GROUP SIZE AND PARALLELISM EFFECTS IN TASKS WITH HETEROGENEOUS LEVELS OF DIFFICULTY: A STOCHASTIC ORDER APPROACH *

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Abstract

We consider tasks where in order to perform them it is sufficient that one member of a group will know how to do it. We are interested in the effect of task difficulty, and variability of that difficulty, on group performance, and in particular on the marginal contribution of an additional number to the performance of groups of different size. We explore the implications of various stochastic orders over task difficulty and variability. Some intuitive conjectures are shown to be false.

Keywords: Management, stochastic comparison

1. Introduction

Consider a group, consisting of members of similar ability, which can perform a task as long as at least one of its members can perform it. The tasks involved vary in degree of difficulty. We are interested in the effect of task difficulty, and variability of that difficulty, on group performance, and in particular on the marginal contribution of an extra group member. A similar issue arises in multiple-engine searches on the web. Apparently a single search engine “picks up” only about 16 percent of the relevant references (Gordon, M. and P. Pathak, 1999), so multiple engines are often used. How is the contribution of an extra engine expected to be influenced by the difficulty, and inner variety, of the search? Other examples include problems, tests and challenges given to geographically separate groups or ones not allowed to communicate so as to preserve individual creativity. Various game shows also have similar features. While in practice groups will not usually consist of members of identical ability, our focus on the implications

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of group size naturally leads to the investigation of homogeneous groups. As there is no division of labor or cooperation per se, so we use the term “group” rather than “team”.

It stands to reason that very easy tasks can be performed by anyone, and thus a one-member group or one search engine is sufficient for performing such tasks. Very difficult tasks are likely to be doable only by large groups. Thus the type of tasks where, say, adding a fourth member to a group of three is likely to be most significant are those of intermediate difficulty. Those are the issues of most interest to us. But it is not clear whether the contribution of an extra member is monotone in the variability of task difficulty. The main results of this paper (Sections 3 to 5) provide answers to such problems under several stochastic orders related to the variability of task difficulty. The conclusions can be used in group size selection as a function of task difficulty and degree of heterogeneity.

The rest of the paper is organized as follows. In Section 2, the problem of interest is defined explicitly. In Sections 3 to 5, the impact of the difficulty level on the contribution of an extra member is investigated under several stochastic orders. The main mathematical tools are stochastic comparison (Ross (1983), Shaked and Shanthikumar (1994), and Stoyan (1983)) and total positivity of functions (Karlin (1968)).

2. Problem Definition

Consider a group consisting of *m* members, where *m* is a nonnegative integer. The group is formed to perform a task, and it is assumed that it can complete the task if at least one of its members can do the job. For concreteness, we envision a test, in which each task is a multiple-choice question administered to a group.

Tasks vary by degree of difficulty. Let X be the probability that any particular member *cannot* perform its tasks, or equivalently, with probability $1–X$ the member can perform the task. Thus X is a random variable in the interval [0,1]. Assume that X has a distribution function $F_X(x)$ and density function $f_X(x)$.

Assume that the performance of individual group members is **consistent**, in the sense that all have an equal probability of success (see Note 2.1 for a model where individual members perform independently once they are selected). The probability that the group of m members will be able to perform the task is given by, for $m\geq 0$,

$$\begin{aligned} p_X(m) &= \int_0^1 (1-x^m)dF_X(x) \\ &= E[1-X^m]. \end{aligned} \tag{2.1}$$

It is readily seen that the sequence $\{p_X(m), m\geq 0\}$ is an increasing concave function of m. Define the marginal contribution of the $(m+1)st$ member as:

$$\begin{aligned} d_X(m) &= p_X(m+1) - p_X(m) \\ &= \int_0^1 x^m(1-x)dF_X(x) \\ &= E[X^m(1-X)]. \end{aligned} \tag{2.2}$$

We are interested in the sequence $\{d_X(m), m\geq 0\}$. Since the function $d_X(m)$ is convex and decreasing in m, the marginal contribution of extra group members is decreasing, in agreement with intuition. It is easy to see that $\{d_X(m), m\geq 0\}$ is in fact a mixture of geometric distributions and is thus a probability mass function, i.e., $\sum_{m=0}^{\infty} d_X(m) = 1$, if $P\{X=1\} <1$. Clearly, thus, a change in X cannot move $d_X(m)$ in the same direction for all m.

