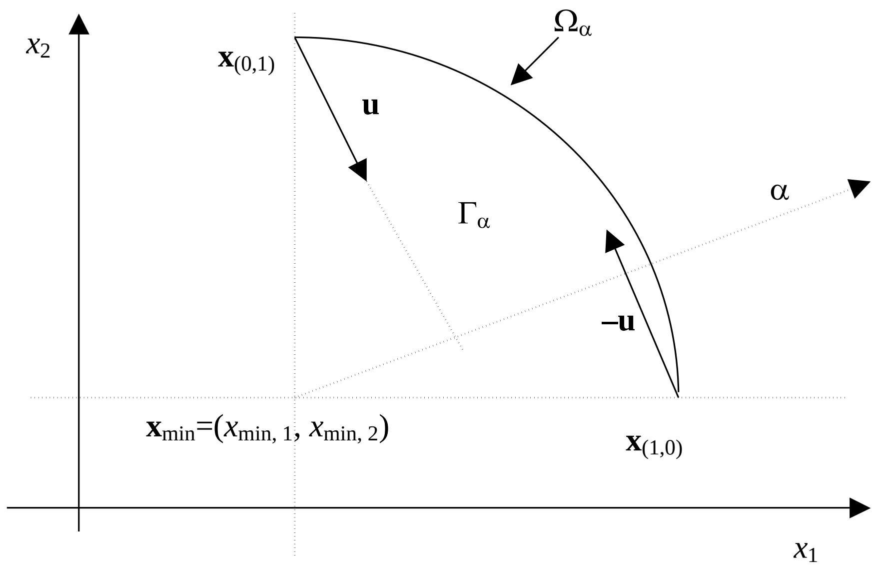


**Figure 2.2** The set $\Omega_\alpha$ when $K$=2

Now, we go back to the mapping $U_1(\mathbf{x}) = \mathcal{A}(\mathbf{x}_{\min}+\mathbf{x})-(\mathbf{x}_{\min}+\mathbf{x})$: $\Omega_\alpha - \mathbf{x}_{\min} \to S_\alpha$. We show that the mappings $U_1$ and $U_2$ are homotopic. Consider the homotopy $h(\lambda, \mathbf{x}) = \lambda U_1(\mathbf{x}) + (1-\lambda)U_2(\mathbf{x})$: $[0, 1]\times(\Omega_\alpha - \mathbf{x}_{\min}) \to S_\alpha$. It is clear that $h(\lambda, \mathbf{x})$ is continuous and $h(0, \mathbf{x}) = U_2(\mathbf{x})$ and $h(1, \mathbf{x}) = U_1(\mathbf{x})$. Also, we claim that $h(\lambda, \mathbf{x})$ has no zero point on $[0, 1]\times(\partial\Omega_\alpha - \mathbf{x}_{\min})$. If the claim is not true, we consider three cases: $\lambda\in(0, 1)$; $\lambda=0$; and $\lambda=1$. If $\lambda\in(0, 1)$, suppose that there exists $(\lambda, \mathbf{x})\in(0, 1)\times(\partial\Omega_\alpha - \mathbf{x}_{\min})$ such that $0 = h(\lambda, \mathbf{x}) = \lambda U_1(\mathbf{x})+(1-\lambda)U_2(\mathbf{x})$. Then $U_1(\mathbf{x}) = -(1/\lambda-1)U_2(\mathbf{x})$, which leads to

$$\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) = \mathbf{x}_{\min}+\mathbf{x} - (1/\lambda - 1)(t_{\mathbf{x}}\alpha^T - \mathbf{x}). \tag{2.20}$$

If $\mathbf{x} \in \partial\Omega_\alpha - \mathbf{x}_{\min}$, then at least one of the elements of $\mathbf{x}$ is zero. Since $\alpha$ is positive and $\lambda$ is less than 1, equation (2.20) implies that $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) \notin \mathbf{x}_{\min}+\mathcal{R}_+^K$, which contradicts the fact that $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) \in \mathbf{x}_{\min}+\mathcal{R}_+^K$. For instance, if $x_1 = 0$, then the first element of the vector $\mathbf{x} - (1/\lambda-1)(t_{\mathbf{x}}\alpha^T - \mathbf{x})$ is the first element of $-(1/\lambda - 1)t_{\mathbf{x}}\alpha^T$, which is negative.

If $\lambda = 0$, $h(0, \mathbf{x})$ is nonzero since $U_2(\mathbf{x})$ is nonzero on $\partial\Omega_\alpha - \mathbf{x}_{\min}$. If $\lambda = 1$, $h(1, \mathbf{x})$ is nonzero since $U_1(\mathbf{x})$ is nonzero on $\partial\Omega_\alpha - \mathbf{x}_{\min}$; otherwise, we have found a larger fixed point of $\mathcal{A}$

on the boundary set. Therefore, we have proved that the homotopy $h(\lambda, \mathbf{x})$ is nonzero on $[0, 1]\times(\partial\Omega_\alpha-\mathbf{x}_{\min})$ if $\mathcal{A}$ is nonzero on $\partial\Omega_\alpha$.

According to degree theory, $U_1(\mathbf{x})$ and $U_2(\mathbf{x})$ must have the same topological degree at the vector zero on $\Omega_\alpha-\mathbf{x}_{\min}$ (with their image in $S_\alpha$), i.e., $r(U_2, \Omega_\alpha-\mathbf{x}_{\min}, 0) = r(U_1, \Omega_\alpha-\mathbf{x}_{\min}, 0) \neq 0$. Then the mapping $U_1$ has at least one zero point on $\Omega_\alpha-\mathbf{x}_{\min}$. Since $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x}) = \mathbf{x}_{\min}+\mathbf{x} + U_1(\mathbf{x})$ for $\mathbf{x} \in\Omega_\alpha$, it is clear that $\mathcal{A}$ has at least one fixed point on $\Omega_\alpha$. This completes the proof of Lemma 2.6.

Intuitively, the mapping $\mathcal{A}$ becomes zero at every point in $\Omega_\alpha$ in the one-dimensional subspace $\{t\alpha, -\infty< t < \infty\}$. The mapping $\mathcal{A}$ becomes zero in the subspace $S_\alpha$ at a point in $\Omega_\alpha$ where the mapping $U_1$ becomes zero. Since the product space $S_\alpha\oplus\{t\alpha, -\infty< t < \infty\}$ is the whole space, we know that $\mathcal{A}$ becomes zero in $\mathbf{x}_{\min}+\mathcal{R}_+^K$ at that point in $\Omega_\alpha$.

**Lemma 2.7** Assume that $\mathcal{A}^{(1)}(\mathbf{x}_{\min})$ is irreducible. Then $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < 1$ if and only if there exists another fixed point $\mathbf{z} \in\mathcal{F}_{ix}$ such that $\mathbf{z} \geq \mathbf{x}_{\min}$ and $\mathbf{z} \neq \mathbf{x}_{\min}$.

**Proof**. It is easy to see that the conclusions hold if $K$=1. Next, we consider cases with $K$>1.

If there exists a fixed point $\mathbf{z} \in\mathcal{F}_{ix}$ such that $\mathbf{z} \geq \mathbf{x}_{\min}$ and $\mathbf{z} \neq \mathbf{x}_{\min}$, then we consider the closed line segment $\mathbf{x}(t) = (1-t)\mathbf{x}_{\min} + t\mathbf{z}$, $0\leq t\leq 1$ and $\beta(t) = \mathbf{v}\mathcal{A}(\mathbf{x}(t))$, where $\mathbf{v}$ is a nonnegative vector to be determined. By the mean-value theorem, we have $\mathbf{v}(\mathbf{z}-\mathbf{x}_{\min}) = \beta(1) - \beta(0) = \beta'(\xi) = \mathbf{v}\mathcal{A}^{(1)}(\mathbf{x}(\xi))(\mathbf{z}-\mathbf{x}_{\min})$ for some $\xi$ between 0 and 1. Choose $\mathbf{v}$ to be the eigenvector of $\mathcal{A}^{(1)}(\mathbf{x}(\xi))$ corresponding to the eigenvalue $sp(\mathcal{A}^{(1)}(\mathbf{x}(\xi)))$. The matrix $\mathcal{A}^{(1)}(\mathbf{x}(\xi))$ is irreducible since $\mathcal{A}^{(1)}(\mathbf{x}(\xi)) \geq \mathcal{A}^{(1)}(\mathbf{x}_{\min})$. Then $\mathbf{v}$ is positive, which implies that $sp(\mathcal{A}^{(1)}(\mathbf{x}(\xi))) = 1$. If $\mathbf{y}$ is in the interior of $\mathcal{R}_+^K(\mathbf{z})$, then we have $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < sp(\mathcal{A}^{(1)}(\mathbf{x}(\xi))) = 1$. Otherwise, some of the components of $\mathbf{z}$ equal their corresponding components in $\mathbf{x}_{\min}$. This case can be dealt with in a way similar to the $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) = 1$ case in Lemma 2.2.

Now, assume that $sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < 1$. We consider the set $\mathbf{x}_{\min} + \mathcal{R}_+^K$. We distinguish two cases: 1) $t^* < \infty$; and 2) $t^* = \infty$. First, suppose that $t^* < \infty$. By Lemma 2.6, $\mathcal{A}$ must have a fixed point larger than $\mathbf{x}_{\min}$.

Second, suppose that $t^* = \infty$. Then there must be some direction $\mathbf{d}$ such that $\beta_{\mathbf{d}}(t) = \alpha\mathcal{A}(\mathbf{x}_{\min}+t\mathbf{d}) < \alpha\mathbf{x}_{\min} + t$ for all $t > 0$. That implies that $\beta'_{\mathbf{d}}(t) = \alpha\mathcal{A}^{(1)}(\mathbf{x}_{\min}+t\mathbf{d})\mathbf{d} < 1$ for all $t>0$. Since $\mathcal{A}^{(1)}(\mathbf{x}_{\min})$ is irreducible, such a direction $\mathbf{d}$ must have at least one zero component. Suppose that $\mathbf{d} = (d_1, \ldots, d_i, 0, \ldots, 0)$ with positive $\{d_1, \ldots, d_i\}$. Since $\alpha$ is positive, $\mathcal{A}^{(1)}(\mathbf{x}_{\min}+t\mathbf{d})$ is independent of $d_1, \ldots$, and $d_i$. Then $\mathcal{A}(\mathbf{x}_{\min}+\mathbf{x})$ is a linear mapping with respect to $\{x_1, \ldots, x_i\}$. Then we have

$$\begin{pmatrix} a_1^*(\mathbf{x}) \\ \vdots \\ a_i^*(\mathbf{x}) \end{pmatrix} = A_1 \begin{pmatrix} x_1 \\ \vdots \\ x_i \end{pmatrix} + \begin{pmatrix} \hat{a}_1^*(x_{i+1}, ..., x_K) \\ \vdots \\ \hat{a}_i^*(x_{i+1}, ..., x_K) \end{pmatrix}. \tag{2.21}$$

See equation (2.8) for the definition of $A_1$. At any fixed point $\mathbf{x}$ of $\mathcal{A}$, we must have

$$\begin{pmatrix} x_1 \\ \vdots \\ x_i \end{pmatrix} = (I - A_1)^{-1} \begin{pmatrix} \hat{a}_1^*(x_{i+1}, ..., x_K) \\ \vdots \\ \hat{a}_i^*(x_{i+1}, ..., x_K) \end{pmatrix}, \tag{2.22}$$

where $I$ is the identity matrix. The matrix $I - A_1$ is invertible since $sp(A_1) \leq sp(\mathcal{A}^{(1)}(\mathbf{x}_{\min})) < 1$. Now, we consider a new mapping $\mathcal{A}_1$ with variables $\{x_{i+1}, \ldots, x_K\}$ only, obtained by replacing $\{x_1, \ldots, x_i\}$ in $\{a_{i+1}^*(\mathbf{x}), ..., a_K^*(\mathbf{x})\}$ with the right hand side of equation (2.22). Apparently, the Perron-Frobenius eigenvalue of the Jacobian of $\mathcal{A}_1$ at $\mathbf{x}_{\min}$ is less than one and every element of the Jacobian is an increasing function with respect to every variable $\{x_{i+1}, \ldots, x_K\}$. Thus, $\mathcal{A}_1$ must have a fixed point larger than $\{x_{\min,i+1}, \ldots, x_{\min,K}\}$. (Note that the induction method is utilized here.) Therefore, $\mathcal{A}$ must have a fixed point larger than $\mathbf{x}_{\min}$. This completes the proof Lemma 2.7.

