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Coxian Representations of Generalized Erlang Distributions

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Abstract This paper studies Coxian representations of generalized Erlang distributions. A nonlinear program is derived for computing the parameters of minimal Coxian representations of generalized Erlang distributions. The nonlinear program is also used to characterize the triangular order and the admissible region of generalized Erlang distributions. It is shown that the admissible region associated with a triangular order may not be convex. For generalized Erlang distributions of *ME*-order 3, a minimal Coxian representation is found explicitly. In addition, an algorithm is developed for computing a special type of ordered Coxian representations - the bivariate Coxian representation - for generalized Erlang distributions.

Keywords Coxian distribution; phase-type distribution; matrix-analytic methods
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1 Introduction

A *generalized Erlang distribution* of order m is defined as the distribution of a nonnegative random variable with probability distribution function

$$F(t) = \alpha_0 + \sum_{i=1}^{m} \alpha_i \Big(1 - \sum_{j=0}^{i-1} \frac{(\lambda t)^j}{j!} e^{-\lambda t}\Big), \qquad t \ge 0, \tag{1}$$

where $\{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ are real numbers with a unit sum (i.e., $\alpha_0+\alpha_1+\ldots+\alpha_m = 1$) and λ is a positive real number. It is easy to see that equation (1) can be written as

$$F(t) = 1 - \alpha \exp\{E_{m,\lambda} t\}\mathbf{e}, \qquad t \ge 0, \tag{2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $E_{m,\lambda}$ is an $m\times m$ matrix with all diagonal elements being $-\lambda$, the $(j+1, j)$-th element λ for $1 \le j \le m-1$, and all other elements 0, and $\mathbf{e}$ is a column vector of ones. The representation $(\alpha, E_{m,\lambda})$ is called a *matrix-exponential* (*ME*) *representation* of $F(t)$ (see Lipsky [14]).

By Theorem 1.1 in O'Cinneide[17] and Theorem 4.1 in O'Cinneide[18], if the density function $F'(t)$ is positive for $t > 0$, the generalized Erlang distribution $F(t)$ is a Coxian distribution and has an *ordered Coxian representation* of the form $(\beta, S(\mathbf{x}))$ of order $m+N$, i.e., $F(t) = 1 - \beta\exp\{S(\mathbf{x})t\}\mathbf{e}$ for $t \ge 0$, where N is a nonnegative integer, $\mathbf{x} = (x_1, x_2, \ldots, x_{m+N})$ is a

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vector of size $m+N$, $x_1 \geq x_2 \geq \ldots \geq x_{m+N} > 0$, β is a substochastic vector of size $m+N$ (i.e., $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\ldots$, $\beta_{m+N} \geq 0$, $\beta_1 + \beta_2 + \ldots + \beta_{m+N} \leq 1$), and

$$S(\mathbf{x}) = \begin{pmatrix} -x_1 & & & \\ x_2 & -x_2 & & \\ & \ddots & \ddots & \\ & & x_{m+N} & -x_{m+N} \end{pmatrix}. \tag{3}$$

The objective of this paper is to find an ordered Coxian representation (β, $S(\mathbf{x})$) of the smallest order for $F(t)$. This issue is interesting because Coxian representations, as special phase-type (PH) representations[15], have a probabilistic interpretation that makes it easy to use generalized Erlang distributions in stochastic modeling.

The generalized Erlang distribution is a natural generalization of the exponential distribution and has been used in the studies of queueing models, risk and insurance models (Asmussen[1,2], Haddad et al.[9], Latouche and Ramaswami[13], Lipsky[14], Neuts[15], Sasaki et al.[21]). The set of generalized Erlang distributions is a subset of *ME*-distributions (Lipsky[14] and Asmussen and Bladt[3]) and is, under a mild condition, a subset of *PH*-distributions and Coxian distributions. Thus, a generalized Erlang distribution always has *ME*-representations and, under a mild condition, also has *PH*-representations and Coxian representations.

As indicated by equations (1) and (2), it is easy to construct an *ME*-representation for a generalized Erlang distribution. Proposition 2.1 of this paper indicates that it is also easy to find an *ME*-representation of the smallest order for a generalized Erlang distribution. Unfortunately, the use of *ME*-representations in stochastic modeling is limited. On the other hand, a Coxian representation or a *PH*-representation of a generalized Erlang distribution links the distribution to the absorption time of a Markov chain. Such a probabilistic interpretation of the *PH* and Coxian representations makes them useful in stochastic modeling. Furthermore, Coxian representations are featured by a bi-diagonal *PH*-generator that has a simple structure and is suitable for numerical computation. Consequently, finding a Coxian representation, especially a Coxian representation of the smallest order, is an interesting issue.

Theorem 4.1 in O'Cinneide[18] implies that a generalized Erlang distribution with a positive density function on $(0, \infty)$ has Coxian representations. An algorithm developed in He and Zhang[12] can be used for computing Coxian representations of the smallest order for generalized Erlang distributions. The algorithm requires solving a series of nonlinear programs. In this paper, a nonlinear program is established for determining parameters in the minimal Coxian representations of generalized Erlang distributions. The algorithm developed in He and Zhang[12] is modified and becomes significantly more efficient for generalized Erlang distributions. Characterization results are obtained for the triangular order of generalized Erlang distributions. It is shown that the admissible region up to a triangular order may not be convex. Yet the triangular order along any ray originated from a stochastic point (vector) is nondecreasing. Explicit results are obtained for a number of special cases as well.

The rest of the paper is organized as follows. In Section 2, preliminary results are presented. In Section 3, a set of necessary and sufficient conditions is identified for a Coxian representation to be minimal. Then an algorithm is developed for computing a minimal Coxian representation. Section 4 is concerned with the characterization of the triangular order. In Section 5, the minimal Coxian representations of generalized Erlang distributions of *ME*-order 3 are obtained explicitly. In Section 6, a special form of Coxian representations - bivariate Coxian representation - is studied.

2 Matrix Representations and Generalized Erlang Distributions

As indicated by equations (1) and (2), a generalized Erlang distribution has an *ME*-representation

$(\alpha,\ E_{m,\lambda})$, which can be constructed easily from its distribution function, its density function, or its Laplace-Stieltjes transform. Since we are interested in representations, without loss of generality, we assume that $\alpha_0 = 0$ (i.e., $\alpha\mathbf{e} = 1$) in the rest of this paper.

It is well known that *ME*-representations of *ME*-distributions, *PH*-representations of *PH*-distributions, and Coxian representations of Coxian distributions are not unique. This fact leads to the concept of minimal representations. A *minimal ME-representation* of an *ME*-distribution is the one of the smallest order. The *ME-order* of an *ME*-distribution is the order of its minimal *ME*-representation. The following properties of the *ME*-representations $(\alpha,\ E_{m,\lambda})$ of a generalized Erlang distribution characterize the minimal *ME*-representations of generalized Erlang distributions.