Our objective is to find how the difficulty level of a task effects the magnitude of $d_X(m)$ for all $m \geq 0$. For that purpose, we consider two random variables X and Y in [0, 1] representing the probabilities that two tasks cannot be performed, respectively. We explore the relationship between $\{d_X(m), m\geq 0\}$ and $\{d_Y(m), m\geq 0\}$, given that certain relationship exists between X and Y. The relative difficulty level of tasks can be modelled by using stochastic orders, such as the likelihood ratio order, stochastically larger order, convex order, and more variable order of random variables. More specifically, we are interested in problems such as, given that X is smaller than Y in certain stochastic order, how many times the sequence $\{d_X(m)–d_Y(m), m\geq 0\}$ changes its sign. Note that a change of sign occurs if the value of $d_X(m)–d_Y(m)$ goes from negative (positive) to positive (negative). A single sign change in response to a change in task variability would imply that the benefit of adding a member to a small group differs from that of adding a member to a large group.

To demonstrate the complexity of the problem of interest we present an example to show that, without imposing proper stochastic order on X and Y, the sequence $\{d_X(m)–d_Y(m), m\geq 0\}$ can change its sign many times.

Example 2.1 We consider two random variables X and Y with distributions $P\{X=0.01\} = 0.05$, $P\{X=0.02\} = 0.1$, $P\{X=0.03\} = 0.1$, $P\{X=0.04\} = 0.1$, $P\{X=0.05\} = 0.15$, $P\{X=0.5\}=0.2$, $P\{X=0.95\} = 0.2$, and $P\{X=0.995\} = 0.1$; $P\{Y=0.02\} = 0.1$, $P\{Y=0.05\} = 0.1$, $P\{Y=0.5\} = 0.4$, and $P\{Y=0.995\} = 0.4$, respectively.

By routine calculations, we have $d_X(0) = 0.59355 > d_Y(0) = 0.395$, $d_X(1) = 0.0763 < d_Y(1) = 0.1087$, $d_X(4) = 0.01488 > d_Y(4) = 0.01446$, and $d_X(41) = 0.001627 < d_X(41) = 0.001628$. Thus, the sign of the sequence $\{d_X(m)–d_Y(m), m\geq 0\}$ changes at least three times. ■

For two random variables X and Y, if $E[f(X)] \leq E[f(Y)]$ holds for any nondecreasing function $f(x)$, then we say that X is stochastically smaller than Y, denoted as $X \leq_{st} Y$. We refer readers to Shaked and Shanthikumar (1994) and Stoyan (1983) for more details about stochastic order of random variables. It is easy to verify that in Example 2.1 $X \leq_{st} Y$. Thus, stochastic orders stronger than the stochastically smaller order are required to ensure that $\{d_X(m)–d_Y(m), m\geq 0\}$ changes its sign only once (or less than a predetermined number).

Note 2.1 A different Conceptualization Assume that individuals in a group solve a problem independently. Given the probabilities that m members cannot perform a task are $\{x_1, x_2, \ldots, x_m\}$, the conditional probability that the group can perform is

$$1 - x_1 x_2 \cdots x_m . \tag{2.3}$$

Then the unconditional probability that a group of size m can perform is given as $1–E[X_1X_2\ldots X_m]$. If member's abilities are drawn from distributions with the same mean, we have $d_X(m) = (E[X_1])^m(1–E[X_1])$. Unlike the previous model, where the distribution of X_1 plays an important role in the analysis of $\{d_X(m), m\geq 0\}$, the mean $E[X_1]$ is the only variable that is important in this model. This model is a special case of the one defined in (2.1) when the value of X is known (i.e., tasks are homogeneous in degree of difficulty). In other words, in our central model task difficulty varies but group member's abilities are equal, while in the model of (2.3) group member's abilities vary, but tasks are homogeneous. Further discussion on the relationship between the two models can be found in Section 3.

3. Random vs. Deterministic Level of Difficulty

In this section, we consider a random variable X defined on [0, 1] and its corresponding deterministic level $Y \equiv \mu = E[X]$. The comparison of the two models is interesting since a) it is well known that X is larger than μ under the convex order (Shaked and Shanthikumar, 1994), and b) the deterministic model is equivalent to the alternative model presented in Note 2.1.