Summarizing the results in this section, we obtain the following relationship between the Perron-Frobenius eigenvalue at a particular fixed point and other fixed points.

**Theorem 2.8** Assume that $\mathcal{F}_{ix}$ is not empty. Consider any fixed point $\mathbf{y} \in \mathcal{F}_{ix}$. If $\mathcal{A}^{(1)}(\mathbf{y})$ is irreducible, then

1) $sp(\mathcal{A}^{(1)}(\mathbf{y})) < 1$ if and only if there exists a larger fixed point of $\mathcal{A}$, i.e., $\exists\ \mathbf{z} \in \mathcal{F}_{ix}$, $\mathbf{z} \geq \mathbf{y}$ and $\mathbf{z} \neq \mathbf{y}$, and $\mathbf{y}$ is the minimal fixed point;
2) $sp(\mathcal{A}^{(1)}(\mathbf{y})) = 1$ if and only if $\mathbf{y}$ is the only fixed point of $\mathcal{A}$;
3) $sp(\mathcal{A}^{(1)}(\mathbf{y})) > 1$ if and only if there exists a smaller fixed point of $\mathcal{A}$, i.e., $\exists\ \mathbf{z} \in \mathcal{F}_{ix}$, $\mathbf{z} \leq \mathbf{y}$ and $\mathbf{z} \neq \mathbf{y}$;

**Proof.** Obvious from Lemmas 2.1 to 2.7.

## 3. MARKOV CHAINS OF *M/G/*1 TYPE WITH A TREE STRUCTURE

From now on, we shall view each string $J \in \aleph$ as a node in a $K$-ary tree. In the $K$-ary tree, each node $J$ has a parent node and $K$ children $\{J+1, J+2, \ldots, J+K\}$, except that the root node $J = 0$ that has no parent node. The node $J+k$ is called a type $k$ node. The following Markov chain

$\{C_n, n \geq 0\}$ of *M/G*/1 type with a tree structure was introduced in Takine, et al. [22]. The transition probabilities of the Markov chain are given as:

$$\begin{aligned} &P\{C_{n+1} = J + H \mid C_n = J + k\} = a(k, H), \quad \text{for} \quad J \in \aleph, H \in \aleph, 1 \leq k \leq K; \\ &P\{C_{n+1} = H \mid C_n = 0\} = b(H), \quad \text{for} \quad H \in \aleph. \end{aligned} \tag{3.1}$$

From the definition, it is clear that in one transition, the Markov chain can move from the current node to its parent node or any descendent node of its parent node. The transition probabilities depend only on the type of the current node. By the law of total probability, we must have

$$\sum_{J \in \aleph} a(k, J) = 1, \quad 1 \leq k \leq K; \qquad \sum_{J \in \aleph} b(J) = 1. \tag{3.2}$$

Such a Markov chain $\{C_n, n \geq 0\}$ is called a Markov chain of *M/G*/1 type with a tree structure. We are interested in the classification of this type of Markov chains. Recall that **e** is a column vector of ones.

**Theorem 3.1** (Theorem 3.2, HE [11]) Assume that the Markov chain $\{C_n, n \geq 0\}$ is irreducible and aperiodic, $\sum_{J \in \aleph} \sum_{j=1}^{K} b(J) N(J, j) < \infty$, and the matrix $\mathcal{A}^{(1)}(\mathbf{e})$ is irreducible. Then the Markov chain is

1) positive recurrent if and only if $sp(\mathcal{A}^{(1)}(\mathbf{e})) < 1$, i.e., there is a fixed point of $\mathcal{A}$ that is larger than **e**;
2) null recurrent if and only if $sp(\mathcal{A}^{(1)}(\mathbf{e})) = 1$, i.e., **e** is the only fixed point of $\mathcal{A}$;
3) transient if and only if $sp(\mathcal{A}^{(1)}(\mathbf{e})) > 1$, i.e., there is a fixed point of $\mathcal{A}$ that is smaller than **e**.

**Note**. In HE [11], the matrix $\mathcal{A}^{(1)}(\mathbf{e})$ was constructed by using the mean-drift method. But the results in HE [11] do not show the relationship between the classification conditions and the existence of the fixed point(s) of $\mathcal{A}$. We also note that, by equation (3.2), $\mathbf{e} \in \mathcal{R}_+^K$ is always a fixed point of the mapping $\mathcal{A}$. For this case, the classification conditions are explicit in terms of the original system parameters because **e** is a natural and explicit fixed point of $\mathcal{A}$. According to equation (2.5), the $(k, j)$th element of $\mathcal{A}^{(1)}(\mathbf{e})$ can be interpreted as the mean number of appearances of the integer $j$ in a transition starting from a type $k$ node. Theorem 3.1 indicates that if the mean drift away from the root node (which is measured by $sp(\mathcal{A}^{(1)}(\mathbf{e}))$) is large enough, the Markov chain becomes transient.

**Proof**. We use Theorem 3.2 in HE [11] and Theorem 2.8 in this paper to prove the theorem. The proof of part 1) is the same as that of Theorem 3.2 in HE [11]. The main tool used in this part of the proof is the mean-drift method (Cohen [2] and Fayolle, et al. [3]). By Theorem 2.8, the condition $sp(\mathcal{A}^{(1)}(\mathbf{e})) < 1$ is equivalent to the existence of another fixed point of $\mathcal{A}$ that is larger than $\mathbf{e}$. Part 2) is obtained from part 1) and part 3). Next, we prove part 3).

According to Takine, et al. [22] (also see HE [12]), the minimal fixed point $\mathbf{x}_{\min}$ equals the vector $\mathbf{G} = (G_1, \cdots, G_K)^T$, where the element $G_k$ is the probability that the Markov chain will eventually reach its parent node from the current type $k$ node. Apparently, at least one of these $\{G_1, \cdots, G_K\}$ is less than one if the Markov chain is transient, i.e., $\mathbf{G} \leq \mathbf{e}$ and $\mathbf{G} \neq \mathbf{e}$. According to Theorem 2.8, $sp(\mathcal{A}^{(1)}(\mathbf{G})) \leq 1$ is always true, which is consistent with Corollary 3.3 in HE [12]. If $sp(\mathcal{A}^{(1)}(\mathbf{e})) > 1$, $\mathbf{G} = \mathbf{x}_{\min} \leq \mathbf{e}$ and $\mathbf{x}_{\min} \neq \mathbf{e}$. Thus, some of $\{G_1, G_2, \ldots, G_K\}$ are less than one. Therefore, the Markov chain is transient. On the other hand, if the Markov chain is transient, $\mathbf{x}_{\min} = \mathbf{G} < \mathbf{e}$ holds in the sense that $G_k < 1$ for $1 \leq k \leq K$, since the Markov chain is irreducible. According to Theorem 2.8, we must have $sp(\mathcal{A}^{(1)}(\mathbf{e})) > 1$. This completes the proof of Theorem 3.1.

According to Theorem 3.1, there are two ways to classify a Markov chain of *M/G/*1 type with a tree structure. The first one is to utilize the fixed point $\mathbf{e}$ and calculate the Perron-Frobenius eigenvalue $sp(\mathcal{A}^{(1)}(\mathbf{e}))$. The second approach is to find another fixed point $\mathbf{y}$ that is larger (smaller) than $\mathbf{e}$. If a larger (smaller) fixed point $\mathbf{y}$ can be found, with some additional conditions, the Markov chain is positive recurrent (transient).

**Note**: Recently, we learned that Theorem 3.1 (or Theorem 3.2 in HE [11]) can be obtained partially from a classical result in the theory about multi type branching processes. According to the result (see Theorem 2, pp 186, Athreya and Ney [1]), the Markov chain of interest is recurrent if and only if $sp(\mathcal{A}^{(1)}(\mathbf{e})) \leq 1$ and the Markov chain is transient if and only if $sp(\mathcal{A}^{(1)}(\mathbf{e})) > 1$. But the classical results do not distinguish the positive recurrent and null recurrent cases and do not offer geometric insight into the classification conditions. Furthermore, it seems unlikely that the classical results can be applied to the matrix case (Section 4) or the *GI/M/*1 case (Section 5).

## 4. GENERALIZATION TO MARKOV CHAINS OF MATRIX *M/G/*1 TYPE WITH A TREE STRUCTURE

In this section, we generalize the results obtained in Sections 2 and 3 to Markov chains of matrix *M/G/*1 type with a tree structure. As shall be shown, there is a fundamental difference between the matrix case and the scalar case. Thus, the generalization is not straightforward. Therefore, we shall only generalize Lemmas 2.1 and 2.2 and the main results in Section 3 to the matrix case.

### 4.1 Properties of the mapping $\mathcal{A}$: $\mathcal{M}_m^K \to \mathcal{M}_m^K$

Let $\mathcal{M}_m$ be the set of all $m \times m$ substochastic matrices (i.e., for $X \in \mathcal{M}_m$, $X\mathbf{e} \leq \mathbf{e}$), where $m$ is a positive integer. For $X_k \in \mathcal{M}_m$, $1 \leq k \leq K$, let

$$\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_K \end{pmatrix}. \tag{4.1}$$

Let $\mathcal{M}_m^K$ be the set of all such **X**, i.e., $\mathcal{M}_m^K = \mathcal{M}_m \times \mathcal{M}_m \times \ldots \times \mathcal{M}_m$, the cross-product of $K$ number of $\mathcal{M}_m$. If $m$=1, it is clear that $\mathcal{M}_m^K = \mathcal{R}_+^K(\mathbf{e})$. If $m$=1, $\mathbf{x} = (0, 0, \ldots, 0)^T$ is the smallest vector and $\mathbf{x} = \mathbf{e}$ is the largest in $\mathcal{R}_+^K(\mathbf{e})$. If $m$>1, $\mathbf{X} = 0$ is still the smallest element in $\mathcal{M}_m^K$. However, there is no single largest element in $\mathcal{M}_m^K$. Let

$$\mathcal{M}_m^K(1) = \{\mathbf{X}\text{: } \mathbf{X} \in \mathcal{M}_m^K \text{, and } X_k \text{ is stochastic (i.e., } X_k\mathbf{e} = \mathbf{e}) \text{ for } 1 \le k \le K\}. \tag{4.2}$$

$\mathcal{M}_m^K$ and $\mathcal{M}_m^K(1)$ are two convex sets. The product of matrices is defined as follows:

$$\mathbf{X}^{(J)} = X_{j_{|J|}} X_{j_{|J|-1}} \cdots X_{j_2} X_{j_1}, \quad \text{for } \mathbf{X} \in \mathcal{M}_m^K,\ J = j_1 j_2 \cdots j_{|J|-1} j_{|J|} \in \aleph. \tag{4.3}$$

Note that the order of the multiplication matters in the matrix case.

Let $\{A(k, J), J \in \aleph$ and $1 \le k \le K\}$ be a set of $m \times m$ matrices in $\mathcal{M}_m$. The matrices $\{A(k, J), 1 \le k \le K$ and $J \in \aleph\}$ satisfy

$$\sum_{J \in \aleph} A(k, J)\mathbf{e} = \mathbf{e}, \quad 1 \le k \le K. \tag{4.4}$$

Then the mapping $\mathcal{A}$: $\mathcal{M}_m^K \to \mathcal{M}_m^K$ is defined as follows:

$$A_k^*(\mathbf{X}) = \sum_{J \in \aleph} A(k, J)\mathbf{X}^{(J)}, \quad \text{for } \mathbf{X} \in \mathcal{M}_m^K,\ 1 \le k \le K;$$
$$\mathcal{A}(\mathbf{X}) = \begin{pmatrix} A_1^*(\mathbf{X}) \\ \vdots \\ A_K^*(\mathbf{X}) \end{pmatrix}. \tag{4.5}$$

For any **X** in $\mathcal{M}_m^K$ with matrices $\{X_1, X_2, \ldots, X_K\}$ and **Y** in $\mathcal{M}_m^K$ with matrices $\{Y_1, Y_2, \ldots, Y_K\}$, if $X_k \le Y_k$ for $1 \le k \le K$, then we say $\mathbf{X} \le \mathbf{Y}$. It is easy to see that the mappings $\{A_k^*(\mathbf{X}), 1 \le k \le K\}$ are continuous and nondecreasing, i.e., for any **X** and **Y** in $\mathcal{M}_m^K$ with $\mathbf{X} \le \mathbf{Y}$, $\mathcal{A}(\mathbf{X}) \le \mathcal{A}(\mathbf{Y})$. The set of fixed points $\mathcal{F}_{ix,m}$ of the mapping $\mathcal{A}$ in $\mathcal{M}_m^K$ is defined as

$$\mathcal{F}_{ix,m} = \{\mathbf{X} : \mathcal{A}(\mathbf{X}) = \mathbf{X} \text{ and } \mathbf{X} \in \mathcal{M}_m^K\}. \tag{4.6}$$

Note that $\mathcal{F}_{ix,m}$ includes only stochastic and substochastic fixed points, which is different from $\mathcal{F}_{ix}$ defined in Section 2. Due to condition (4.4), both $\mathcal{M}_m^K(1)$ and $\mathcal{M}_m^K$ are invariant under the continuous mapping $\mathcal{A}$. By Brouwer's fixed point theorem, $\mathcal{A}$ has at least one *stochastic* fixed point. Thus, $\mathcal{F}_{ix,m}$ is not empty. We denote by $\mathbf{X}_{\min}$ the minimal fixed point (if exists) in $\mathcal{F}_{ix,m}$. Next, we show the relationship between the minimal fixed point and other fixed points. Let $\mathcal{M}_m^K(\mathbf{X}) = \{\mathbf{Y}\text{: } \mathbf{Y}\in\mathcal{M}_m^K \text{ and } \mathbf{Y}\le\mathbf{X}\}$.