Proposition 2.1. *If $(\alpha,\ E_{m,\lambda})$ represents a probability distribution, then the first and the last nonzero elements in α must be positive. The ME-representation $(\alpha,\ E_{m,\lambda})$ is a minimal ME-representation if and only if that $(\alpha,\ E_{m,\lambda})$ is an ME-representation and $\alpha_m > 0$.*

Proof. The first part of the proposition is obtained directly from equation (1) due to the fact that the density function of the generalized Erlang distribution must be nonnegative. The second part can be obtained from the fact that, if $\alpha_m > 0$, the term $t^{m-1}\exp\{-\lambda t\}$ is in the density function of $(\alpha,\ E_{m,\lambda})$, which cannot be in the density function of generalized Erlang distributions of order $m-1$ or less. Then $(\alpha,\ E_{m,\lambda})$ has no equivalent *ME*-representation of order $m-1$ or less. Therefore, the *ME*-representation $(\alpha,\ E_{m,\lambda})$ is minimal. This completes the proof of Proposition 2.1.

If β is a sub-stochastic vector and $S(\mathbf{x})$ is defined in (3), where $\mathbf{x}= (x_1,\ x_2,\ \ldots,\ x_N)$ with all elements being positive real numbers, then $(\beta,\ S(\mathbf{x}))$ is called a *Coxian representation* of order *N* of a *Coxian distribution* (Note: This definition of Coxian distributions does not include the Coxian distributions whose density function has positive roots. This definition is slightly different from the one given in Cox[5]). If, in addition, $x_1 \geq x_2 \geq \ldots \geq x_N$, then $(\beta,\ S(\mathbf{x}))$ is called an *ordered Coxian representation* of order *N*. A minimal Coxian representation of a Coxian distribution is the Coxian representation of the smallest order of that Coxian distribution. The order of a minimal Coxian representation is called the *triangular order* of the Coxian distribution[19]). More details on Coxian representations and Coxian distributions can be found in Cox[5,6], Cumani[7], Dehon and Latouche[8], He and Zhang[11,12], and O'Cinneide[18,18–20].

A generalized Erlang distribution may not be a *PH*-distribution or a Coxian distribution, since its density function may have positive roots. Consequently, it may not have Coxian or *PH* representations. By Theorem 1.1 in O'Cinneide[17] and Theorem 4.1 in O'Cinneide[18], the positivity of the density function ensures that a generalized Erlang distribution is a Coxian distribution and a *PH*-distribution and, consequently, has Coxian and *PH* representations. That condition can be stated in the following form, which is assumed throughout this paper.

Assumption 1. *The polynomial function $g_\alpha(t) = \sum_{i=1}^{m} \alpha_i \frac{t^{i-1}}{(i-1)!}$ has no positive root.*

The following *spectral polynomial algorithm* introduced in He and Zhang[10] is used in the paper. Denote by $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_N)$ a vector of size *N*, with all elements being nonzero. For an *ME*-representation $(\alpha,\ T,\ \mathbf{u})$ (i.e., $1 - \alpha\exp\{Tt\}\mathbf{u}$, for $t \geq 0$, is a probability distribution function), define

$$\begin{cases} \mathbf{p}_1 = -T\mathbf{u}/x_1; \\ \mathbf{p}_n = (x_{n-1}I + T)\mathbf{p}_{n-1}/x_n, \qquad 2 \leq n \leq N; \\ \mathbf{p}_{N+1} = (x_N I + T)\mathbf{p}_N, \end{cases} \tag{4}$$

where I is the identity matrix. Let $P(\mathbf{x}) = (\mathbf{p}_1,\ \mathbf{p}_2,\ \ldots,\ \mathbf{p}_N)$. If $\mathbf{p}_{N+1} = 0$ and T is invertible, then equation (4) can be rewritten as $TP(\mathbf{x}) = P(\mathbf{x})S(\mathbf{x})$ and it can be shown $P(\mathbf{x})\mathbf{e} = \mathbf{u}$. By Theorem 2.2 in He and Zhang[12], the bi-diagonal representation $(\beta(\mathbf{x}),\ S(\mathbf{x}))$ with $\beta(\mathbf{x})$

$= \alpha P(\mathbf{x})$ is equivalent to (α, T, $\mathbf{u}$), i.e., they represent the same probability distribution. A condition for $\mathbf{p}_{N+1} = 0$ is that all eigenvalues (counting multiplicities) of T are in the vector $\mathbf{x}$.

Suppose that (γ, T, $\mathbf{u}$) is an *ME*-representation of a generalized Erlang distribution. Based on the spectral polynomial algorithm, a simple method is developed in He and Zhang[12] for computing a minimal *ME*-representation of the form (α, $E_{m,\lambda}$).

The above results indicate that, if a generalized Erlang distribution is defined in the form of probability distribution function, probability density function, Laplace-Stieltjes transform, or *ME*-representation, a minimal *ME*-representation of the form (α, $E_{m,\lambda}$) can be found. Therefore, in the rest of the paper, we assume that a minimal *ME*-representation (α, $E_{m,\lambda}$) of a generalized Erlang distribution is available.

3 Minimal Coxian Representations

Given a minimal *ME*-presentation (α, $E_{m,\lambda}$) of a generalized Erlang distribution, if α is nonnegative and $\alpha_m>0$, then (α, $E_{m,\lambda}$) is a minimal Coxian representation. If α is not nonnegative, an algorithm developed in He and Zhang[12] can be used for computing an equivalent minimal Coxian representation. The idea is to find $\mathbf{x}$ such that ($\beta(\mathbf{x})$, $S(\mathbf{x})$), obtained by applying the spectral polynomial algorithm to (α, T=$E_{m,\lambda}$, $\mathbf{u}$=$\mathbf{e}$), is an equivalent Coxian representation. In He and Zhang[12], a set of necessary and sufficient conditions on $\mathbf{x}$ has been identified for ($\beta(\mathbf{x})$, $S(\mathbf{x})$) to be a minimal Coxian representation. In this section, that set of conditions is simplified, which leads to a significantly more efficient algorithm and a characterization of the admissible region of triangular order.

Choose $\mathbf{x} = (x_1, x_2, \ldots, x_N, \lambda, \ldots, \lambda)$ of size m+N and with positive elements. Since all eigenvalues of $E_{m,\lambda}$ (counting multiplicities) are in the vector $\mathbf{x}$, the representations (α, $E_{m,\lambda}$) and ($\beta(\mathbf{x})$, $S(\mathbf{x})$) are equivalent. By Lemma 5 in He and Zhang[12], a minimal Coxian representation is a Coxian representation ($\beta(\mathbf{x})$, $S(\mathbf{x})$) such that $\beta(\mathbf{x}) = \alpha P(\mathbf{x}) \geq 0$ holds and m+N is the triangular order. Therefore, finding a minimal Coxian representation of the generalized Erlang distribution, if it exists, is equivalent to finding the minimal N and the corresponding $\mathbf{x}$ such that $\alpha P(\mathbf{x}) \geq 0$.