The basic question to be answered in this section is how many times $d_X(m)–d_Y(m)$ changes its sign. We would like to show that $d_X(m)–d_Y(m)$ changes its sign at most once. For that purpose, we first prove a monotonicity property of the function $E[X^{t+s}(1–X^h)]/E[X^t(1–X^h)]$ with respect to t. By Hölder inequality (Chow and Teicher, 1988), we have

$$E[X^{t+s}(1-X^h)] \le \sqrt{E[X^{t+2s}(1-X^h)]E[X^t(1-X)]}, \quad (3.1)$$

which is equivalent to

$$\frac{E[X^{t+s}(1-X^h)]}{E[X^t(1-X^h)]} \le \frac{E[X^{t+s+s}(1-X^h)]}{E[X^{t+s}(1-X^h)]}. \quad (3.2)$$

Inequality (3.2) implies that $E[X^{t+s}(1–X^h)]/E[X^t(1–X^h)]$ may be increasing with respect to t, which is proved in the following lemma. Denote by $p=P\{X=1\}$.

Lemma 3.1 *If* $P\{X=1\} + P\{X=0\} = 1$, *then* $E[X^t(1–X^h)] = 0$ *and* $E[X^t(1–X)] – p^t(1–p) = –p^t(1–p) < 0$ *for all* $t>0$. *If* $P\{X=1\} + P\{X=0\} < 1$, *then* $E[X^t(1–X^h)] > 0$ *and* $E[X^{t+s}(1–X^h)]/E[X^t(1–X^h)]$ *is strictly increasing in t for all fixed positive s and h.*

Proof. The first part of the lemma is straightforward. We show the second part. Suppose that $t>u$. We consider functions $E[X^{t+s}(1–X^h)]/E[X^u(1–X^h)]$ and $E[X^{u+s}(1–X^h)]/E[X^t(1–X^h)]$. First, we have the following calculations: ($f(x)$: the density of X)

$$\begin{aligned} &E[X^{t+s}(1-X^h)]E[X^u(1-X^h)] \\ &= \int_0^1\int_0^1 x^{t+s}(1-x^h)y^u(1-y^h)f(x)f(y)dxdy \\ &= \frac{1}{2}\int_0^1\int_0^1 (x^{t+s}y^u + y^{t+s}x^u) \\ &\qquad \cdot(1-x^h)(1-y^h)f(x)f(y)dxdy. \end{aligned} \quad (3.3)$$

To prove the lemma, it is sufficient to show that $E[X^{t+s}(1–X^h)]/E[X^u(1–X^h)] > E[X^{u+s}(1–X^h)]/E[X^t(1–X^h)]$, which is equivalent to (by equation (3.3))

$$\begin{aligned} &\frac{1}{2}\int_0^1\int_0^1 (x^{t+s}y^u + y^{t+s}x^u - x^{u+s}y^t - y^{u+s}x^t) \\ &\qquad \cdot(1-x^h)(1-y^h)f(x)f(y)dxdy \\ &= \frac{1}{2}\int_0^1\int_0^1 x^u y^t\left(\left(\frac{x}{y}\right)^{t-u} - 1\right)(x^s - y^s) \\ &\qquad \cdot(1-x^h)(1-y^h)f(x)f(y)dxdy > 0. \end{aligned} \quad (3.4)$$

Since $t>u$, $(x^s– y^s)((x/y)^{t-u}–1) > 0$. Therefore, the integral in equation (3.4) is positive. This completes the proof of Lemma 3.1. ■

Let $\eta(t) = E[X^t]–\mu^t$. Next we show that the function $\eta(t)$ is unimodal in [1, ∞).

Lemma 3.2 *Assume that* $P\{X=1\}+P\{X=0\} < 1$. *The function* $\eta(t)$ *is negative in* (0,1) *and positive in* (1, ∞) *with* $\eta(0) = \eta(1) = 0$. *The function* $\eta(t)$ *is unimodal in the interval* $[1, \infty)$, i.e., $\eta(t)$ *is increasing and then decreasing in* $[1, \infty)$. *The function* $\eta(t)$ *is unimodal in the interval* [0, 1], i.e., $\eta(t)$ *is decreasing and then increasing in* [0, 1].