**Lemma 4.1** Assume equation (4.4) holds. Then $\mathcal{M}_m^K$ is invariant under the mapping $\mathcal{A}$ and $\mathcal{F}_{ix,m}$ is not empty. If $\mathbf{X}\in\mathcal{F}_{ix,m}$, the subset $\mathcal{M}_m^K(\mathbf{X})$ is invariant under the mapping $\mathcal{A}$. In addition, the subset $\mathcal{M}_m^K(1)$ is invariant under the mapping $\mathcal{A}$. Thus, the mapping $\mathcal{A}$ has at least one stochastic fixed point.

**Proof**. The conclusions are obtained by the monotonicity of the mapping $\mathcal{A}$ and equation (4.4). This completes the proof of Lemma 4.1.

The minimal nonnegative fixed point of $\mathcal{A}$ can be computed by using the following iterative method: let $\mathbf{X}[0]$ have $X_k = 0$ (zero matrix), $1\le k\le K$, and $\mathbf{X}[n+1] = \mathcal{A}(\mathbf{X}[n])$, $n\ge 1$. Then the sequence $\{\mathbf{X}[n], n\ge 0\}$ converges monotonically to $\mathbf{X}_{\min}$. A stochastic fixed point of $\mathcal{A}$ in $\mathcal{M}_m^K(1)$ can be calculated by using the following iterative method: let $\mathbf{X}[0]$ have $X_k = \mathbf{e}\mathbf{e}^T/m$, $1\le k\le K$, and $\mathbf{X}[n+1] = \mathcal{A}(\mathbf{X}[n])$, $n\ge 1$. Then, if the sequence $\{\mathbf{X}[n], n\ge 0\}$ converges, it converges to a stochastic fixed point of $\mathcal{A}$.

**Lemma 4.2** There exists a minimal fixed point $\mathbf{X}_{\min}$ such that $X_{\min,\, k} \le X_k$, $1\le k\le K$, for any fixed point $\mathbf{X} = (X_1^T, X_2^T, \ldots, X_K^T)^T \in \mathcal{F}_{ix,m}$, i.e., $\mathbf{X}_{\min} \le \mathbf{X}$.

**Proof**. The proof is similar to that of Lemma 2.1. This completes the proof.

Similar to equation (2.5), we define the following matrices.

$$
\begin{aligned}
&N(0,k,\mathbf{X}) = 0, \quad 1\le k\le K,\ J\in\aleph,\ \mathbf{X}\in\mathcal{M}_m^K;\\
&N(J,k,\mathbf{X}) = \delta_{\{j_{|J|}=k\}}I + \sum_{n=1}^{|J|-1}\delta_{\{j_n=k\}}X_{j_{|J|}}X_{j_{|J|-1}}\cdots X_{j_{n+1}},\ 1\le k\le K,\ J\in\aleph,\ J\ne 0, \mathbf{X}\in\mathcal{M}_m^K; \quad (4.7)\\
&A_{k,j}^{*(1)}(\mathbf{X}) = \sum_{J\in\aleph:\, J\ne 0} A(k,J)N(J,j,\mathbf{X}), \quad 1\le k, j\le K,\ \mathbf{X}\in\mathcal{M}_m^K,
\end{aligned}
$$

where $\delta_{\{.\}}$ is an indicator function. The function $N(J, k, \mathbf{X})$ is similar to $N(J, k)$ for $m$=1. We assume that all the summations in equation (4.7) are finite. Define an $mK\times mK$ matrix $\mathcal{A}^{(1)}(\mathbf{X})$ with its $(k, j)$th block $A_{k,j}^{*(1)}(\mathbf{X})$. We call $\mathcal{A}^{(1)}(\mathbf{X})$ the differentiation matrix of $\mathcal{A}$ at $\mathbf{X}$. Apparently, every element of $\mathcal{A}^{(1)}(\mathbf{X})$ is nondecreasing with respect to every element of $\mathbf{X}$. Note that $A_{k,j}^{*(1)}(\mathbf{X})$ is different from the $a_{k,j}^{*(1)}(\mathbf{x})$ even when $m$=1. The two definitions are consistent only if $m$=1 and **x = e**.

**Lemma 4.3** Consider any stochastic fixed point $\mathbf{Y} \in \mathcal{F}_{ix,m}$. Assume that $\mathcal{A}^{(1)}(\mathbf{Y})$ is irreducible. If $sp(\mathcal{A}^{(1)}(\mathbf{Y})) \le 1$, $\mathbf{X}_{\min} = \mathbf{Y}$. In this case, $\mathbf{X}_{\min} = \mathbf{Y}$ is the only fixed point of the mapping $\mathcal{A}$ in $\mathcal{M}_m^K(\mathbf{Y})$ (and $\mathcal{M}_m^K$). If $sp(\mathcal{A}^{(1)}(\mathbf{Y})) > 1$, $\mathbf{X}_{\min} \le \mathbf{Y}$ and $\mathbf{X}_{\min} \ne \mathbf{Y}$. In this case, the mapping $\mathcal{A}$ has at least two fixed points in $\mathcal{M}_m^K(\mathbf{Y})$ (and $\mathcal{M}_m^K$).

**Proof**. The proof is similar to that of Lemma 2.2. If $sp(\mathcal{A}^{(1)}(\mathbf{Y})) < 1$, suppose that there exists at least one other fixed point in $\mathcal{M}_m^K(\mathbf{Y})$. We denote that fixed point as **X**. Then we have the following calculations.

$$\begin{aligned}\left(A_k^*(\mathbf{X}) - A_k^*(\mathbf{Y})\right)\mathbf{e} &= \sum_{J\in\aleph} A(k,J)\left(\mathbf{X}^{(J)} - \mathbf{Y}^{(J)}\right)\mathbf{e} \\ &= \sum_{J\in\aleph:\, J\neq 0} A(k,J)\left\{\sum_{n=1}^{|J|} X_{j_{|J|}}\cdots X_{j_{n+1}}\left(X_{j_n} - Y_{j_n}\right)Y_{j_{n-1}}\cdots Y_{j_1}\mathbf{e}\right\}.\end{aligned} \tag{4.8}$$

Since the mapping $\mathcal{A}$ is nondecreasing and $\mathbf{X} \le \mathbf{Y} \in \mathcal{M}_m^K$ and **Y** is stochastic (i.e., $Y_k\mathbf{e} = \mathbf{e}$, $1 \le k \le K$), we have, for $1 \le k \le K$,

$$\begin{aligned}\left(A_k(\mathbf{X}) - A_k(\mathbf{Y})\right)\mathbf{e} &= \sum_{J\in\aleph:\, J\neq 0} A(k,J)\left\{\sum_{n=1}^{|J|} X_{j_{|J|}}\cdots X_{j_{n+1}}\left(X_{j_n} - Y_{j_n}\right)\mathbf{e}\right\} \\ &= \sum_{j=1}^{K}\left(\sum_{J\in\aleph:\, J\neq 0} A(k,J)\left\{\delta_{\{j_{|J|}=j\}} + \sum_{n=1}^{|J|-1}\delta_{\{j_n=j\}} X_{j_{|J|}}\cdots X_{j_{n+1}}\right\}\left(X_j - Y_j\right)\mathbf{e}\right) \\ &= \sum_{j=1}^{K} A_{k,j}^{*(1)}(\mathbf{X})(X_j - Y_j)\mathbf{e}.\end{aligned} \tag{4.9}$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_K)$ be the eigenvector of $\mathcal{A}^{(1)}(\mathbf{Y})$ corresponding to the eigenvalue $sp(\mathcal{A}^{(1)}(\mathbf{Y}))$. The vector $\alpha$ is normalized by $\alpha\mathbf{e}=1$, where $\alpha_k$ is a vector of the size $m$, $1 \le k \le K$. Every element of the vector $\alpha$ is positive since $\mathcal{A}^{(1)}(\mathbf{Y})$ is irreducible. Equation (4.9) leads to

$$\begin{aligned} 0 &< \sum_{k=1}^{K}\alpha_k\left(A_k^*(\mathbf{Y}) - A_k^*(\mathbf{X})\right)\mathbf{e} = \sum_{k=1}^{K}\alpha_k\sum_{j=1}^{K} A_{k,j}^{*(1)}(\mathbf{X})(Y_j - X_j)\mathbf{e} \\ &\le \sum_{j=1}^{K}\left(\alpha\mathcal{A}^{(1)}(\mathbf{Y})\right)_j(Y_j - X_j)\mathbf{e} = \alpha\mathcal{A}^{(1)}(\mathbf{Y})(\mathbf{Y}-\mathbf{X})\mathbf{e} \\ &\Leftrightarrow 0 < \alpha(\mathbf{Y}-\mathbf{X})\mathbf{e} \le sp(\mathcal{A}^{(1)}(\mathbf{Y}))\alpha(\mathbf{Y}-\mathbf{X})\mathbf{e}\end{aligned} \tag{4.10}$$

The last inequality in equation (4.10) implies that $sp(\mathcal{A}^{(1)}(\mathbf{Y})) \geq 1$, which contradicts $sp(\mathcal{A}^{(1)}(\mathbf{Y})) < 1$. Therefore, $\mathbf{Y}$ is the only fixed point in $\mathcal{M}_m^K(\mathbf{Y})$ and $\mathbf{X}_{\min} = \mathbf{Y}$. Since $\mathbf{Y}$ can be any stochastic fixed point in $\mathcal{F}_{ix,m}$, then $\mathcal{F}_{ix,m}$ has only one element.

If $sp(\mathcal{A}^{(1)}(\mathbf{Y})) = 1$, suppose that there is another fixed point $\mathbf{X}$. If $\mathbf{X}$ is an interior point of $\mathcal{M}_m^K(\mathbf{Y})$, then $\mathcal{A}^{(1)}(\mathbf{X}) \leq \mathcal{A}^{(1)}(\mathbf{Y})$, $\mathcal{A}^{(1)}(\mathbf{X}) \neq \mathcal{A}^{(1)}(\mathbf{Y})$, and the vector $(Y_i - X_i)\mathbf{e}$ is nonzero and positive. Then the inequalities in equation (4.10) becomes strict, which leads to $sp(\mathcal{A}^{(1)}(\mathbf{Y})) > 1$. Therefore, there is no fixed point in the interior of $\mathcal{M}_m^K(\mathbf{Y})$. This implies that there is no fixed point in the interior of $\mathcal{M}_m^K$. If $X_i\mathbf{e} = Y_i\mathbf{e}$ for some $i$, then $X_i = Y_i$ since $X_i \leq Y_i$. Applying the same method used in Lemma 2.2 in dealing with the case with $sp(\mathcal{A}^{(1)}(\mathbf{Y})) = 1$, it can be proved that there is no fixed point $\mathbf{X}$ that is smaller than $\mathbf{Y}$.