Lemma 3.1. *For fixed N, that $\alpha P(\boldsymbol{x}) \geq 0$ holds for $\boldsymbol{x} = (x_1, x_2, \ldots, x_N, \lambda, \ldots, \lambda)$ with min$\{x_1, x_2, \ldots, x_N\}>\lambda$ if and only if there exist positive numbers $\{\eta_n, 1\leq n\leq N\}$ such that $\alpha A(\eta) \geq 0$, where $\eta = (\eta_1, \ldots, \eta_N)$ and $A(\eta)$ is an $m\times(m+N)$ matrix whose elements $\{a_{i,n}, 1\leq i\leq m, 1\leq n\leq m+N\}$ are given as follows: $a_{1,1} = 1$, $a_{i,1} = 0$, $2\leq i\leq m$, and, for $2\leq n\leq N+1$,*

$$a_{1,n} = 1; \quad a_{i,n} = \eta_{n-1}a_{i-1,n-1} + a_{i,n-1}, \qquad 2 \leq i \leq m; \tag{5}$$

for $N+2\leq n\leq m+N$,

$$\begin{aligned} a_{i,n} &= 0, \qquad 1 \leq i \leq n-N-1; \\ a_{i,n} &= a_{i-1,n-1}, \qquad n-N \leq i \leq m. \end{aligned} \tag{6}$$

If a solution to $\alpha A(\eta) \geq 0$ exists, we have $\eta_n = \lambda/(x_n-\lambda)$ or $x_n = \lambda + \lambda/\eta_n$, $1 \leq n \leq N$. In addition, if η is a positive solution to $\alpha A(\eta) \geq 0$ and $\eta_i \geq \eta_{i+1}$, then $\eta' = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \eta_i, \eta_{i+2}, \ldots, \eta_N)$ satisfies $\alpha A(\eta') \geq 0$.

Proof. Let $p_{i,n}$ be the i-th element in the vector $\mathbf{p}_n$, the n-th column of $P(\mathbf{x})$, $1\leq i\leq m$ and $1\leq n\leq m+N$. By equation (4), it is easy to obtain, $p_{1,1} = \lambda/x_1$, $p_{i,1} = 0$, $2\leq i\leq m$, and, for $2\leq n\leq m+N$,

$$\begin{aligned} p_{1,n} &= \frac{(x_{n-1}-\lambda)}{x_n}p_{1,n-1}; \\ p_{i,n} &= \frac{\lambda}{x_n}p_{i-1,n-1} + \frac{(x_{n-1}-\lambda)}{x_n}p_{i,n-1}, \qquad 2 \leq i \leq m. \end{aligned} \tag{7}$$

Let $\eta_n = \lambda/(x_n - \lambda)$, $1 \le n \le N$. Then we have $p_{1,1} = \eta_1/(1+\eta_1)$, $p_{i,1} = 0$, $2\le i\le m$. Equation (7) becomes, for $2\le n\le N$,

$$\begin{aligned} p_{1,n} &= \frac{\eta_n}{(1+\eta_n)\eta_{n-1}} p_{1,n-1}; \\ p_{i,n} &= \frac{\eta_n}{(1+\eta_n)\eta_{n-1}} (\eta_{n-1}p_{i-1,n-1} + p_{i,n-1}), \qquad 2 \le i \le m; \end{aligned} \tag{8}$$

$p_{1,N+1} = p_{1,N}/\eta_N$, $p_{i,N+1} = (\eta_N p_{i-1,N} + p_{i,N})/\eta_N$, $2 \le i \le m$; and, for $N+2\le n\le m+N$,

$$\begin{aligned} p_{i,n} &= 0, \qquad 1 \le i \le n - N - 1; \\ p_{i,n} &= p_{i-1,n-1}, \qquad n - N \le i \le m. \end{aligned} \tag{9}$$

Combining equations (5), (6), (8) and (9), we obtain

$$P(\mathbf{x}) = A(\eta)\mathrm{diag}\Big(\frac{\eta_1}{1+\eta_1}, \frac{\eta_2}{\prod_{i=1}^{2}(1+\eta_i)}, \cdots, \frac{\eta_N}{\prod_{i=1}^{N}(1+\eta_i)}, \frac{1}{\prod_{i=1}^{N}(1+\eta_i)}, \cdots, \frac{1}{\prod_{i=1}^{N}(1+\eta_i)}\Big), \tag{10}$$

where *diag*(.) is for a diagonal matrix. If $\{\eta_n,\ 1\le n\le N\}$ are positive, then $\alpha P(\mathbf{x}) \ge 0$ if and only if $\alpha A(\eta) \ge 0$. The first part of the lemma follows.

Suppose that $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ is a positive solution to $\alpha A(\eta) \ge 0$ with $\eta_i \ge \eta_{i+1}$. Let $\mathbf{a}_n(\eta)$ be the n-th column of the matrix $A(\eta)$. For $\eta' = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \eta_i, \eta_{i+2}, \ldots, \eta_N)$, we have, for $1\le i\le N-1$,

$$\begin{aligned} &\alpha\,\mathbf{a}_n(\eta) = \alpha\,\mathbf{a}_n(\eta'), \quad \text{for } 1 \le n \le N+m,\ n \ne i+1; \\ &\alpha\,\mathbf{a}_{i+1}(\eta) = \alpha \\ &\mathbf{a}_i(\eta) + \eta_i K_i, \quad \alpha\,\mathbf{a}_{i+1}(\eta') = \alpha\,\mathbf{a}_i(\eta) + \eta_{i+1}K_i, \end{aligned} \tag{11}$$

where K_i is a function associated with variables $\{\eta_1, \eta_2, \cdots, \eta_{i-1}\}$. Since $\alpha\mathbf{a}_i(\eta) \ge 0$ and $\alpha\mathbf{a}_{i+1}(\eta)\ge 0$, it is easy to see that $\alpha\mathbf{a}_i(\eta)\ge 0$ and $\alpha\mathbf{a}_{i+1}(\eta')\ge 0$ holds if $\eta_i \ge \eta_{i+1}$. Consequently, we have $\alpha A(\eta')\ge 0$. This completes the proof of Lemma 3.1.

Now, we are ready to state the main result of this section.

Theorem 3.2. *For a generalized Erlang distribution (α, $E_{m,\lambda}$) for which Assumption 1 holds, its triangular order equals the smallest integer N for which the nonlinear system $\alpha A(\eta)\ge 0$ has a positive and finite solution. If $\{\eta_n,\ 1\le n\le N\}$ is a positive and finite solution to $\alpha A(\eta)\ge 0$, then ($\beta(\boldsymbol{x})$, $S(\boldsymbol{x})$) is a minimal Coxian representation, where $\boldsymbol{x}= \lambda(1+1/\eta_1,\ 1+1/\eta_2,\ \ldots,\ 1+1/\eta_N,\ 1,\ \ldots,\ 1)$, $\beta(\boldsymbol{x}) = \alpha P(\boldsymbol{x})$, and $P(\boldsymbol{x})$ is obtained from $\boldsymbol{x}$ by using equation (4). Furthermore, if elements in $\boldsymbol{x}$ are not in descending order, $\boldsymbol{x}$ can be sorted into descending order and ($\beta(\boldsymbol{x})$, $S(\boldsymbol{x})$) is a minimal ordered Coxian representation.*

Proof. Note that i) By Lemma 6 in He and Zhang[12], in the minimal Coxian representation (if it exists), we must have $\min\{x_1, x_2, \ldots, x_N\}>\lambda$; ii) If $\alpha P(\mathbf{x})\ge 0$ for $\mathbf{x}$, by the last part of Lemma 3.1, $\alpha P(\mathbf{x})\ge 0$ holds if $\mathbf{x}$ is sorted in descending order. The proof of Theorem 3.2 is consequently completed.

To find a solution to $\alpha A(\eta)\ge 0$, nonlinear programs can be utilized. For instance, we have used the following nonlinear program for numerical experimentations:

$$\begin{aligned} \min \quad & \sum_{n=1}^{N} \eta_n \\ \text{s.t.} \quad & \alpha\,A(\eta) \ge 0, \\ & 0 < \eta_1 \le \eta_2 \le \cdots \le \eta_N. \end{aligned} \tag{12}$$

Note that the last constraint in (12) is due to the last part of Lemma 3.1. It is clear that the determination of η is independent of the value of λ. Equation (10) indicates that the determination of $\beta(\mathbf{x}) = \alpha P(\mathbf{x})$ is also independent of the value of λ. Thus, finding a minimal Coxian representation and determining the triangular order of a generalized Erlang distribution are independent of the value of λ.