Proof. By Jensen's inequality (Chow and Teicher, 1988), the first part of the lemma is obtained immediately. Note that $\eta(t)$ is increasing at t=1. If $\eta(t)$ is not unimodal in $[1, \infty)$, then there exist t_1 and t_2 such that: $1< t_1< t_2$, $\eta(t_1)$ is a local maximum and $\eta(t_2)$ a local minimum. Choose δ, s_1 and s_2 such that $1< s_1< t_1< s_1+\delta < s_2 < t_2< s_2 + \delta$, and $\eta(s_1) = \eta(s_1+\delta)$ and $\eta(s_2) = \eta(s_2+\delta)$. From $\eta(s_1) = \eta(s_1+\delta)$, we obtain

$$\begin{aligned} E[X^{s_1}]-\mu^{s_1} &= E[X^{s_1+\delta}]-\mu^{s_1+\delta} \\ \Rightarrow E[X^{s_1}(1-X^{\delta})] &= \mu^{s_1}(1-\mu^{\delta}). \end{aligned} \tag{3.5}$$

A similar equation can be established at s_2. Then we have

$$\frac{E[X^{s_2}(1-X^{\delta})]}{E[X^{s_1}(1-X^{\delta})]}=\mu^{s_2-s_1}. \tag{3.6}$$

Now, we consider $s_1+\varepsilon$, $s_1+\delta+\varepsilon$, $s_2+\varepsilon$, and $s_2+\delta+\varepsilon$,. We choose ε and δ small enough so that $\eta(s_1+\varepsilon) > \eta(s_1+\delta+\varepsilon)$ and $s_1+\varepsilon < t_1 < s_1+\delta+\varepsilon$, since t_1 is a maximum point, and $\eta(s_2+\varepsilon) < \eta(s_2+\delta+\varepsilon)$ and $s_2+\varepsilon < t_2 < s_2+\delta+\varepsilon$, since t_2 is a minimum point. Then we must have

$$E[X^{s_1+\varepsilon}(1-X^{\delta})] > \mu^{s_1+\varepsilon}(1-\mu^{\delta}) \tag{3.7}$$

$$E[X^{s_2+\varepsilon}(1-X^{\delta})] < \mu^{s_2+\varepsilon}(1-\mu^{\delta})$$

Then we have

$$\begin{aligned} &\frac{E[X^{s_2+\varepsilon+s_2-s_1}(1-X^{\delta})]}{E[X^{s_1+\varepsilon}(1-X^{\delta})]} \\ &\le \mu^{s_2-s_1}=\frac{E[X^{s_2-s_1}(1-X^{\delta})]}{E[X^{s_1}(1-X^{\delta})]}. \end{aligned} \tag{3.8}$$

which contradicts the result obtained in Lemma 3.1.

Similarly, it can be also shown that $\eta(t)$ is negative and unimodal in (0, 1). This completes the proof of Lemma 3.2. ■

The implications of the above results on the original problem can be summarized as follows. Since $\eta(t)$ is unimodal in $[1, \infty)$, we have, for $t>s$, if $\eta(s) > \eta(s+1)$, then $\eta(t) > \eta(t+1)$. Since $d_X(t)-d_Y(t) = \eta(t)-\eta(t+1)$, we have, if $d_X(s) > d_Y(s)$, then $d_X(t) > d_Y(t)$ for $t>s$. Note that $d_X(1)-d_Y(1) = \eta(1)-\eta(2) = -\eta(2) < 0$ and there exists at least one point $t>1$ such that $d_X(t)-d_Y(t) = \eta(t)-\eta(t+1) > 0$. Therefore, the functions $d_X(t)$ and $d_Y(t)$ have exactly one intersection in $[1, \infty)$. Since $d_X(0)-d_Y(0) = \eta(0)-\eta(1) = 0$, the sequence $d_X(m)-d_Y(m)$ changes its sign exactly once. Some interesting special cases are given as follows.