If $sp(\mathcal{A}^{(1)}(\mathbf{Y})) > 1$, define $\mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon) = \{\mathbf{X}: \ \alpha\mathbf{X}\mathbf{e} \leq 1-\varepsilon, \mathbf{X} \in \mathcal{M}_m^K(\mathbf{Y})\}$ and $\mathcal{M}_m^K(\mathbf{Y}, = \varepsilon) = \{\mathbf{X}: \ \alpha\mathbf{X}\mathbf{e} = 1-\varepsilon, \mathbf{X} \in \mathcal{M}_m^K(\mathbf{Y})\}$. It is easy to see that both $\mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon)$ and $\mathcal{M}_m^K(\mathbf{Y}, = \varepsilon)$ are convex sets. We would like to show that, if $\varepsilon$ is small enough, the subset $\mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon)$ is invariant under the mapping $\mathcal{A}$. The geometric intuition of this proof is similar to that of Lemma 2.2. For any $\mathbf{X} \in \mathcal{M}_m^K(\mathbf{Y}, = \varepsilon)$, by equation (4.9), we have

$$\begin{aligned}\alpha\mathcal{A}(\mathbf{X})\mathbf{e} - \alpha\mathcal{A}(\mathbf{Y})\mathbf{e} &= \alpha\mathcal{A}^{(1)}(\mathbf{X})(\mathbf{X}-\mathbf{Y})\mathbf{e} \\ &= \alpha\mathcal{A}^{(1)}(\mathbf{Y})(\mathbf{X}-\mathbf{Y})\mathbf{e} + \alpha\left[\mathcal{A}^{(1)}(\mathbf{X}) - \mathcal{A}^{(1)}(\mathbf{Y})\right](\mathbf{X}-\mathbf{Y})\mathbf{e} \\ &= sp(\mathcal{A}^{(1)}(\mathbf{Y}))\alpha(\mathbf{X}-\mathbf{Y})\mathbf{e} + \alpha\left[\mathcal{A}^{(1)}(\mathbf{X}) - \mathcal{A}^{(1)}(\mathbf{Y})\right](\mathbf{X}-\mathbf{Y})\mathbf{e}.\end{aligned} \tag{4.11}$$

Note that $\mathbf{Y}$ is a fixed point with $Y_k\mathbf{e} = \mathbf{e}$, $1 \leq k \leq K$, and $\alpha\mathbf{Y}\mathbf{e} - \alpha\mathbf{X}\mathbf{e} = 1 - \alpha\mathbf{X}\mathbf{e} = \varepsilon$. Equation (4.11) leads to

$$\begin{aligned}\alpha\mathcal{A}(\mathbf{X})\mathbf{e} &= 1 - sp(\mathcal{A}^{(1)}(\mathbf{Y}))\ \alpha(\mathbf{Y}-\mathbf{X})\mathbf{e} + \alpha\left[\mathcal{A}^{(1)}(\mathbf{X}) - \mathcal{A}^{(1)}(\mathbf{Y})\right](\mathbf{X}-\mathbf{Y})\mathbf{e} \\ &\leq 1 - \varepsilon - \left[sp(\mathcal{A}^{(1)}(\mathbf{Y})) - 1\right]\varepsilon + \alpha\left[\mathcal{A}^{(1)}(\mathbf{Y}) - \mathcal{A}^{(1)}(\mathbf{X})\right](\mathbf{Y}-\mathbf{X})\mathbf{e}.\end{aligned} \tag{4.12}$$

Since $\mathcal{A}^{(1)}(\mathbf{X})$ is continuous in $\mathbf{X}$ and every element of $\alpha$ is positive, we can choose small enough $\varepsilon$ such that

$$\alpha\left[\mathcal{A}^{(1)}(\mathbf{Y}) - \mathcal{A}^{(1)}(\mathbf{X})\right] \leq \left[sp(\mathcal{A}^{(1)}(\mathbf{Y})) - 1\right]\alpha \tag{4.13}$$

for any $\mathbf{X} \in \mathcal{M}_m^K(\mathbf{Y}, = \varepsilon)$. Equations (4.12) and (4.13) lead to $\alpha\mathcal{A}(\mathbf{X})\mathbf{e} \leq 1-\varepsilon$ for any $\mathbf{X} \in \mathcal{M}_m^K(\mathbf{Y}, = \varepsilon)$, i.e., $\mathcal{A}(\mathbf{X}) \in \mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon)$ for any $\mathbf{X} \in \mathcal{M}_m^K(\mathbf{Y}, = \varepsilon)$. For any $\mathbf{X} \in \mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon)$, consider the closed line segment $\mathbf{X}(t) = (1-t)\mathbf{X} + t\mathbf{Y}$, $0 \leq t \leq 1$. Similar to the proof of Lemma 2.2, it can be shown that there exists $t_1$ such that $0 \leq t_1 \leq 1$, $\mathbf{X}(t_1) \in \mathcal{M}_m^K(\mathbf{Y}, = \varepsilon)$ and $\mathbf{X} \leq \mathbf{X}(t_1)$. This implies that $\mathcal{A}(\mathbf{X}) \leq \mathcal{A}(\mathbf{X}(t_1)) \in \mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon)$. Thus, we have shown that the subset $\mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon)$

is invariant under the mapping $\mathcal{A}$. Therefore, there is a fixed point in $\mathcal{M}_m^K(\mathbf{Y}, \leq \varepsilon)$, which is not stochastic. Thus, the minimal fixed point is not equal to **Y**. This completes the proof.

**Note**: The proof of Lemma 4.3 seems simpler than that of Lemma 2.2, but it works only when a *stochastic* fixed point **Y** is considered.

## 4.2 Markov chains of matrix *M/G*/1 type with a tree structure

Now, we consider a Markov chain $\{(C_n, \eta_n), n\geq 0\}$, where $C_n$ takes values in $\aleph$ and $\eta_n$ takes integer values from 1 to $m$. The random variable $\eta_n$ is an auxiliary variable. The transition probabilities of the Markov chain are given as, for $J$ and $H$ in $\aleph$, $1\leq k\leq K$,

$$\begin{aligned} &P\{C_{n+1} = J + H, \eta_{n+1} = j \mid C_n = J + k, \eta_n = i\} = (A(k, H))_{i,j}, 1 \leq i, j \leq m; \\ &P\{C_{n+1} = H, \eta_{n+1} = j \mid C_n = 0, \eta_n = i\} = (B(H))_{i,j}, 1 \leq i, j \leq m, \end{aligned} \tag{4.14}$$

where matrices $\{A(k, J), J\in\aleph, 1\leq k\leq K\}$ satisfy equation (4.4) and matrices $\{B(J), J\in\aleph\}$ satisfy $\sum_{J\in\aleph} B(J)\mathbf{e} = \mathbf{e}$. Then $\{(C_n, \eta_n), n\geq 0\}$ is called a Markov chain of matrix *M/G*/1 type with a tree structure, which was introduced in Takine, et al. [22].

According to Theorem 3.2 in HE [12], the classification of such Markov chains is determined completely by the Perron-Frobenius eigenvalue of $\mathcal{A}^{(1)}(\mathbf{Y})$ of any stochastic fixed point **Y** of the mapping $\mathcal{A}$. The results are summarized in the following theorem.

**Theorem 4.4** (Theorem 3.2, HE [12]) Assume that $\{(C_n, \eta_n), n\geq 0\}$ is irreducible and aperiodic, $\sum_{J\in\aleph}\sum_{k=1}^{K} B(J)N(J, k, \mathbf{Y}) < \infty$, and the matrix $\mathcal{A}^{(1)}(\mathbf{Y})$ is irreducible for a stochastic fixed point **Y** of the mapping $\mathcal{A}$. Then the Markov chain is

1) positive recurrent if and only if $sp(\mathcal{A}^{(1)}(\mathbf{Y})) < 1$;
2) null recurrent if and only if $sp(\mathcal{A}^{(1)}(\mathbf{Y})) = 1$;
3) transient if and only if $sp(\mathcal{A}^{(1)}(\mathbf{Y})) > 1$, i.e., there exists another fixed point of the mapping $\mathcal{A}$ that is smaller than **Y**.

**Proof.** The matrix $\mathcal{A}^{(1)}(\mathbf{Y})$ was first constructed in HE [12] by using the mean-drift method. The matrix $\mathcal{A}^{(1)}(\mathbf{Y})$ was denoted as $P(\mathbf{Y})$ in HE [12]. Thus, this theorem is equivalent to Theorem 3.2 in HE [12]. Therefore, we refer to Theorem 3.2 in HE [12] for a proof. We like to point out that by using Lemmas 4.1, 4.2, and 4.3, an alternative and shorter proof of part 3) can be obtained in a way similar to the proof of part 3) of Theorem 3.1. Details are omitted. This completes the proof of Theorem 4.4.

For the matrix *M/G/*1 case ($m$>1), there is *no* natural and explicit fixed point in $\mathcal{M}_m^K$ to provide information about the ergodicity of the Markov chain. Thus, we need to find a stochastic fixed point in a smaller subset $\mathcal{M}_m^K(1)$ - the set of stochastic matrices first. Then information about the classification problem can be recovered.

Similar to the scalar case, the minimal fixed point $\mathbf{X}_{\min}$ can be interpreted as the probability of the first passage from a node to its parent node (see Takine, et al. [22]). Usually, the minimal fixed point $\mathbf{X}_{\min}$ is denoted as $\mathbf{G}$ with $m\times m$ matrices $\{G_1, \ldots, G_K\}$ in matrix analytic methods. Lemma 4.3 implies that $sp(\mathcal{A}^{(1)}(\mathbf{G})) \le 1$, a result that was proved first in HE [12]. This property of $\mathbf{G}$ distinguishes it from other (possible) fixed points.

Theorem 4.4 can be used to find stability conditions for various queueing models. For instance, we consider a discrete time queueing system with multiple types of customers and a last-come-first-served general preemptive resume (LCFS-GPR) service discipline. In general, this queueing system is not work-conserving. Thus, the usual traffic intensity fails to provide information about system stability. Under some conditions, the queueing process of this model can be formulated as an *M/G/*1 type Markov chain with a tree structure. Then Theorem 4.4 can be used to find whether or not such a queueing system is stable. Theorem 4.4 can also used to find whether or not the queueing model considered in HE and Alfa [13] and the communication system considered in Van Houdt and Blondia [23] are stable. Details are omitted.

## 5. MARKOV CHAINS of *GI/M/*1 TYPE WITH A TREE STRUCTURE

In this section, we consider a Markov chain of *GI/M/*1 type with a tree structure $\{C_n, n\ge 0\}$ that was introduced in Yeung and Sengupta [24]. We only consider the scalar case. Thus, we shall use notation introduced in Section 2 as well as the results obtained in Section 2. The Markov chain $\{C_n, n\ge 0\}$ is defined on $\aleph$. The transition probabilities of the Markov chain are given as:

$$\begin{aligned} &P\{C_{n+1} = J+k \mid C_n = J+H\} = a(k,H), \quad \text{for} \quad J \ge 0, H \ge 0, 1 \le k \le K; \\ &P\{C_{n+1} = 0 \mid C_n = J\} = b(J), \quad \text{for} \quad J \ge 0. \end{aligned} \tag{5.1}$$

Note that the probabilistic interpretations of the nonnegative numbers $\{a(k, J), b(J), J\in\aleph, 1\le k\le K\}$ are different from that of Section 3. From the definition, it is clear that in one transition, the Markov chain can move from the current node to one of its children or any node that is an immediate child of an ancestor of the current node. The transition probabilities depend on the type of the targeted node. If $J = j_1\cdots j_n$, denote by $f(J, i) = j_{n-i+1}\cdots j_n$, for $1\le i\le n$, and $f(J, 0) = 0$, i.e., $f(J, i)$ is the sub-string of $J$ consisting of the last $i$ numbers of $J$. By the law of total probability, we have

$$b(J) + \sum_{i=0}^{|J|}\sum_{k=1}^{K} a(k, f(J,i)) = 1, \quad \text{for any} \quad J \in \aleph. \tag{5.2}$$

Note that at least one of $\{a(k, 0), 1 \le k \le K\}$ is positive. An example of the transition structure of a *GI/M/*1 type Markov chain with a tree structure is shown in Figure 5.1 for $K$=2.

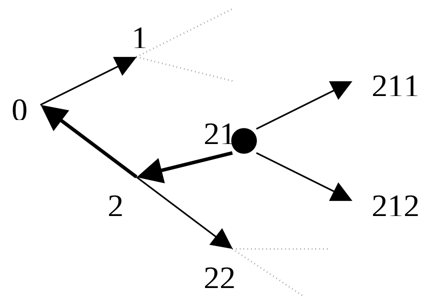


**Figure 5.1** Nodes reachable in one transition from the node $J$=21.

Figure 5.1 shows all nodes reachable in one transition from the node $J$=21: node 0 with probability $b(21)$, 1 with $a(1, 21)$, 2 with $a(2, 21)$, 21 with $a(1, 1)$, 22 with $a(2, 1)$, 211 with $a(1,0)$, and 212 with $a(2, 0)$.