Based on Theorem 3.2 and the nonlinear program (12), the following computational procedure can be used to find a minimal Coxian representation for a generalized Erlang distribution.

Step 1. Find a minimal *ME*-representation $(\alpha,\ E_{m,\lambda})$. If the minimal *ME*-representation is a Coxian representation, a solution is found and the search process is terminated. Otherwise, check if the polynomial function $g_\alpha(t)$ (defined in Assumption 1) has no positive roots. If it is true, then set N to 1 and go to Step 2; Otherwise, there is no solution and the search process is terminated.

Step 2. Solve the nonlinear program (12) to find $\{\eta_1,\ \eta_2,\ \ldots,\ \eta_N\}$.

Step 3. If a solution is found, go to Step 4; otherwise, set $N =: N{+}1$ and go to Step 2.

Step 4. Let $\mathbf{x}= \lambda(1+1/\eta_1, 1+1/\eta_2, \cdots, 1+1/\eta_N, 1, \cdots, 1)$ and use the spectral polynomial algorithm to find $(\beta(\mathbf{x}),\ S(\mathbf{x}))$, which is a minimal Coxian representation.

By O'Cinneide[18], the algorithm will be terminated in a finite number of iterations if Assumption 1 holds.

The nonlinear system $\alpha A(\eta)=0$ and the nonlinear program (12) have N positive variables and $m{+}N$ polynomial functions of degree $m{-}1$ or less. Although the number of variables increases as N increases, the degrees of the polynomial functions are always no more than $m{-}1$. On the other hand, the nonlinear program in He and Zhang[12], in general, has polynomial functions of degree $m{+}N$. Therefore, the specialized algorithm in this paper is more efficient than the one given in He and Zhang[12].

Using the nonlinear system $\alpha A(\eta)\geq 0$, explicit results on the triangular order and the minimal Coxian representation can be obtained for some special cases.

Corollary 3.3. *Assume that $(\alpha,\ E_{m,\lambda})$ of a generalized Erlang distribution satisfying Assumption 1 is a minimal ME-representation.*

a) The triangular order is m or $m{+}1$ if and only if

$$\max_{2\leq n\leq m}\left\{\frac{\alpha_{n-1}^-}{\alpha_n^+}\right\} \leq \min_{2\leq n\leq m}\left\{\frac{\alpha_{n-1}^+}{\alpha_n^-}\right\}, \tag{13}$$

with the convenience $a_{n-1}^+/0 = \infty$ and $0/a_n^+ = 0$, where $\alpha_n^+ = \max\{0,\ \alpha_n\}$ and $\alpha_n^- = \max\{0,\ -\alpha_n\}$. For any η between the left and right hand sides of equation (13), an ordered Coxian representation can be constructed from vector $\mathbf{x}= \lambda(1+1/\eta, 1, \cdots, 1)$ by using the spectral polynomial algorithm.

b) Let $N^* = \max\{n :\ \exists\ i,\ \min\{\alpha_i,\ \alpha_{i+1}, \cdots, \alpha_{i+n-1}\} < 0, \max\{\alpha_i, \alpha_{i+1}, \cdots, \alpha_{i+n-1}\} \leq 0,\ n \geq 1\}$. Then the triangular order is greater than or equal to $N^* + m$.

Proof. a) The result holds if the triangular order is m. For $N = m{+}1$, the nonlinear system $\alpha A(\eta) \geq 0$ becomes

$$\begin{aligned}&\alpha_1 \geq 0;\\ &\alpha_n + \alpha_{n+1}\eta \geq 0, \qquad 1 \leq n \leq m-1;\\ &\alpha_m \geq 0.\end{aligned} \tag{14}$$

Thus, if $\alpha_{n+1} \geq 0$, then $\eta \geq (-\alpha_n)/\alpha_{n+1}$; or if $\alpha_{n+1} < 0$, then $\eta \leq \alpha_n/(-\alpha_{n+1})$. The result is obtained immediately.

b) This part is obtained directly from the observation that, in each inequality in the nonlinear system $\alpha A(\eta) \geq 0$, there are at most N consecutive elements from the sequence $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ involved. This completes the proof of Corollary 3.3.

Example 3.1. Consider the generalized Erlang distribution (α, $E_{m,\lambda}$) with α = (1, –0.3, 0.2, –0.2, 0.3), m=5, and $\lambda = 1$. It can be verified that Assumption 1 is satisfied. Thus, this generalized Erlang distribution has Coxian representations of finite order. It is easy to check that the condition in part a) of Corollary 3.3 is not satisfied. Thus, the triangular order is greater than 6. For $N = 2$, the nonlinear system $\alpha A(\eta) \geq 0$ becomes

$$\begin{aligned}
&1 \geq 0;\\
&1 - 0.3\eta_1 \geq 0;\\
&1 - 0.3(\eta_1 + \eta_2) + 0.2\eta_1\eta_2 \geq 0;\\
&-0.3 + 0.2(\eta_1 + \eta_2) - 0.2\eta_1\eta_2 \geq 0;\\
&0.2 - 0.2(\eta_1 + \eta_2) + 0.3\eta_1\eta_2 \geq 0;\\
&-0.2 + 0.3(\eta_1 + \eta_2) \geq 0;\\
&0.3 \geq 0.
\end{aligned} \tag{15}$$

The set of solutions of the nonlinear system (15) are not empty and are plotted in the red colored area in Figure 3.1. For example, $\eta_1 = 3$ and $\eta_2 = 0.65$ satisfy all inequalities in (15) and lead to a Coxian representation corresponding to $\mathbf{x}$= (12.6923, 6.5625, 5, 5, 5, 5, 5). An ordered Coxian representation ($\beta(\mathbf{x})$, $S(\mathbf{x})$) of order 7 with $\beta(\mathbf{x})$ = (0.3939, 0.3717, 0.0377, 0.0078, 0.0078, 0.1378, 0.0433) can be obtained. Hence, the triangular order of the example is 7.