Example 3.1 In this example, we assume that X has a uniform distribution on [0,1] and Y = 0.5. It is easy to verify that $g(t) = E[X^t]-E[Y^t] = 1/(t+1) - 0.5^t$ for $t \ge 0$. By Jensen's inequality, the function $g(t)$ is nonnegative, i.e., $E[X^t] \ge (E[X])^t = (E[Y])^t$. According to Lemma 3.2, $g(t)$ is unimodal. Therefore, $d_X(m)-d_Y(m)$ changes its sign only once. ■

Example 3.2 In this example, we assume that $Y \equiv \mu$ and X is define as: $P\{X=\mu+\varepsilon\} = P\{X=\mu-\varepsilon\} = 0.5$, where $0\le\mu-\varepsilon\le\mu+\varepsilon\le1$, and μ, $\varepsilon \ge 1$. For this case, $g(t)=0.5[(\mu+\varepsilon)^t + (\mu-\varepsilon)^t] - \mu^t$. Again, by Jensen's inequality, the function $g(t)$ is nonnegative for $t\ge0$. By Lemma 5.2, $g(t)$

is unimodal in [1, ∞] and $d_X(m)–d_Y(m)$ changes its sign only once. ■

Next example actually generalizes the results in Lemma 3.2.

Example 3.3 Consider a random variable X defined on [0, 1]. Let

$$Y=\begin{cases} X, & a \le X \le b; \\ \mu_X, & \text{otherwise}, \end{cases} \quad (3.9)$$

where $\mu_X = E[X]$ and a and b are so chosen that $0 \le a \le b \le 1$ and $E[Y] = \mu_X$. Intuitively, X is more variable than Y. First, we have the following calculations:

$$\begin{aligned} E[X^t - Y^t] &= \int_0^a x^t dF_X(x) + \int_b^1 x^t dF_X(x) \\ &\quad - \mu_X^t P\{\{X \le a\} \cup \{X \ge b\}\} \end{aligned} \quad (3.10)$$

$$= P\{\{X \le a\} \cup \{X \ge b\}\} \cdot \left[\frac{\int_0^a x^t dF_X(x) + \int_b^1 x^t dF_X(x)}{P\{\{X \le a\} \cup \{X \ge b\}\}} - \mu_X^t \right]$$

$$= P\{\{X \le a\} \cup \{X \ge b\}\}(E[Z^t] - \mu_X^t),$$

where Z is a random variable defined on $[0, a] \cup [b, 1]$ with density function

$$f_Z(x) = \begin{cases} 0, & a \le x \le b; \\ \dfrac{f_X(x)}{P\{\{X \le a\} \cup \{X \ge b\}\}}, & \text{otherwise}. \end{cases} \quad (3.11)$$

It can be verified that $E[Z] = \mu_X$. By Lemmas 3.1 and 3.2, $E[Z^t]–(\mu_X)^t$ is unimodal in t in [1, ∞). By equation (3.11), $E[X^t–Y^t]$ is unimodal in [1, ∞]. Thus, $d_X(m)–d_Y(m)$ changes its sign at most once.

A specific example is given as follows. Let X be a uniform distribution in [0, 1], a=0.25, and b=0.75. Then the function

$$E[X^t - Y^t] = \frac{1}{t+1} - (0.5)^{t+1} - \frac{(0.75)^{t+1} - (0.25)^{t+1}}{t+1} \quad (3.12)$$

is unimodal in [1, ∞], which is not straightforward to verify directly. ■

4. Effect Under the Likelihood Ratio Order

In this section, we consider the effect under the likelihood ratio order. For two independent random variables X and Y, if the ratio $f_X(x)/f_Y(x)$ is non-increasing over the union of the supports of X and Y, then X is said to be smaller than Y in the likelihood ratio order, denoted as $X \le_{lr} Y$.

Since $X \le_{lr} Y$, we have $E[X^m] \le E[Y^m]$ and $p_X(m) \ge p_Y(m)$ for all $m \ge 0$. That is, the distribution $\{d_X(m), m\ge0\}$ is stochastically smaller than $\{d_Y(m), m\ge0\}$. (Note: With a slight abuse of notation, we shall say that the distributions $\{d_X(m), m\ge0\}$ and $\{d_Y(m), m\ge0\}$ are stochastically ordered, though stochastic orders are defined for random variables.) That implies that a group with ability $1–X$ has a larger chance to solve a problem than a group with ability $1–Y$, regardless of the group size. Further properties of $\{d_X(m)–d_Y(m), m\ge0\}$ can be obtained from Lemma 4.1. For mathematical convenience, we extend the function $d_X(m)$ to $d_X(t)$ for all real nonnegative t, i.e.,

$$d_X(t) = E[X^t(1-X)]. \quad (4.1)$$

Since X takes values in [0,1], the function $d_X(t)$ is finite, continuous, and non-increasing for $t \ge 0$.