Let us call the node $J$ a level $|J|$ node. It is clear that, in one transition, a Markov chain of *M/G/*1 type can move away from the root node for any arbitrary number of levels, but it can only move one level closer to the root node. On the contrary, in one transition, a Markov chain of *GI/M/*1 type can move several levels closer to the root node, but it can only move one level away from the root node. Also for the *GI/M/*1 case, the transition probabilities $\{a(k, 0), 1 \le k \le K\}$ are the same for any (current) node.

Because of the above difference between the *M/G/*1 and *GI/M/*1 cases, the analyses of the two types of Markov chains are dramatically different (Neuts [18, 19], Takine, et al. [22], and Yeung and Sengupta [24]). That difference extends to the fixed points of the mapping $\mathcal{A}$ defined in Section 2. For the *M/G/*1 case, **e** is always a fixed point, which leads to information for the classification problem. But **e** is not, in general, a fixed point for the *GI/M/*1 case if $m$>1. Thus, to find classification conditions, we need to identify a fixed point of $\mathcal{A}$ (possibly different from the minimal fixed point) first.

## 5.1 A fixed point with elements summing to one

Let

$$\mathcal{R}^K_{+,S=1} = \left\{ \mathbf{x} : \sum_{i=1}^{K} x_i = 1, \mathbf{x} \in \mathcal{R}^K_+ \right\} \quad \text{and} \quad \mathcal{R}^K_{+,S\le 1} = \left\{ \mathbf{x} : \sum_{i=1}^{K} x_i \le 1, \mathbf{x} \in \mathcal{R}^K_+ \right\}. \qquad (5.3)$$

It is easy to see that $\mathcal{R}^K_{+,\Sigma=1}$ and $\mathcal{R}^K_{+,\Sigma\le 1}$ are convex sets. An example of the sets $\mathcal{R}^K_{+,\Sigma=1}$ and $\mathcal{R}^K_{+,\Sigma\le 1}$ are shown in Figure 5.2 for $K$=2.

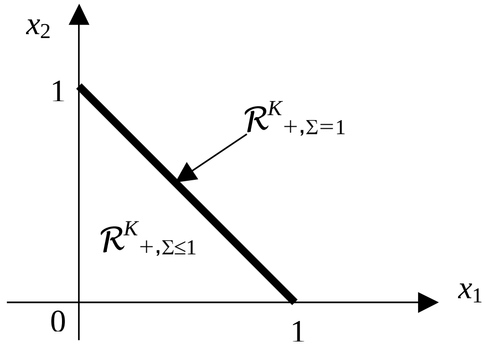


**Figure 5.2** The sets $\mathcal{R}^K_{+,\Sigma=1}$ and $\mathcal{R}^K_{+,\Sigma\le 1}$ for $K$=2.

We first focus on the subset $\mathcal{R}^K_{+,\Sigma=1}$. We show that the subset $\mathcal{R}^K_{+,\Sigma=1}$ is invariant under $\mathcal{A}$ for the *GI/M/*1 case.

**Lemma 5.1** If equation (5.2) holds and $\lim_{n\to\infty} \max_{J\in\aleph:\,|J|=n} \{b(J)\} = 0$, the subset $\mathcal{R}^K_{+,\Sigma=1}$ is invariant under $\mathcal{A}$. Thus, there exists at least one fixed point of $\mathcal{A}$ in the subset $\mathcal{R}^K_{+,\Sigma=1}$. The subset $\mathcal{R}^K_{+,\Sigma\le 1}$ is also invariant under $\mathcal{A}$. Thus, the minimal fixed point $\mathbf{x}_{\min}$ of $\mathcal{A}$ is in $\mathcal{R}^K_{+,\Sigma\le 1}$. Any fixed point in $\mathcal{R}^K_{+,\Sigma=1}$ is larger than or equal to the minimal fixed point.

**Proof**. First, we show that the subset $\mathcal{R}^K_{+,\Sigma=1}$ is invariant under $\mathcal{A}$. For any $\mathbf{x} \in \mathcal{R}^K_{+,\Sigma=1}$, we need to prove that $\mathcal{A}(\mathbf{x}) \in \mathcal{R}^K_{+,\Sigma=1}$. That is equivalent to proving that for any nonnegative vector $\mathbf{x}$ with $\sum_{k=1}^{K} x_k = 1$, $\sum_{k=1}^{K} a_k^*(\mathbf{x}) = 1$ holds. First, we assume that the transition probability $a(k, J) = 0$ for any string $J$ with $|J|>N$, where $N$ is a positive integer. By equation (5.2), we have, for any $J\in\aleph$ and $1\le j\le K$,

$$b(J) = b(j+J) + \sum_{k=1}^{K} a(k, j+J)\,. \tag{5.4}$$

If $a(k, J) = 0$ for any string $J$ with $|J|>N$, then $b(J) = 0$ for $|J| \ge N$. We rewrite the sum of the vector $\mathcal{A}(\mathbf{x})$ in the following way.

$$
\begin{aligned}
\sum_{k=1}^{K} a_k^*(\mathbf{x}) &= \sum_{k=1}^{K}\sum_{J\in\aleph} a(k,J)\mathbf{x}^{(J)} = \sum_{n=0}^{N}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} \\
&= \sum_{n=0}^{N-1}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} + \sum_{J\in\aleph:|J|=N}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} \\
&= \sum_{n=0}^{N-1}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} + \sum_{J\in\aleph:|J|=N-1}\left\{\sum_{j=1}^{K}\left(\sum_{k=1}^{K} a(k,j+J)\right)x_j\right\}\mathbf{x}^{(J)} \\
&= \sum_{n=0}^{N-1}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} + \sum_{J\in\aleph:|J|=N-1}\left\{\sum_{j=1}^{K} b(J)x_j\right\}\mathbf{x}^{(J)} \\
&= \sum_{n=0}^{N-1}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} + \sum_{J\in\aleph:|J|=N-1} b(J)\mathbf{x}^{(J)},
\end{aligned}
\tag{5.5}
$$

in which we have used $x_1 + x_2 + \ldots + x_K = 1$. Now, we move to level $N$–1, $N$–2, …, and carry out the following calculations:

$$
\begin{aligned}
\sum_{k=1}^{K} a_k^*(\mathbf{x}) &= \sum_{n=0}^{N-2}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} + \sum_{J\in\aleph:|J|=N-2}\left\{\sum_{j=1}^{K}\left(b(j+J)+\sum_{k=1}^{K} a(k,j+J)\right)x_j\right\}\mathbf{x}^{(J)} \\
&= \sum_{n=0}^{N-2}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} + \sum_{J\in\aleph:|J|=N-2}\left\{\sum_{j=1}^{K} b(J)x_j\right\}\mathbf{x}^{(J)} \\
&= \sum_{n=0}^{N-2}\sum_{J\in\aleph:|J|=n}\left\{\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{x}^{(J)}\right\} + \sum_{J\in\aleph:|J|=N-2} b(J)\mathbf{x}^{(J)} = \cdots = \sum_{k=1}^{K} a(k,0) + b(0) = 1.
\end{aligned}
\tag{5.6}
$$

Therefore, the subset $\mathcal{R}_{+,\Sigma=1}^{K}$ is invariant under the mapping $\mathcal{A}$. Since $\mathcal{R}_{+,\Sigma=1}^{K}$ is convex and $\mathcal{A}$ is continuous, $\mathcal{A}$ has at least one fixed point in $\mathcal{R}_{+,\Sigma=1}^{K}$.

For the general case, we prove the result by taking $N$ to infinity. Since $\{a(k, J), J\in\aleph, 1\le k\le K\}$ and $\mathbf{x}$ are all nonnegative, by the *monotone convergence theorem*, we have

$$
\begin{aligned}
\sum_{k=1}^{K} a_k^*(\mathbf{x}) &= \sum_{k=1}^{K}\sum_{J\in\aleph} a(k,J)\mathbf{x}^{(J)} = \sum_{k=1}^{K}\left[\sum_{J\in\aleph:|J|\le N} a(k,J)\mathbf{x}^{(J)} + \sum_{J\in\aleph:|J|>N} a(k,J)\mathbf{x}^{(J)}\right] \\
&= \lim_{N\to\infty}\sum_{k=1}^{K}\left(\sum_{J\in\aleph:|J|\le N} a(k,J)\mathbf{x}^{(J)}\right).
\end{aligned}
\tag{5.7}
$$

By equations (5.6) and (5.7), we have

$$\sum_{k=1}^{K} a_k^*(\mathbf{x}) = \lim_{N\to\infty}\left\{\sum_{k=1}^{K}\left(\sum_{J\in\aleph:\,|J|\le N} a(k,J)\mathbf{x}^{(J)}\right) + \sum_{J\in\aleph:\,|J|=N} b(J)\mathbf{x}^{(J)} - \sum_{J\in\aleph:\,|J|=N} b(J)\mathbf{x}^{(J)}\right\}$$
$$= 1 - \lim_{N\to\infty}\left[\sum_{J\in\aleph:\,|J|=N} b(J)\mathbf{x}^{(J)}\right] \ge 1 - \lim_{N\to\infty}\left[\max_{J\in\aleph:\,|J|=N}\{b(J)\}\right] = 1. \tag{5.8}$$

Equation (5.8) also shows that $\sum_{k=1}^{K} a_k^*(\mathbf{x}) \le 1$. Thus, the subset $\mathcal{R}_{+,\Sigma=1}^{K}$ is invariant under $\mathcal{A}$. By Brouwer's fixed point theorem, $\mathcal{A}$ has at least one fixed point in $\mathcal{R}_{+,\Sigma=1}^{K}$.

For any nonzero point $\mathbf{x}$ in $\mathcal{R}_{+,\Sigma\le 1}^{K}$, $\mathbf{x}/(x_1+ x_2+\ldots+ x_K)$ is in $\mathcal{R}_{+,\Sigma=1}^{K}$. Since $x_1+ x_2+\ldots+ x_K \le 1$, we have $\mathbf{x} \le \mathbf{x}/(x_1+ x_2+\ldots+ x_K)$. It is then easy to see that $\mathcal{A}(\mathbf{x}) \le \mathcal{A}(\mathbf{x}/(x_1+ x_2+\ldots+ x_K)) \in \mathcal{R}_{+,\Sigma=1}^{K} \subset \mathcal{R}_{+,\Sigma\le 1}^{K}$. Thus, the subset $\mathcal{R}_{+,\Sigma\le 1}^{K}$ is invariant under $\mathcal{A}$. It is apparent that the minimal fixed point is in $\mathcal{R}_{+,\Sigma\le 1}^{K}$ and the fixed point in $\mathcal{R}_{+,\Sigma=1}^{K}$ is larger than or equal to the minimal fixed point. This completes the proof of Lemma 4.1.

According to Theorem 2.8, any fixed point in $\mathcal{R}_{+,\Sigma=1}^{K}$ provides information about the minimal fixed solutions in the subset $\mathcal{R}_{+,\Sigma\le 1}^{K}$. Thus, we need to find a fixed point in $\mathcal{R}_{+,\Sigma=1}^{K}$. For that purpose, we propose the following simple computational procedure. Let $\mathbf{x}[0] = \mathbf{e}/K$, and $\mathbf{x}[n+1] = \mathcal{A}(\mathbf{x}[n])$ for $n\ge 0$. Then the sequence $\{\mathbf{x}[n], n\ge 0\}$ must have a convergent subsequence since the subset $\mathcal{R}_{+,\Sigma=1}^{K}$ is bounded and tight. If the sequence $\{\mathbf{x}[n], n\ge 0\}$ converges, we denote the limit as $\mathbf{y}^*$, which is a fixed point with $\sum_{k=1}^{K} y_k^* = 1$ and $\mathbf{y}^* = \mathcal{A}(\mathbf{y}^*)$. The fixed point $\mathbf{y}^*$ plays an important role in classifying Markov chains of *GI*/*M*/1 type with a tree structure.

For later use, we prove the following interesting result.