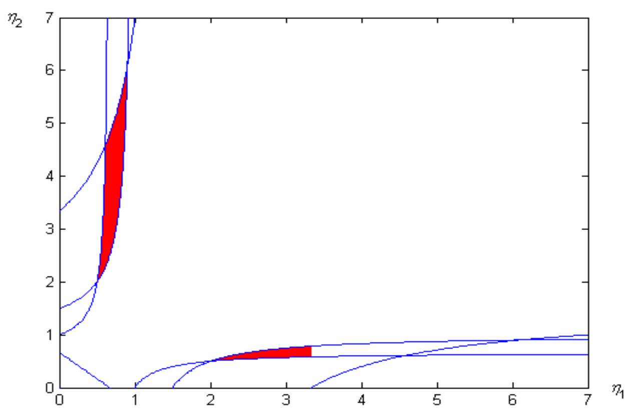

Figure 3.1. The feasible regions (red colored areas) of the nonlinear system (15)

4 Characterization of the Triangular Order

In this section, we use the nonlinear system $\alpha A(\eta) \geq 0$ to characterize the triangular order. Since the parameter λ does not play a role in this nonlinear system, the characterization of the triangular order is independent of the value of λ. Define

$$\Omega_{(m,N)} = \{\alpha : \quad \alpha\mathbf{e} = 1, \ \alpha \in R^m, \exists\, \eta = (\eta_1, ..., \eta_N) > 0, \ \alpha A(\eta) \geq 0\}, \tag{16}$$

the set in which the triangular order of $(\alpha, E_{m,\lambda})$ is $m+N$ or less. We call $\Omega_{(m,N)}$ the *admissible region* of triangular order $m+N$. It can be shown that $\Omega_{(m,N)} \subseteq \Omega_{(m,N+1)}$ for $0 \le N < \infty$ (see Proposition 5.1 in [10]). Then $\Omega_m^* = \cup_{N>0}\Omega_{(m,N)}$ is the set of all generalized Erlang distributions that are also Coxian. It is well-known that the set Ω_m^* is convex for all positive integer m. For $m=3$, it has been shown in O'Cinneide[17] that $\Omega_{(3,N)}$ is convex for $N \ge 0$ (see Figure 5.1). Unfortunately, in general, the set $\Omega_{(m,N)}$ is not convex.

Example 4.1. For $m = 4$, consider $\alpha = (0.8, -0.4, 0.2, 0.4)$ and $\beta = (0.4, 0.2, -0.4, 0.8)$. By part a) of Corollary 3.3, the triangular orders of $(\alpha, E_{(4,\lambda)})$ and $(\beta, E_{(4,\lambda)})$ are 5. Let $\gamma = (\alpha + \beta)/2 = (0.6, -0.1, -0.1, 0.6)$. By part b) of Corollary 3.3, the triangular order of $(\gamma, E_{(4,\lambda)})$ is at least 6. In fact, it can be verified that the triangular order of $(\gamma, E_{(4,\lambda)})$ is 6. Thus, $\Omega_{(4,5)}$ is not convex.

Although the set $\Omega_{(m,N)}$ is in general not convex, a number of geometric properties of $\Omega_{(m,N)}$ can be identified.

Lemma 4.1. *Let $\alpha(t) = \mathbf{v}_0 + t\mathbf{v}_d$ for $t \ge 0$, where $\mathbf{v}_0 \in \Omega_{(m,0)}$ (Note that $\Omega_{(m,0)}$ denotes the probability polytope of dimension m), $\mathbf{v}_d = \alpha - \mathbf{v}_0$, and $\alpha \in \Omega_{(m,N)}$. Then $\alpha(t) \in \Omega_{(m,N)}$ for $0 \le t \le 1$.*

Proof. By definition, $\alpha(t)A(\eta) = \mathbf{v}_0 A(\eta) + t\mathbf{v}_d A(\eta)$. Since $\mathbf{v}_0 \ge 0$ and $A(\eta) \ge 0$, then $\mathbf{v}_0 A(\eta) \ge 0$. Consequently, $\alpha(1)A(\eta) \ge 0$ implies $\alpha(t)A(\eta) \ge 0$ for $0 \le t \le 1$. This completes the proof.

Based on Lemma 4.1, several observations on the triangular order and $\Omega_{(m,N)}$ can be made.

1) For any half line $\mathbf{v}_0 + t\mathbf{v}_d$ starting from $\mathbf{v}_0 \in \Omega_{(m,0)}$ and $\mathbf{v}_d\mathbf{e} = 0$, if $\mathbf{v}_0 + t^*\mathbf{v}_d \in \Omega_{(m,N)}$ for $t^* > 0$, then $\mathbf{v}_0 + t\mathbf{v}_d \in \Omega_{(m,N)}$ for $0 \le t \le t^*$.

2) The set $\Omega_{(m,N)}$ is the union of a set of pyramids that have the probability polytope $\Omega_{(m,0)}$ as their basis.

3) On any half line $\mathbf{v}_0 + t\mathbf{v}_d$, $t \ge 0$, starting from $\mathbf{v}_0$, where $\mathbf{v}_0 \in \Omega_{(m,0)}$ and $\mathbf{v}_d\,\mathbf{e} = 0$, the triangular order of generalized Erlang distribution $(\alpha(t), E_{m,\lambda})$ is a nondecreasing function of t, provided that $(\alpha(t), E_{m,\lambda})$ is a generalized Erlang distribution.

Next, we characterize the boundary of $\Omega_{(m,N)}$. Mainly, we are interested in the border separating α with triangular order $m+N$ and α with triangular order greater than $m+N$. Define

$L_{(m,N)} = \{\alpha$: The triangular order of $(\alpha, E_{m,\lambda})$ is $m+N$. There exists a nonzero vector γ such that $\gamma\mathbf{e} = 0$ and, for any $t > 0$, if $(\alpha+t\gamma, E_{m,\lambda})$ also represents a Coxian distribution, the triangular order is greater than $m+N$; otherwise, $(\alpha+t\gamma, E_{m,\lambda})$ does not represents a Coxian distribution.$\}$

The set $L_{(m,N)}$ is characterized partially as follows.

Theorem 4.2. *For generalized Erlang distributions, we have*

$$\{\alpha : \ \alpha \in \Omega_{(m,N)} \textit{ and solution to } \alpha A(\eta) \ge 0 \textit{ and } \eta > 0 \textit{ is unique}\} \subset L_{(m,N)}. \tag{17}$$

Proof. Suppose that the solution to $\alpha A(\eta) \ge 0$ is unique for a given α in $\Omega_{(m,N)}$. Then there exists n such that $\alpha\mathbf{a}_n(\eta) = 0$. If that is not true, then $\alpha\mathbf{a}_n(\eta) > 0$ for all $1 \le n \le m+N$. Consider $\eta+t\theta$ for any nonnegative vector θ. By Taylor expansion of the function $A(\eta)$, we have $\alpha A(\eta + t\theta) = \alpha(A(\eta) + tA_1 + \ldots + t^N A_N) > 0$ if t is sufficiently small, which is a contradiction.

Choose a stochastic vector $\beta \in \Omega_{(m,0)}$ and define $\gamma = \alpha - \beta$. Consider $\alpha(t) = \alpha + t\gamma = \beta + (1+t)\gamma$. If α is not on the boundary, then $\alpha(t) \in \Omega_{(m,N)}$ if t is sufficiently small. By Lemma 4.1, there exists η' such that $(\alpha + t\gamma)A(\eta') = 0$ if t is sufficiently small. Since $\alpha A(\eta') = 0$ and the solution is unique, we must have $\eta = \eta'$. Since $(\alpha + t\gamma)A(\eta) = \alpha A(\eta) + t\gamma A(\eta)$, for n such that $\alpha\mathbf{a}_n(\eta) = 0$, we must have $\gamma\mathbf{a}_n(\eta) = 0$, which leads to $\alpha\mathbf{a}_n(\eta) = \beta\mathbf{a}_n(\eta)$ for all stochastic vectors β. However, this cannot be true for all stochastic vectors β, since the vector $\mathbf{a}_n(\eta)$ is nonzero and nonnegative. Therefore, the vector α must be on the boundary, i.e., $\alpha \in L_{(m,N)}$. This completes the proof.