Lemma 4.1 *Assume that* $X \le_{lr} Y$ *and* $X \ne Y$.

Then the function $d_X(t)$–$d_Y(t)$ changes its sign exactly once for all $t \geq 0$. In fact, if $d_X(t_0) = d_Y(t_0)$ at some positive t_0, then we have

$$\begin{cases} E[X^s(1-X)] \\ \qquad > E[Y^s(1-Y)], \quad 0 \leq s < t_0; \\ E[X^s(1-X)] \\ \qquad < E[Y^s(1-Y)], \quad s > t_0. \end{cases} \tag{4.2}$$

Proof. Since $X \leq_{lr} Y$ and $X \neq Y$, we have $d_X(0) > d_Y(0)$. If $d_X(t) \geq d_Y(t)$ for all $t \geq 0$, then $d_X(m) \geq d_Y(m)$ for all $m \geq 1$, and $d_X(0) > d_Y(0)$, which contradicts the fact that both $\{d_X(m), m\geq 0\}$ and $\{d_Y(m), m\geq 0\}$ are proper probability distributions. Therefore, $d_X(t)$–$d_Y(t)$ must change its sign at least once for $t \geq 0$. Next, we show that $d_X(t)$–$d_Y(t)$ can change its sign at most once.

Suppose that $d_X(t)$–$d_Y(t)$ changes its sign at t_0. We have $d_X(t_0)$=$d_X(t_0)$. Let

$$w_X(x) = \frac{x^{t_0}(1-x) f_X(x)}{d_X(t_0)}, \text{ for } 0 \leq x \leq 1. \tag{4.3}$$

The function $w_X(x)$ is nonnegative and is a density function in the interval [0,1]. Suppose W_X is a random variable with density function $w_X(x)$. Similarly, we define a function $w_Y(x)$ and its corresponding random variable w_Y for the random variable Y. By definition (4.3) and the definition of t_0, we have

$$\frac{w_X(x)}{w_Y(x)} = \frac{f_X(x)}{f_Y(x)}, \quad \text{for} \quad 0 \leq x \leq 1. \tag{4.4}$$

By equation (4.4) and the assumption $X \leq_{lr} Y$, we have $W_X \leq_{lr} W_Y$ and $W_X \neq W_Y$. Since the function x^s is increasing in x for x>0 and s>0, we have

$$\begin{aligned} E[(W_X)^s] &= \int_0^1 x^s w_Y(x) dx \\ &= \frac{E[X^{s+t_0}(1-X)]}{E[X^{t_0}(1-X)]} \\ &< E[(W_Y)^s] = \frac{E[Y^{s+t_0}(1-Y)]}{E[Y^{t_0}(1-Y)]}. \end{aligned} \tag{4.5}$$

Equation (4.5) is obtained from equation (4.4) and the fact $d_X(t_0) = d_Y(t_0)$.

Since equation (4.2) holds for any t_0 satisfying $d_X(t_0) = d_Y(t_0)$, there can be at most one such point t_0. Thus, the function $d_X(t)$–$d_Y(t)$ changes its sign at t_0 and only at t_0. Therefore, the function $d_X(t)$–$d_Y(t)$ changes its sign exactly once for all $t \geq 0$. This completes the proof of Lemma 4.1. ■

Since $d_X(t)$–$d_Y(t)$ changes its sign exactly once, the sequence $\{d_X(m)$–$d_Y(m), m\geq 0\}$ changes its sign exactly once. If $m \leq t_0$, $d_X(m) \geq d_Y(m)$; otherwise $d_X(m) \leq d_Y(m)$. Thus we have shown that the sequence $\{d_X(m)$–$d_Y(m), m\geq 0\}$ changes its sign exactly once if $X \leq_{lr} Y$ and $X \neq Y$.

For the group size problem, the implications of the above analysis can be summarized as follows. If a task is easier, it is more likely for a group to solve the problem (i.e., $\{d_X(m), m\geq 0\} \geq_{st} \{d_Y(m), m\geq 0\}$, which is intuitive. The marginal contribution of an additional group member for an easier task is: a) larger if the group is small; and, b) smaller if the group is large.