**Lemma 5.2** For $\{a(k, J), b(J), J\in\aleph, 1\le k\le K\}$ satisfying equation (5.2), $\lim_{n\to\infty} \max_{J\in\aleph:\,|J|=n}\{b(J)\} = 0$, and any $\mathbf{x}\in \mathcal{R}_{+,\Sigma=1}^{K}$, we have

$$b(J) = \sum_{H\in\aleph:\,H\ne 0} \mathbf{x}^{(H)} \sum_{k=1}^{K} a(k, H+J), \quad J\in\aleph. \tag{5.9}$$

**Proof**. By equation (5.4), $b(J) = b(j+J) + \sum_{k=1}^{K} a(k, j+J)$, for $J\in\aleph$ and $1\le j\le K$. For any $\mathbf{x} \in \mathcal{R}_{+,\Sigma=1}^{K}$ and $J\in\aleph$, we have

$$
\begin{aligned}
b(J) &= \left(\sum_{j=1}^{K} x_j\right) b(J) = \sum_{j=1}^{K} x_j b(J) = \sum_{j=1}^{K} x_j \left[ b(j+J) + \sum_{k=1}^{K} a(k, j+J) \right] \\
&= \sum_{j=1}^{K} x_j b(j+J) + \sum_{j=1}^{K}\sum_{k=1}^{K} x_j a(k, j+J) \\
&= \sum_{i=1}^{K}\sum_{j=1}^{K} x_i x_j b(i+j+J) + \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K} x_i x_j a(k, i+j+J) + \sum_{j=1}^{K}\sum_{k=1}^{K} x_j a(k, j+J) \\
&= \cdots \\
&= \sum_{H\in\aleph:\, H\neq 0} \mathbf{x}^{(H)} \left(\sum_{k=1}^{K} a(k, H+J)\right).
\end{aligned} \tag{5.10}
$$

Note that we used the assumption $\lim_{n\to\infty} \max_{J\in\aleph:\,|J|=n} \{b(J)\} = 0$ in equation (5.10). This completes the proof of Lemma 5.2.

**Note**: Lemma 5.1 can be obtained from Lemma 5.2. Letting $J$=0 in equation (5.9) and adding $\{a(k, 0), 1\leq k\leq K\}$ on both sides of equation (5.9), it is easy to see that $\mathcal{A}(\mathbf{x})$ is in $\mathcal{R}^K_{+,\Sigma=1}$ for any $\mathbf{x} \in \mathcal{R}^K_{+,\Sigma=1}$. Thus, the set $\mathcal{R}^K_{+,\Sigma=1}$ is invariant under the mapping $\mathcal{A}$. Therefore, $\mathcal{A}$ has a fixed point in $\mathcal{R}^K_{+,\Sigma=1}$.

## 5.2 Classification of Markov chains of *GI/M*/1 type with a tree structure

We denote the minimal fixed point of the mapping $\mathcal{A}$ as $\mathbf{R} = (R_1, R_2, \ldots, R_K)^T$, i.e., $\mathbf{R} = \mathbf{x}_{\min}$. According to Yeung and Sengupta [24], $R_k$ is the mean number of visits to the node $J+k$, before the Markov chain returns to the node $J$, given that the Markov chain was initially in the node $J$. Such an interpretation of $\mathbf{R}$ is obtained from the *GI/M*/1 structure of the Markov chain. In addition, $\mathbf{R}^{(W)}$ can be interpreted as the mean number of visits to the node $J+W$ ($J\in\aleph$, $W\in\aleph$, and $W\neq 0$), before the Markov chain returns to the node $J$, given that the Markov chain was initially in the node $J$.

To prove the main result of this section, we shall use the subinvariant (invariant) measures of Markov chains. Next, we introduce subinvariant and invariant measures of the Markov chains of *GI/M*/1 type with a tree structure. We refer to Chapter 5 of Seneta [21] for some classical results about the (sub)invariant measures of Markov chains.

A measure $\pi = \{\pi(J)\colon 0 \leq \pi(J) < \infty \text{ for } J\in\aleph\}$ is a subinvariant measure of the Markov chain $\{C_n, n\geq 0\}$ if $\pi(J+k) \geq \sum_{H\in\aleph} \pi(J+H) a(k,H)$ for all $J\in\aleph$ and $1\leq k\leq K$, and $\pi(0) \geq \sum_{J\in\aleph} \pi(J) b(J)$. If $\pi(J+k) = \sum_{H\in\aleph} \pi(J+H) a(k,H)$ for all $J\in\aleph$ and $1\leq k\leq K$, and

$\pi(0) = \sum_{J\in\aleph} \pi(J) b(J)$, the measure π is called an invariant measure. If $\sum_{J\in\aleph} \pi(J)$ is finite for a (sub)invariant measure π, then π is called a finite (sub)invariant measure.

According to Theorem 5.3 in Seneta [21], if the Markov chain $\{C_n, n\geq 0\}$ is irreducible, $\pi(\mathbf{R}) = \{\pi_{\mathbf{R}}(J) = \mathbf{R}^{(J)}, J\in\aleph\}$ is a subinvariant measure (note that $\pi_{\mathbf{R}}(0) = \mathbf{R}^{(0)} = 1$). The measure $\pi(\mathbf{R})$ is the minimal subinvariant measure, i.e., for any subinvariant measure π of $\{C_n, n\geq 0\}$ with $\pi(0) = 1$, we must have $\pi(J) \geq \pi_{\mathbf{R}}(J)$ for all $J\in\aleph$. We first show the following relationship between the (sub)invariant measures and the fixed point(s) of the mapping $\mathcal{A}$.

**Lemma 5.3** Assume that the Markov chain $\{C_n, n\geq 0\}$ is irreducible and aperiodic, $\lim_{n\to\infty} \max_{J\in\aleph: |J|=n} \{b(J)\} = 0$, and $\mathcal{A}^{(1)}(\mathbf{y}^*)$ is irreducible. If $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) > 1$, then $\pi(\mathbf{R})$ is a finite invariant measure. If $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) = 1$, then $\pi(\mathbf{R})$ is an infinite invariant measure. If $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) < 1$, then $\pi(\mathbf{R})$ is a subinvariant measure but not an invariant measure. In this case, let **z** be the fixed point of $\mathcal{A}$ that is larger than **R**, then $\pi(\mathbf{z}) = \{\pi(J) = \mathbf{z}^{(J)}, J\in\aleph\}$ is an infinite invariant measure.

**Proof**. If $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) > 1$, by Theorem 2.8, we have $\mathbf{R} \leq \mathbf{y}^*$ and $\mathbf{R} \neq \mathbf{y}^*$. For the measure $\pi(\mathbf{R})$, it is easy to verify that $\pi_{\mathbf{R}}(J+k) = \sum_{H\in\aleph} \pi_{\mathbf{R}}(J+H) a(k,H)$ holds for all $J\in\aleph$ and $1\leq k\leq K$. We only need to show $1 = \pi_{\mathbf{R}}(0) = \sum_{J\in\aleph} b(J)\mathbf{R}^{(J)}$. Let $r = R_1+R_2+ \ldots +R_K$. Then $0<r<1$ and $\mathbf{R}/r$ is in the subset $\mathcal{R}^K_{+,\Sigma=1}$. By Lemma 5.2, we have

$$\begin{aligned}
\sum_{J\in\aleph} \pi_{\mathbf{R}}(J) b(J) &= \sum_{J\in\aleph} \mathbf{R}^{(J)} b(J) = \sum_{J\in\aleph} \mathbf{R}^{(J)} \sum_{H\in\aleph:\, H\neq 0} (\mathbf{R}/r)^{(H)} \sum_{k=1}^{K} a(k, H+J) \\
&= \sum_{J\in\aleph} \left( \sum_{k=1}^{K} a(k,J) \right) \sum_{J_1, J_2\in\aleph:\, J_2\neq 0,\, J_1+J_2=J} \mathbf{R}^{(J_1)} (\mathbf{R}/r)^{(J_2)} \\
&= \sum_{J\in\aleph} \left( \sum_{k=1}^{K} a(k,J) \right) \mathbf{R}^{(J)} \sum_{n=1}^{|J|} (1/r)^n = \sum_{J\in\aleph} \left( \sum_{k=1}^{K} a(k,J) \right) \mathbf{R}^{(J)} \left( \frac{\left[1-(1/r)^{|J|}\right](1/r)}{1-1/r} \right) \\
&= \frac{1}{r-1} \sum_{J\in\aleph} \left( \sum_{k=1}^{K} a(k,J) \right) \mathbf{R}^{(J)} \left(1-(1/r)^{|J|}\right) \\
&= \frac{1}{r-1} \left( \sum_{J\in\aleph} \left( \sum_{k=1}^{K} a(k,J) \right) \mathbf{R}^{(J)} - \sum_{J\in\aleph} \left( \sum_{k=1}^{K} a(k,J) \right) (\mathbf{R}/r)^{(J)} \right) \\
&= \left( \frac{1}{r-1} \right) (r-1) = 1.
\end{aligned} \tag{5.11}$$

Thus, $\pi(\mathbf{R})$ is an invariant measure. Since $r<1$, $\sum_{J\in\aleph} \pi_{\mathbf{R}}(J) = \sum_{J\in\aleph} \mathbf{R}^{(J)} = 1/(1-r) < \infty$. Therefore, $\pi(\mathbf{R})$ is a finite invariant measure.

If $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) = 1$, $\mathbf{R} = \mathbf{y}^*$ is the only fixed point of the mapping $\mathcal{A}$. Again, for the measure $\pi(\mathbf{R})$, it is easy to see that $\pi_{\mathbf{R}}(J+k) = \sum_{H\in\aleph}\pi_{\mathbf{R}}(J+H)a(k,H)$ holds for all $J\in\aleph$ and $1\le k\le K$. To show $1 = \pi_{\mathbf{R}}(0) = \sum_{J\in\aleph} b(J)\mathbf{R}^{(J)}$, we introduce another Markov chain $\{C_{t,n},\ n\ge 0\}$ for $0<t\le 1$ as follows:

$$
\begin{aligned}
&a_t(k,0) = a(k,0), \qquad \text{for} \quad 1\le k\le K;\\
&a_t(k,J) = ta(k,J), \qquad \text{for} \quad J\in\aleph,\ J\ne 0 \ \text{ and } \ 1\le k\le K;\\
&b_t(J) = (1-t)b(0) + tb(J), \quad \text{for} \quad J\in\aleph.
\end{aligned}
\tag{5.12}
$$

The Markov chain $\{C_{t,n},\ n\ge 0\}$ is irreducible and aperiodic. Since the transition probability from any node $J$ ($\ne 0$) to the node 0 is $b_t(J) > (1-t)b(0)>0$, the Markov chain $\{C_{t,n},\ n\ge 0\}$ is positive recurrent for $0<t<1$. Consider the mapping $\mathcal{A}_t$ associated with transition probabilities $\{a_t(k, J),\ J\in\aleph$ and $1\le k\le K\}$ defined by equation (2.2). For any $\mathbf{x}\in\mathcal{R}^K_{+,\Sigma\le 1}$, we have $\mathcal{A}_t(\mathbf{x}) \le \mathcal{A}(\mathbf{x}) \in \mathcal{R}^K_{+,\Sigma\le 1}$ (note that $0<t\le 1$). Thus, the set $\mathcal{R}^K_{+,\Sigma\le 1}$ is invariant under the mapping $\mathcal{A}_t$. By Brouwer's fixed point theorem, the mapping $\mathcal{A}_t$ has a fixed point in the subset $\mathcal{R}^K_{+,\Sigma\le 1}$, denoted as $\mathbf{R}_t$. Since the Markov chain is positive recurrent for $0<t<1$, by Theorem 5,3 in Seneta [20], $\pi(\mathbf{R}_t) = \{\pi_{\mathbf{R}_t}(J) = \mathbf{R}_t^{(J)},\ J\in\aleph\}$ is a finite invariant measure, which implies that $1 = \pi_{\mathbf{R}_t}(0) = \sum_{J\in\aleph} b_t(J)\mathbf{R}_t^{(J)}$. Since $\mathbf{R}_t$ is the minimal fixed point of $\mathcal{A}_t$, we can use equation (2.4) to obtain it. Then it is easy to see that $\mathbf{R}_t$ is nondecreasing in $t$. This further implies that $\mathbf{R}_t$ converges to $\mathbf{R}$ when $t$ goes to 1. Therefore, we have

$$
\begin{aligned}
1 = \pi_{\mathbf{R}_t}(0) &= \sum_{J\in\aleph} b_t(J)\mathbf{R}_t^{(J)}\\
&= t\sum_{J\in\aleph} b(J)\mathbf{R}_t^{(J)} + (1-t)b(0)\sum_{J\in\aleph}\mathbf{R}_t^{(J)}\\
&= t\sum_{J\in\aleph} b(J)\mathbf{R}_t^{(J)} + (1-t)b(0)\frac{1}{1-(R_{t,1}+\cdots+R_{t,K})}.
\end{aligned}
\tag{5.13}
$$