Combining Lemma 4.1 and Theorem 4.2, it is possible to plot the boundary $L_{(m,N)}$. The idea is to find the triangular order for α on the half lines starting from a point in $\Omega(m, 0)$. For given m and each $N = 1$, the set $L_{(m,N)}$ can be identified as follows.

i) For given α, solve the nonlinear program (12) to find a solution η^*, if it exists.

ii) Solve system (12) again with a modified objective function

$$\rho = \max \frac{\sum_{n=1}^{N} |\eta_n - \eta_n^*|}{1 + \sum_{n=1}^{N} |\eta_n - \eta_n^*|}. \tag{18}$$

iii) If ρ is zero for the optimal solution (i.e., the solution to the nonlinear system $\alpha A(\eta) \geq 0$ is unique), the solution η^* is in $L_{(m,N)}$.

To see boundary points, we consider the following half line:

$$\alpha(t) = \mathbf{v}_0 + t\mathbf{v}_d, \text{ for } t > 0,$$

where $\mathbf{v}_0 \in \Omega_{(m,0)}$ and $\mathbf{v}_d$ satisfies $\mathbf{v}_d\mathbf{e} = 0$. Then we plot $\rho(t)$ defined in equation (18) for $\alpha(t)$. For $\mathbf{v}_0 = (0.25, 0.25, 0.25, 0.25)$ and $\mathbf{v}_d = (0.1, -0.4, -0.1, 0.4)$, $\rho(t)$ is plotted in Figure 4.1.

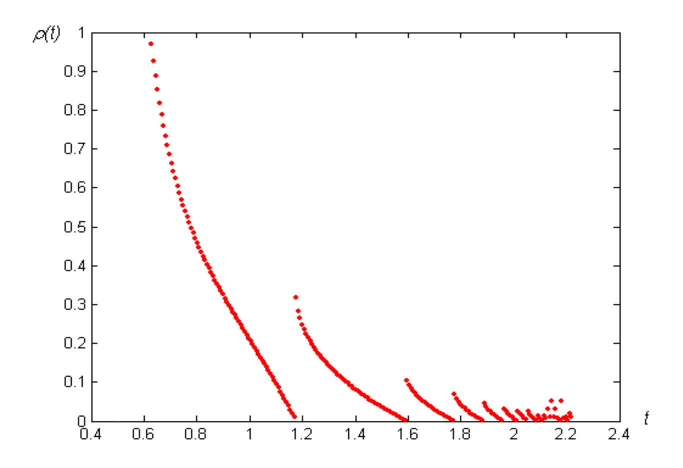

Figure 4.1. The function $\rho(t)$.

Figure 4.1 shows that the function $\rho(t)$ has a number of discontinuous points. Those are the points at which the triangular order of generalized Erlang distribution $(\alpha(t), E_{m,\lambda})$ is increased by one, i.e., $\alpha(t)$ is on a boundary. More specifically, we have

1) $t <$ 0.625, the triangular order of $(\alpha(t), E_{(4,\lambda)})$ is 4 or less;

2) 0.625$< t <$1.1741, the triangular order of $(\alpha(t), E_{(4,\lambda)})$ is 5;

3) 1.1767$< t <$1.5907, the triangular order of $(\alpha(t), E_{(4,\lambda)})$ is 6;

4) 1.5933$< t <$1.7751, the triangular order of $(\alpha(t), E_{(4,\lambda)})$ is 7;

5) 1.7751< t <1.8862, the triangular order of ($\alpha(t)$, $E_{(4,\lambda)}$) is 8;

6) 1.8912< t <1.9594, the triangular order of ($\alpha(t)$, $E_{(4,\lambda)}$) is 9;

7) 1.9619< t <2.0124, the triangular order of ($\alpha(t)$, $E_{(4,\lambda)}$) is 10;

8) 2.0149< t <2.0515, the triangular order of ($\alpha(t)$, $E_{(4,\lambda)}$) is 11;

9) 2.0520< t <2.0625, the triangular order of ($\alpha(t)$, $E_{(4,\lambda)}$) is 12.

For $t > 2.0625$, $\alpha(t)$ is getting close to the boundary of Ω_m^* and the triangular order of ($\alpha(t)$, $E_{(4,\lambda)}$) increases rapidly. For $t > 2.35$, ($\alpha(t)$, $E_{(4,\lambda)}$) does not represent a probability distribution. The boundaries $\{L_{(m,N)},\ 1 \le N < \infty\}$ can be generated by choosing different $(\mathbf{v}_0, \mathbf{v}_d)$.

Remark 4.1. The characterization of Ω_m^* given in this section holds for the triangular order of Coxian distributions and *PH*-generators with only real eigenvalues.

5 Explicit Solutions for Generalized Erlang Distributions of *ME*-Order 3

By Proposition 2.1, for a minimal *ME*-representation (α, $E_{m,\lambda}$) of order 3, we must have $\alpha_1 \ge 0$ and $\alpha_3 > 0$. Theorem 7.5 in O'Cinneide[19] states that the triangular order of (α, $E_{3,\lambda}$) is 3, if α_2 is nonnegative, and is $3 + \lceil \alpha_2^2/(2\alpha_1\alpha_3 - \alpha_2^2) \rceil$, if α_2 is negative and $\alpha_2^2 < 2\alpha_1\alpha_3$, where $\lceil x \rceil$ represents the smallest integer that is greater than or equal to x. Otherwise, (α, $E_{3,\lambda}$) does not represent a Coxian distribution. In this section, we find a minimal ordered Coxian representation explicitly, if $\alpha_2^2 < 2\alpha_1\alpha_3$.

The boundary line $L_{(3,N)} = \{(\alpha_1,\ \alpha_2,\ \alpha_3)\colon\ \alpha_2^2 = 2N\alpha_1\alpha_3/(N+1),\ \alpha_1 > 0,\ \alpha_2 < 0$, and $\alpha_3 > 0,\ \alpha_1 + \alpha_2 + \alpha_3 = 1\}$ for $N \ge 1$, which is identified explicitly as

$$L_{(3,N)} = \left\{ (\alpha_1, \alpha_2, \alpha_3) : \left\{ \begin{array}{l} 0 < \alpha_1 < 2 - \frac{2}{N+2};\ \alpha_3 = 1 - \alpha_1 - \alpha_2 > 0;\ \alpha_2 < 0; \\ \alpha_2 = -\Big[\dfrac{N\alpha_1 \pm \sqrt{2N(N+1)\alpha_1 - N^2\alpha_1^2 - 2N\alpha_1^2}}{N+1}\Big]. \end{array} \right\} \right\}, \tag{19}$$

and is plotted in Figure 5.1.