5. Effect Under the "More Variable" Stochastic Order

Consider two independent random variables X and Y with the same mean $E[X] = E[Y]$, having distribution function $F(x)$ and $G(x)$

with density functions $f(x)$ and $g(x)$, respectively. We say that X is *more variable* than Y, denoted as $X \geq_{var} Y$, if the function $f(x)–g(x)$ changes its sign exactly twice with sign sequence {+, –, +}. In fact, $X \geq_{var} Y$ is a condition for $X \geq_c Y$ (Stoyan, 1983). In this section, we show that if $X \geq_{var} Y$ and $X \neq Y$, then $d_X(m)–d_Y(m)$ changes its sign exactly twice. By definition, we have

$$d_X(m) - d_Y(m) = \int_0^1 x^m (1-x)(f(x) - g(x))dx. \quad (5.1)$$

According to a result in Karlin (1968), if the function $x^m(1–x)$ is totally positive, than the number of sign changes of $d_X(m)–d_Y(m)$ is not larger than the number of sign changes of $f(x)–g(x)$. Next, we use that result to determine the number of sign changes of $d_X(m)–d_Y(m)$. A function $K(x, y)$ is said to be *totally positive* if for any positive integer r, for all $x_1< x_2<...< x_r$ and $y_1< y_2<...< y_r$,

$$\det \begin{pmatrix} K(x_1, y_1) & \cdots & K(x_1, y_r) \\ \vdots & \vdots & \vdots \\ K(x_r, y_1) & \cdots & K(x_r, y_r) \end{pmatrix} \geq 0, \quad (5.2)$$

where det(.) is for the determinant of a matrix.

Lemma 5.1 *Let* $K(m, x) = x^m(1–x)$ *for* $m\geq 0$ *and* $0\leq x\leq 1$. *The function* $K(m, x)$ *is totally positive*.

Proof. For $x_1< x_2<...< x_r$ and $m_1< m_2<...< m_r$,

$$\eta = \det \begin{pmatrix} K(m_1, x_1) & \cdots & K(m_1, x_r) \\ \vdots & \vdots & \vdots \\ K(m_r, x_1) & \cdots & K(m_r, x_r) \end{pmatrix}$$

$$= \left(\prod_{i=1}^{r} (1-x_i) x_i^{m_1} \right) \det \begin{pmatrix} x_1^{m_1-m_1} & \cdots & x_r^{m_1-m_1} \\ \vdots & \vdots & \vdots \\ x_1^{m_r-m_1} & \cdots & x_r^{m_r-m_1} \end{pmatrix} \quad (5.3)$$

$$= \left(\prod_{i=1}^{r} (1-x_i) x_i^{m_1} \right) \det(A),$$

where A is a generalized Vandermond matrix. The positivity of det(A) is well known (see Polya and Szego (1976), Problem 48). Therefore, the function $K(m, x)$ is totally positive. This completes the proof of Lemma 5.1 ■

By equation (5.1) and Lemma 5.1, we have the following interesting result.

Theorem 5.2 *If* $X \geq_{var} Y$ *and* $X \neq Y$, *then* $d_X(m)–d_Y(m)$ *changes its sign exactly twice in the form* {+, –, +}. *Consequently*, $\{d_X(m), m\geq 0\} \geq_c \{d_Y(m), m\geq 0\}$.

Proof. Since $X \geq_{var} Y$, $f(x)–g(x)$ changes its sign exactly twice. By a basic property of totally positive functions - the variation diminishing property - and equation (5.1), the function $d_X(m)–d_Y(m)$ changes its sign at most twice. If $d_X(m)–d_Y(m)$ changes its sign only once, then $\{d_X(m), m\geq 0\}$ is stochastically larger than $\{d_Y(m), m\geq 0\}$. Since $E[X] = E[Y]$, that implies that $d_X(m) = d_Y(m)$, for all m, which leads to $X=Y$. But $X=Y$ contradicts the assumption $X \neq Y$. Therefore, $d_X(m)–d_Y(m)$ changes its sign exactly twice, which implies $\{d_X(m), m\geq 0\} \geq_c \{d_Y(m), m\geq 0\}$. This completes the proof of Theorem 5.2. ■

There are many examples for which $f(x)–g(x)$ changes its sign exactly twice (see Karlin 1968). Thus, Theorem 5.2 can be quite useful.

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