Using equation (2.2) and equation (2.5), we can find the differentiations of $\mathbf{R}_t$ with respect to $t$ as:

$$
\begin{pmatrix} \dfrac{dR_{t,1}}{dt} \\ \vdots \\ \dfrac{dR_{t,K}}{dt} \end{pmatrix} = (I - \mathcal{A}_t^{(1)}(\mathbf{R}_t))^{-1} \begin{pmatrix} \sum_{J\in\aleph, J\ne 0}\mathbf{R}_t^{(J)}a(1,J) \\ \vdots \\ \sum_{J\in\aleph, J\ne 0}\mathbf{R}_t^{(J)}a(K,J) \end{pmatrix}.
\tag{5.14}
$$

It is clear that $sp(\mathcal{A}^{(1)}(\mathbf{R}_t))$ converges to $sp(\mathcal{A}^{(1)}(\mathbf{R})) = 1$. Thus, the sum of the left hand side of equation (5.14) (that is $d(R_{t,1})/dt +\ldots+ d(R_{t,K})/dt$) goes to infinity when $t$ increases to 1. Then we have, by the *l'Hospital's Rule*,

$$\lim_{t\to 1-}(1-t)b(0)\frac{1}{1-(R_{t,1}+\cdots+R_{t,K})} = \frac{-b(0)}{-\left.\dfrac{d(R_{t,1}+\cdots+R_{t,K})}{dt}\right|_{t=1-}} = \frac{b(0)}{\infty} = 0. \qquad (5.15)$$

Since $\mathbf{R}_t$ is nondecreasing in $t$, it is easy to see that $t\sum_{J\in\aleph} b(J)\mathbf{R}_t^{(J)}$ converges monotonically to $\sum_{J\in\aleph} b(J)\mathbf{R}^{(J)}$ as $t$ increases to 1. Combining equations (5.13) and (5.15), we have proved that $1=\pi_{\mathbf{R}}(0)=\sum_{J\in\aleph} b(J)\mathbf{R}^{(J)}$, which implies that $\pi(\mathbf{R})$ is an infinite invariant measure if $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) = 1$.

Lastly, if $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) < 1$, $\mathbf{R} = \mathbf{y}^*$ and there exists another larger fixed point $\mathbf{z}$ of the mapping $\mathcal{A}$. Similar to the $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) > 1$ case, it can be proved that $\pi(\mathbf{z})$ is an infinite invariant measure (by utilizing the point $\mathbf{z}/(z_1+\ldots+z_K) \in \mathcal{R}^K_{+,\Sigma=1}$). For the measure $\pi(\mathbf{R})$, it is easy to see that $\pi_{\mathbf{R}}(J+k)=\sum_{H\in\aleph}\pi_{\mathbf{R}}(J+H)a(k,H)$ holds for all $J\in\aleph$ and $1\le k\le K$, and $\sum_{J\in\aleph} b(J)\mathbf{R}^{(J)} < \sum_{J\in\aleph} b(J)\mathbf{z}^{(J)} = 1 = \pi_{\mathbf{R}}(0)$. Therefore, $\pi(\mathbf{R})$ is a subinvariant measure but not an invariant measure.

This completes the proof of Lemma 5.3.

Now, we state and prove the main result for the Markov chain of *GI/M*/1 type with a tree structure.

**Theorem 5.4** Assume that the Markov chain of *GI/M*/1 type with a tree structure $\{C_n, n\geq 0\}$ is irreducible and aperiodic, $\lim_{n\to\infty}\max_{J\in\aleph:|J|=n}\{b(J)\}=0$, and the matrix $\mathcal{A}^{(1)}(\mathbf{y}^*)$ is irreducible. Then the Markov chain is

1) positive recurrent if and only if $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) > 1$, i.e., there exists a fixed point of $\mathcal{A}$ that is smaller than $\mathbf{y}^*$;
2) null recurrent if and only if $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) = 1$, i.e., $\mathbf{y}^*$ is the only fixed point of $\mathcal{A}$;
3) transient if $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) < 1$, i.e., $\mathbf{y}^*$ is the smallest fixed point and there exists another larger fixed point of $\mathcal{A}$.

**Proof**. We shall prove part 1) and part 3). Part 2) is obtained from part 1), part 3), and Theorem 2.8.

First, we prove part 1). If the Markov chain is positive recurrent, according to Theorems 5.3 and 5.5 in Seneta [20] and by the *GI/M/*1 structure, $\pi(\mathbf{R})$ is a finite invariant measure. Then we must have $r = R_1+R_2+\ldots+R_K < 1$ and $\sum_{J\in\aleph} b(J)\mathbf{R}^{(J)} = 1$. Furthermore, $\mathbf{R}$ is a fixed point of $\mathcal{A}$. Then, $\mathbf{y}^*$ is not the minimal fixed point. According to Theorem 2.8, $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) > 1$. On the other hand, if $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) > 1$, according to Lemma 5.3, the measure $\pi(\mathbf{R})$ is a *finite* invariant measure. Thus, the Markov chain is positive recurrent. In fact, $\{\pi(J) = (1-r)\mathbf{R}^{(J)}$ for $J\in\aleph\}$ is the stationary distribution of the Markov chain.

Now, we prove part 3). If $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) < 1$, according to Theorem 2.8, $\mathcal{A}$ has two fixed points: $\mathbf{y}^*$ and another $\mathbf{z}$, where $\mathbf{y}^* \le \mathbf{z}$ and $\mathbf{y}^* \ne \mathbf{z}$. Let $z = z_1+ z_2+\ldots+ z_K$. Then $z>1$ holds. According to Lemma 5.3, we have two subinvariant measures: $\pi(\mathbf{R})$ and $\pi(\mathbf{z})$. Since $\mathbf{R} \ne \mathbf{z}$, $\pi(\mathbf{R})$ and $\pi(\mathbf{z})$ are two *different* subinvariant measures, i.e., $\pi(\mathbf{R}) = t\,\pi(\mathbf{z})$ does not hold for any positive $t$. By Theorem 5.4 in Seneta [21], a Markov chain (irreducible and aperiodic) is transient if and only if it has at least two different subinvariant measures. Thus, the Markov chain of interest must be transient. On the other hand, if the Markov chain is transient, then by Lemma 5.5 and its corollary in Seneta [21], $\pi(\mathbf{R})$ is a subinvariant measure but not an invariant measure. According to Lemma 5.3, $sp(\mathcal{A}^{(1)}(\mathbf{y}^*))$ must be smaller than 1. By Theorem 2.8, this is equivalent to the existence of a fixed point that is larger than $\mathbf{y}^*$. This completes the proof of Theorem 5.4.

Theorem 5.4 finds applications in queueing theory. For example, we consider a *GI/G*[*K*]/1/LCFS preemptive repeat queue. In that queue, customers arrive in the queueing system according to a renewal process. When a customer arrives, it is marked as a type $k$ customer with probability $p_k$, $1\le k\le K$. For a type $k$ customer, its service time $s_k$ has a general distribution, $1\le k\le K$. All customers join a single queue. The service discipline for all customers is last-come-first-served preemptive repeat. The queueing process in this system can be formulated as a *GI/M/*1 type Markov chain with a tree structure. Then Theorem 5.4 can be used to find whether or not the queueing system is stable. Details are omitted.

To classify a Markov chain of *GI/M/*1 type with a tree structure, according to Theorem 5.4, we need to find the fixed point $\mathbf{y}^*$ first. That requires significant computational efforts. Next, we identify a sufficient condition for transience. Let $\aleph_\infty = \{\ldots j_n \ldots j_2 j_1: 1\le j_n\le K,\ n\ge 1\}$. For $J = \ldots j_n \ldots j_2 j_1 \in \aleph_\infty$, let

$$\hat{a}_J(n) = \sum_{k=1}^{K} a(k, f(J,n)),\ n \ge 0. \tag{5.16}$$

Intuitively, $J\in\aleph_\infty$ is a path from $j_1$ at "*infinity*" to the root node 0. By definition, $\{\hat{a}_J(n), n \ge 0\}$ is a probability distribution for any $J\in\aleph_\infty$. Let

$$\rho(J) = \sum_{n=0}^{\infty} n\hat{a}_J(n), \quad J \in \aleph_\infty; \qquad \rho_{\max} = \max_{J\in\aleph_\infty}\{\rho(J)\}. \tag{5.17}$$

**Theorem 5.5** Assume that Markov chain of *GI/M*/1 type with a tree structure $\{C_n, n\geq 0\}$ is irreducible and aperiodic. If $\rho_{max} < 1$, then the Markov chain is transient.

**Proof.** See Appendix.

Intuitively, the condition $\rho_{max} < 1$ ensures that in each transition, the Markov chain tends to move away from the root node. The condition $\rho_{max} < 1$ is not necessary for transience of the Markov chain. Compared to Theorem 5.4, Theorem 5.5 can be easily implemented. For instance, we consider a *GI/M/*1 type Markov chain with $K$=2 and transition probabilities:

| | | |
|---|---|---|
| $a(1, 0) = 0.3$, | $a(2, 0) = 0.5$, | $b(0) = 0.2$; |
| $a(1, 1) = 0.05$, | $a(2, 1) = 0.05$, | $b(1) = 0.1$; |
| $a(1, 2) = 0.1$, | $a(2, 2) = 0$, | $b(2) = 0.1$; |
| $a(1, 11) = 0.05$, | $a(2, 11) = 0.05$, | $b(11) = 0$; |
| $a(1, 12) = 0.05$, | $a(2, 12) = 0.05$, | $b(12) = 0$; |
| $a(1, 21) = 0.1$, | $a(2, 21) = 0$, | $b(21) = 0$; |
| $a(1, 22) = 0.1$, | $a(2, 22) = 0$, | $b(22) = 0$. |

It can be calculated that $\rho_{max} = 0.3 < 1$. Therefore, the Markov chain is transient.

Theorems 5.4 and 5.5 characterize the classification conditions of Markov chains of *GI/M*/1 type with a tree structure. The two theorems lead to better understanding of such Markov chains and to possible directions for future research.

**Note**: We assume $\lim_{n\to\infty} \max_{J\in\aleph:|J|=n}\{b(J)\} = 0$ to ensure that there is a fixed point in the subset $\mathcal{R}^{K}_{+,\Sigma=1}$. If $\lim_{n\to\infty} \max_{J\in\aleph:|J|=n}\{b(J)\} > 0$, there exists a sequence of strings $\{J(n), n\geq 1\}$ such that $b(J(n)) > b(\infty) > 0$. Intuitively, that means the Markov chain can return to the root node from a "*remote*" node in one transition. Thus, the Markov chain should be positive recurrent. In fact, this is true for some special cases such as the case with $K$=1 or the case with $b(J) > b(\infty) > 0$ for all string $J$. However, a general treatment of the case with $\lim_{n\to\infty} \max_{J\in\aleph:|J|=n}\{b(J)\} > 0$ can be very complicated. We leave it for future research.

## 6. SUMMARY AND DISCUSSIONS

In this paper, we have shown some useful properties associated with some fixed points of the nonlinear mapping $\mathcal{A}$. Fixed point theory and degree theory were utilized to establish a relationship between the fixed points, the matrices of partial differentiation (Jacobian) at the fixed points, and the minimal fixed point of $\mathcal{A}$. These properties were then used to find classification conditions for Markov chains of *M/G*/1 type and *GI/M*/1 type with a tree structure. It was shown that the Perron-Frobenius eigenvalue of the differentiation matrix $\mathcal{A}^{(1)}(\mathbf{x})$ of any special fixed point **x** and the fixed point **x** itself provide information for a complete classification of the Markov chains of interest.