We consider a special type of ordered Coxian representations for which $\mathbf{x} = (y,\ y,\ \ldots,\ y,\ \lambda,\ \lambda,\ \lambda)$, which is called *a bivariate Coxian representation*, where y appears N times in the vector. Then the nonlinear system $\alpha A(\eta) \ge 0$ is reduced to, for $z = \eta_1 = \eta_2 = \ldots = \eta_N = \lambda/(y - \lambda)$,

$$\begin{aligned} &\alpha_1 + \alpha_2(n-1)z + \alpha_3(n-1)(n-2)z^2/2 \ge 0, \qquad 1 \le n \le N+1; \\ &\alpha_2 + \alpha_3 N z \ge 0, \qquad n = N+2; \\ &\alpha_3 \ge 0, \qquad n = N+3. \end{aligned} \tag{20}$$

Lemma 5.1. *Assume that α_2 is negative (and $\alpha_1 \ge 0$ and $\alpha_3 > 0$). The nonlinear system (20) has a positive solution z if and only if $\alpha_2^2 \le 2N\alpha_1\alpha_3/(N+1)$. Consequently, the minimal N for which there exists such a solution to the nonlinear system (20) satisfies $2\frac{(N-1)}{N}\alpha_1\alpha_3 < \alpha_2^2 \le 2\frac{N}{(N+1)}\alpha_1\alpha_3$. For this case, $z = -\alpha_2/(N\alpha_3)$ is a solution.*

Proof. Suppose that $\alpha_2^2 \le 2N\alpha_1\alpha_3/(N+1)$ holds. Let $z = -\alpha_2/(N\alpha_3)$. We verify that z satisfies the nonlinear system (20). It is easy to see that inequality in (20) holds for $n = 1, N+2$,

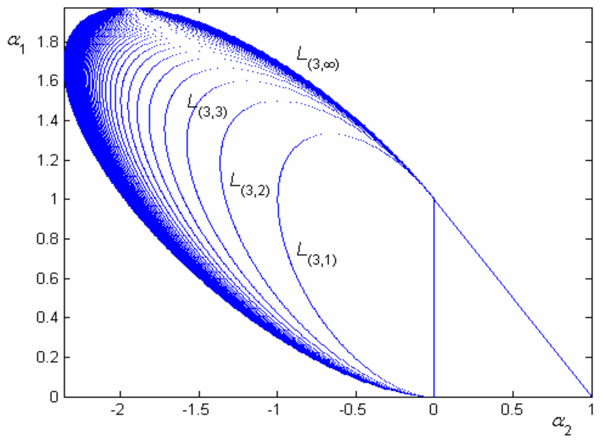

Figure 5.1. Boundaries $\{L_{(3,N)}, N \geq 1\}$ of $\{\Omega_{(3,N)}, N \geq 1\}$.

and $N + 3$. For other cases, we have the following calculations:

$$\begin{aligned}
&\alpha_1 + \alpha_2(n-1)\frac{(-\alpha_2)}{N\alpha_3} + \alpha_3\frac{(n-1)(n-2)}{2}\frac{(-\alpha_2)^2}{N^2\alpha_3^2} \\
=&\frac{2N^2\alpha_1\alpha_3 - (n-1)(2N-n+2)\alpha_2^2}{2N^2\alpha_3} \\
\geq&\frac{2N^2\alpha_1\alpha_3 - (n-1)(2N-n+2)2N\alpha_1\alpha_3/(N+1)}{2N^2\alpha_3} \\
\geq&\alpha_1\Big(1 - \frac{(n-1)(2N-n+2)}{N(N+1)}\Big) \\
\geq&0
\end{aligned} \tag{21}$$

It is easy to see that $(n–1)(2N–n+2)$, as a function of n, is maximized at either $n = N+1$ or $n = N+2$. For both cases, the last inequality in equation (21) holds. Therefore, we have found a positive solution to the nonlinear system (20).

On the other hand, suppose that there is a positive solution z to the nonlinear system (20). We need to show that $\alpha_2^2 \leq 2N\alpha_1\alpha_3/(N+1)$. For that purpose, we first show that the sets of roots of the quadratic functions $f_n(z) = \alpha_1 + \alpha_2(n-1)z + \alpha_3(n-1)(n-2)z^2/2$ interlacing each other for $n>2$. It is easy to obtain the roots of $f_n(z)$ as

$$z_{\pm}(n) = \frac{1}{n-2}\Big(-\frac{\alpha_2}{\alpha_3} \pm \sqrt{\frac{\alpha_2^2}{\alpha_3^2} - 2\frac{\alpha_1}{\alpha_3}\frac{(n-2)}{(n-1)}}\Big). \tag{22}$$

It can be verified that the real part of $z_+(n)$ is decreasing in n. By routine calculations, we obtain

$$f_n(z_{\pm}(n+1)) = \frac{\alpha_3}{(n-1)}\sqrt{\frac{\alpha_2^2}{\alpha_3^2} - 2\frac{\alpha_1}{\alpha_3}\frac{(n-1)}{n}}\Big(\pm\frac{\alpha_2}{\alpha_3} - \sqrt{\frac{\alpha_2^2}{\alpha_3^2} - 2\frac{\alpha_1}{\alpha_3}\frac{(n-1)}{n}}\Big). \tag{23}$$

If $z_+(n)$ is non-real for some n such that $3 \leq n \leq N+1$, we must have $\alpha_2^2 < 2(n–2)\alpha_1\alpha_3/(n–1) \leq 2N\alpha_1\alpha_3/(N+1)$. Thus, the result is obtained. Otherwise, both $z_+(n)$ and $z_-(n)$ are real for $3 \leq n \leq N+1$. Equation (23) shows that $f_n(z_+(n+1)) < 0$ and $f_n(z_-(n+1)) > 0$. Since

$\alpha_3 > 0$, the function $f_n(z)$ is convex, for $3 \le n \le N{+}1$. Therefore, those roots are interlacing, i.e., $z_-(n{+}1) \le z_-(n) \le z_+(n{+}1) \le z_+(n)$. Consequently, the nonlinear system (20) does not hold in the interval $(z_-(N{+}1),\ z_+(3))$. By the inequality in (20) for $n = 2$, we obtain $z \le -\alpha_1/\alpha_2$. By the inequality in (20) for $n = N + 2$, we obtain $z \ge -\alpha_2/(N\alpha_3)$. Thus, to ensure the existence of a solution to nonlinear system (20), we must have

$$\text{either} \quad \frac{\alpha_1}{(-\alpha_2)} > z_+(3) \quad \text{or} \quad z_-(N+1) > \frac{(-\alpha_2)}{N\alpha_3}, \tag{24}$$

which leads to $\alpha_2^2 \le 2N\alpha_1\alpha_3/(N+1)$. This completes the proof of Lemma 5.1.

Note 5.1: The necessity of Lemma 5.1 can be proved by using Theorem 7.4 in O'Cinneide[19].

Lemma 5.1 concludes that there is always a minimal ordered Coxian representation of the special form for the case with *ME*-order 3. Combining Lemma 5.1 of this paper and Theorem 7.3 in O'Cinneide[19], an explicit and complete solution for the minimal ordered Coxian representation problem of generalized Erlang distributions of *ME*-order 3 can be obtained.

Theorem 5.2. *Consider a representation $(\alpha,\ E_{m,\lambda})$ with $m = 3$ and $\lambda > 0$.*

a) If $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$ are nonnegative, then the triangular order of $(\alpha,\ E_{3,\lambda})$ is 3 or less.

b) If $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 > 0$, and $\alpha_2^2 < 2\alpha_1\alpha_2$, then the triangular order of $(\alpha,\ E_{3,\lambda})$ is $3{+}N$, where $N{=}\lceil \alpha_2^2/(2\alpha_1\alpha_3 - \alpha_2^2)\rceil$ and a minimal ordered Coxian representation $(\beta,\ S(\mathbf{x}))$ can be constructed from $\mathbf{x} = (y,\ y,\ \ldots,\ y,\ \lambda,\ \lambda,\ \lambda)$ by using the spectral polynomial algorithm defined in equation (4), where $y = \lambda(1{-}N\alpha_3/\alpha_2)$.

c) If $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 > 0$, and $\alpha_2^2 = 2\alpha_1\alpha_2$, then $(\alpha,\ E_{3,\lambda})$ is a matrix-exponential distribution, but is neither a *PH*-distribution nor a Coxian distribution.

d) For all other cases, $(\alpha,\ E_{3,\lambda})$ does not represent a probability distribution.