More specifically, the classification problem of interest was divided into four subproblems: 1) the classification of Markov chains of *M/G*/1 type with a tree structure; 2) the classification of Markov chains of matrix *M/G*/1 type with a tree structure; 3) the classification of Markov chains of *GI/M*/1 type with a tree structure; and 4) the classification of Markov chains of matrix *GI/M*/1 type with a tree structure. For case 1), explicit classification conditions were found. The reason for the explicit conditions is that a fixed point **e** of the mapping $\mathcal{A}$ is readily obtained. Unfortunately, no such an explicit fixed point exists for the other three cases. Thus no explicit classification condition was found for these cases. Nonetheless, for cases 2) and 3), we were able to prove that a fixed point exists in certain subsets. Then explicit conditions were found in terms of that fixed point. For case 4), no fixed point has been identified in general. Thus, the classification problem of Markov chains of matrix *GI/M*/1 type with a tree structure is still open for future research. The results obtained in this paper indicate that identifying a fixed point might be the key to solve the problem. Generalizations to the matrix cases are challenging problems, especially for the matrix *GI/M*/1 case. But they are interesting future topics.

## APPENDIX The proof of Theorem 5.5

According to a classical result (see Cohen [2] and Fayolle, et al. [3]), the Markov chain of interest is transient if and only if there exists a non-constant and bounded solution $\{h(J), J \in \aleph\}$ that satisfies

$$h(J) = b(J)h(0) + \sum_{k=1}^{K}\sum_{n=0}^{|J|} a(k, f(J,n))h(J - f(J,n) + k), \quad \text{for} \quad J \in \aleph,\ J \neq 0. \quad \text{(A.1)}$$

Next, we construct such a solution $\{h(J), J \in \aleph\}$ if $\rho_{max} < 1$. Let $h(0) = 0$. Define the following transforms for $\mathbf{z} = (z_1, z_2, \ldots, z_K)^T \in \mathcal{R}_+^K(\mathbf{y}^*)$:

$$\begin{aligned} h^*(\mathbf{z},k) &\equiv \sum_{J\in\aleph} h(J+k)\mathbf{z}^{(J+k)}, \quad \text{for} \ 1 \le k \le K; \\ h^*(\mathbf{z}) &\equiv h^*(\mathbf{z},1) + \cdots + h^*(\mathbf{z},K) = \sum_{J\in\aleph} h(J)\mathbf{z}^{(J)}. \end{aligned} \quad \text{(A.2)}$$

Combining equations (A.1) and (A.2), we have

$$
\begin{aligned}
h^*(\mathbf{z}) &= \sum_{J\in\aleph:\, J\neq 0}\left(\sum_{k=1}^{K}\sum_{n=0}^{|J|} a(k,\, f(J,n))h(J - f(J,n)+k)\right)\mathbf{z}^{(J)} \\
&= \sum_{J\in\aleph:\, J\neq 0}\left(\sum_{k=1}^{K}\sum_{n=0}^{|J|} a(k,\, f(J,n))\mathbf{z}^{(f(J,n))}h(J - f(J,n)+k)\mathbf{z}^{(J-f(J,n))}\right) \\
&= \sum_{k=1}^{K}\frac{1}{z_k}\left[\sum_{J,H\in\aleph:\, J+H\neq 0} a(k,J)\mathbf{z}^{(J)}h(H+k)\mathbf{z}^{(H+k)}\right] \\
&= \sum_{k=1}^{K}\frac{1}{z_k}\left[\sum_{J,H\in\aleph} a(k,J)\mathbf{z}^{(J)}h(H+k)\mathbf{z}^{(H+k)} - a(k,0)h(k)z_k\right] \\
&= \sum_{k=1}^{K}\frac{1}{z_k}\left[a_k^*(\mathbf{z})h^*(\mathbf{z},k) - a(k,0)h(k)z_k\right].
\end{aligned}
\tag{A.3}
$$

Let $h(k)=1$ for $1\le k\le K$ and $h(J+k)=w(J)$ for $1\le k\le K$ for all $J\in\aleph$. Define, for $\mathbf{z} = (z_1, z_2, \ldots, z_K)^{\mathrm{T}} \in \mathcal{R}_+^K(\mathbf{y}^*)$:

$$
\begin{aligned}
& w^*(\mathbf{z}) \equiv \sum_{J\in\aleph} w(J)\mathbf{z}^{(J)}; \\
& a^*(\mathbf{z}) \equiv \sum_{k=1}^{K} a_k^*(\mathbf{z}) = \sum_{k=1}^{K}\sum_{J\in\aleph} a(k,J)\mathbf{z}^{(J)} = \sum_{J\in\aleph}\left(\sum_{k=1}^{K} a(k,J)\right)\mathbf{z}^{(J)} \equiv \sum_{J\in\aleph}\tilde{a}(J)\mathbf{z}^{(J)}.
\end{aligned}
\tag{A.4}
$$

Then equation (A.3) leads to

$$
\left[a^*(\mathbf{z}) - \left(\sum_{k=1}^{K} z_k\right)\right]w^*(\mathbf{z}) = \tilde{a}(0). \tag{A.5}
$$

Since $sp(\mathcal{A}^{(1)}(\mathbf{y}^*)) < 1$, by Lemma 2.2, there is no fixed point in $\mathcal{R}_+^K(\mathbf{y}^*)$ except $\mathbf{y}^*$. Therefore, we can find $w^*(\mathbf{z})$ by $w^*(\mathbf{z}) = \tilde{a}(0)\left[a^*(\mathbf{z}) - \left(\sum_{k=1}^{K} z_k\right)\right]^{-1}$ for $\mathbf{z} \in \mathcal{R}_+^K(\mathbf{y}^*)$ and $\mathbf{z} \neq \mathbf{y}^*$. We shall use this expression to prove that $\{w(J), J\in\aleph\}$ are bounded. First, we rewrite (A.5) as follows.

$$
w^*(\mathbf{z}) = \frac{\tilde{a}(0)}{\left[1 - \left(\sum_{k=1}^{K} z_k\right)\right]}\frac{\left[1 - \left(\sum_{k=1}^{K} z_k\right)\right]}{\left[a^*(\mathbf{z}) - \left(\sum_{k=1}^{K} z_k\right)\right]} \equiv \frac{\tilde{a}(0)}{\left[1 - \left(\sum_{k=1}^{K} z_k\right)\right]}\left(\sum_{J\in\aleph} c(J)\mathbf{z}^{(J)}\right). \tag{A.6}
$$

Part of equation (A.6) is evaluated as follows.

$$\left[1-\left(\sum_{k=1}^{K} z_k\right)\right]^{-1}\left[a^*(\mathbf{z})-\left(\sum_{k=1}^{K} z_k\right)\right]=\left[\sum_{n=0}^{\infty}\left(\sum_{k=1}^{K} z_k\right)^n\right]\left[a^*(\mathbf{z})-\left(\sum_{k=1}^{K} z_k\right)\right]$$

$$=\left[\sum_{n=0}^{\infty}\left(\sum_{k=1}^{K} z_k\right)^n\right]a^*(\mathbf{z})-\sum_{n=1}^{\infty}\left(\sum_{k=1}^{K} z_k\right)^n \tag{A.7}$$

$$=a^*(\mathbf{z})\left[\sum_{J\in\aleph}\mathbf{z}^{(J)}\right]-\sum_{J\in\aleph:\,J\neq 0}\mathbf{z}^{(J)}=\tilde{a}(0)-\sum_{J\in\aleph:\,J\neq 0}\mathbf{z}^{J}\left(1-\sum_{n=0}^{|J|}\tilde{a}(J-f(J,n))\right).$$

Then equations (A.6) and (A.7) lead to

$$1=\left\{\tilde{a}(0)-\sum_{J\in\aleph:\,J\neq 0}\mathbf{z}^{(J)}\left(1-\sum_{n=0}^{|J|}\hat{a}(J-f(J,n))\right)\right\}\left(\sum_{J\in\aleph}c(J)\mathbf{z}^{(J)}\right)$$

$$=\tilde{a}(0)c(0)+\sum_{J\in\aleph}\mathbf{z}^{(J)}\left\{\tilde{a}(0)c(J)-\sum_{H,F\in\aleph:\,H+F=J\ H\neq 0}\left(1-\sum_{i=0}^{|H|}\tilde{a}(H-f(H,i))\right)c(F)\right\} \tag{A.8}$$

$$=\tilde{a}(0)c(0)+\sum_{J\in\aleph}\mathbf{z}^{(J)}\left\{\tilde{a}(0)c(J)-\sum_{n=1}^{|J|}\left(1-\sum_{i=0}^{|J-f(J,n)|}\tilde{a}(f(f(J,n),i))\right)c(J-f(J,n))\right\}.$$

According to equation (A.6), we choose $\{c(J),\ J\in\aleph\}$ in the following manner:

$$1=\tilde{a}(0)c(0);$$

$$\tilde{a}(0)c(J)=\sum_{n=1}^{|J|}\left(1-\sum_{i=0}^{|f(J,n)|}\tilde{a}(f(f(J,n),i))\right)c(J-f(J,n)) \tag{A.9}$$

$$=\sum_{n=1}^{|J|}b(f(J,n))c(J-f(J,n))$$

Therefore, all $\{c(J),\ J\in\aleph\}$ are positive. Then equation (A.6) leads to

$$w^*(\mathbf{z})=\tilde{a}(0)\left[\sum_{n=0}^{\infty}\left(\sum_{k=1}^{K} z_k\right)^n\right]\left(\sum_{J\in\aleph}c(J)\mathbf{z}^{(J)}\right)=\tilde{a}(0)\left(\sum_{J\in\aleph}\mathbf{z}^{(J)}\right)\left(\sum_{J\in\aleph}c(J)\mathbf{z}^{(J)}\right)$$

$$=\tilde{a}(0)\sum_{J\in\aleph}\mathbf{z}^{(J)}\left\{\sum_{n=0}^{|J|}c(f(J,n))\right\}. \tag{A.10}$$

Next, we prove that there exists $M$ (>0) such that for any $J\in\aleph_\infty$ and $n$>0,

$$\sum_{i=0}^{n}c(f(J,i))=c(0)+c(j_1)+c(j_2 j_1)+...+c(j_n j_{n-1}\cdots j_1)<M\ . \tag{A.11}$$

Note that $\hat{a}(0) = \tilde{a}(0)$. Again, by equation (A.9), we have, for any $J \in \aleph_\infty$,

$$\begin{aligned} c(f(J,n)) &= \frac{\max_{1\le i\le n}\{c(J - f(J,i))\}}{\tilde{a}(0)} \sum_{i=1}^{n} b(f(J,i)) \\ &\le \frac{\max_{1\le i\le n}\{c(J - f(J,i))\}}{\hat{a}(0)} \sum_{t=1}^{\infty} (t-1)\hat{a}_J(t) \\ &= \frac{\max_{1\le i\le n}\{c(J - f(J,i))\}}{\hat{a}(0)} \left( \sum_{t=1}^{\infty} t\hat{a}_J(t) - \sum_{t=1}^{\infty} \hat{a}_J(t) \right) \\ &= \frac{\max_{1\le i\le n}\{c(J - f(J,i))\}}{\hat{a}(0)} \left( \sum_{t=1}^{\infty} t\hat{a}_J(t) - 1 + \hat{a}(0) \right) \\ &\le \max_{1\le i\le n}\{c(J - f(J,i))\} \left( 1 - \frac{(1-\rho_{\max})}{\hat{a}(0)} \right) \\ &\le \left( 1 - \frac{(1-\rho_{\max})}{\hat{a}(0)} \right)^2 \max_{2\le i\le n}\{c(J - f(J,i))\} \\ &\le \cdots \le \left( 1 - \frac{(1-\rho_{\max})}{\hat{a}(0)} \right)^n c(0) = \left( 1 - \frac{(1-\rho_{\max})}{\hat{a}(0)} \right)^n \frac{1}{\hat{a}(0)}, \end{aligned} \tag{A.12}$$

where $0 < 1 - \frac{(1-\rho_{\max})}{\hat{a}(0)} < 1$. Thus, we must have, for any $J \in \aleph$ and $1 \le k \le K$,

$$\begin{aligned} h(J+k) = w(J) &= \tilde{a}(0) \sum_{i=0}^{|J|} c(f(J,i)) \\ &\le \tilde{a}(0) c(0) \left( 1 - \left( 1 - \frac{(1-\rho_{\max})}{\hat{a}(0)} \right) \right)^{-1} \\ &= \frac{\hat{a}(0)}{1-\rho_{\max}} \equiv M. \end{aligned} \tag{A.13}$$

Thus, we have found a set of non-constant and bounded $\{h(J),\ J \in \aleph\}$ for equation (A.1). Therefore, the Markov chain is transient. This completes the proof of Theorem 5.5.


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