6 Bivariate Coxian Representations

Section 5 shows that there is a minimal Coxian representation with $\mathbf{x} = (y,\ y,\ \ldots,\ y,\ \lambda,\ \lambda,\ \lambda)$, a bivariate Coxian representation, for a generalized Erlang distribution of *ME*-order 3, if Assumption 1 is satisfied. A natural question is whether or not a minimal Coxian representation with $\mathbf{x} = (y,\ y,\ \ldots,\ y,\ \lambda,\ \ldots,\ \lambda)$ exists for all generalized Erlang distributions satisfying Assumption 1. Unfortunately, Example 3.1 has no such a solution for $N = 2$ (see Figure 3.1), since no feasible solution is on the diagonal line in Figure 3.1. Thus, for Example 3.1, there is no bivariate Coxian representation of the triangular order.

Although there is no bivariate Coxian representation of the triangular order for some cases, our numerical experimentations show that many do. More importantly, if Coxian representations exist, bivariate Coxian representations exist and it is much more efficient to compute bivariate Coxian representations. Therefore, it is worth to investigate solutions of this form. In this section, we show that for any generalized Erlang distribution with Coxian representations, there are always bivariate Coxian representations. Such solutions may not be of the triangular order, though.

Let $z = \eta_1 = \eta_2 = \ldots = \eta_N$. Then the nonlinear system $\alpha A(z) \ge 0$ is reduced to

$$\alpha \mathbf{a}_n(z) = \begin{cases} \displaystyle\sum_{i=1}^{\min\{n,m\}} \alpha_i \begin{pmatrix} n-1 \\ \\ i-1 \end{pmatrix} z^{i-1}, & 1 \le n \le N; \\ \displaystyle\sum_{i=n-N}^{\min\{n,m\}} \alpha_i \begin{pmatrix} N \\ i-n+N \end{pmatrix} z^{i-n+N}, & N+1 \le n \le m+N. \end{cases} \tag{25}$$

The next theorem shows that if a generalized Erlang distribution is also a *PH*-distribution, then it has bivariate Coxian representations.

Theorem 6.1. *For a generalized Erlang distribution, it has a bivariate ordered Coxian representation if and only if Assumption 1 holds.*

Proof. The result is equivalent to: the nonlinear system (25) has a positive solution for some N if and only if Assumption 1 holds. If a generalized Erlang distribution has a bivariate ordered Coxian representation, it represents a *PH*-distribution. Consequently, the polynomial function defined in Assumption 1 has no positive root.

On the other hand, suppose that the polynomial function defined in Assumption 1 has no positive root. Let $(\alpha,\ E_{m,\lambda})$ be a minimal *ME*-representation. We need to show that the nonlinear system (25) has a finite positive solution. For all cases with $n \le N$, the left hand side of equation (25), i.e., the polynomial function $\alpha\mathbf{a}_n(z)$, is positive, if z is positive and sufficiently small, since the first nonzero element in α must be positive. For all cases with $n > N$, the polynomial function $\alpha\mathbf{a}_n(z)$ is positive, if z is sufficiently large, since the last nonzero element in α is positive. Next, we show that the two regions are overlapping.

Since the first nonzero element in α must be positive, without loss of generality, we assume that $\alpha_1 > 0$. For $n \le N{+}1$, denote by x_n the smallest positive solution of $\alpha\mathbf{a}_n(z) = 0$. By equation (25), we have, for $N{>}m$ and $n{\le}\ N{+}1$,

$$
\begin{aligned}
0 =& \sum_{i=1}^{m} \alpha_i \frac{n(n-1)\cdots(n-i+1)}{(i-1)!} x_n^{i-1} \sum_{i=1}^{m} \frac{\alpha_i (nx_n)^{i-1}}{(i-1)!}\Big(1-\frac{1}{n}\Big)\cdots\Big(1-\frac{i-1}{n}\Big)\\
=&(nx_n)^{m-1}\Big(\alpha_m + \sum_{i=1}^{m-1} \frac{\alpha_i}{(nx_n)^{m-i}(i-1)!}\Big(\frac{1}{(1-i/n)(1-(i+1)/n)\cdots(1-(m-1)/n)}\Big)\Big). \qquad (26)
\end{aligned}
$$

If $\lim\limits_{n\to\infty}\sup(nx_n) = \infty$, equation (26) becomes $\alpha_m = 0$, which is a contradiction (Proposition 2.1). Therefore, we must have either $\lim\limits_{n\to\infty}\sup(nx_n) < \infty$ or x_n is infinite for large enough n (i.e., $\alpha\mathbf{a}_n(z)$ has no positive root). If $\lim\limits_{n\to\infty}\sup\{nx_n\} = 0$, then equation (26) leads to $\alpha_1 = 0$, which is a contradiction to the assumption $\alpha_1 > 0$. Then we must have $0 < \lim\limits_{n\to\infty}\sup(nx_n) < \infty$ and a subsequence of $\{nx_n,\ n{\ge}1\}$ that converges to a positive finite number. By equation (26), that positive finite number is a root of the polynomial function defined in Assumption 1, which is a contradiction. Therefore, if n is sufficiently large, $\alpha\mathbf{a}_n(z)$ has no positive root. That is: $\alpha\mathbf{a}_n(z)$ is positive in $(0,\infty)$ if n is sufficiently large.

For $N{+}1 < n \le N{+}m$, we have, for any positive $z > 0$,

$$
\begin{aligned}
\alpha\mathbf{a}_n(z) &= \binom{N}{N+m-n}\left[\alpha_m z^{m-1} + \sum_{i=N+m-n+1}^{m-1} \alpha_i z^{i-1} \frac{\binom{N}{N-n+i}}{\binom{N}{N+m-n}}\right]\\
&= \binom{N}{N+m-n}\left[\alpha_m z^{m-1} + \sum_{i=N+m-n+1}^{m-1} \alpha_i z^{i-1} \frac{(m+N-n)!(n-m)!}{(N-n+i)!(n-i)!}\right]\\
&\overset{N\to\infty}{\longrightarrow} \binom{N}{N+m-n}\alpha_m z^{m-1} > 0. \qquad (27)
\end{aligned}
$$

Therefore, the function $\alpha\mathbf{a}_n(z)$ is positive for any given positive z if N is sufficiently large.

Combining the above two cases, there exists a z such that all $\alpha\mathbf{a}_n(z)$ is positive if N is sufficiently large. Consequently, the generalized Erlang distribution has bivariate Coxian representations. This completes the proof of Theorem 6.1.

It is clear that the minimal bivariate Coxian representation corresponds to the smallest N such that the nonlinear system (25) has a positive solution. The order of this solution is an upper bound of the triangular order. If a bivariate Coxian representation exists, the nonlinear program (12) can be solved by finding all the roots of the polynomial in (25) and the intersections of the intervals for which the functions are nonnegative. This approach is numerically efficient. Details are omitted.

Example 6.1 (Example 3.1 continued). For this example, a bivariate ordered Coxian representation $(\beta(\mathbf{x}), S(\mathbf{x}))$ of order 8 can be found: $z = 1$, $\mathbf{x} = (10, 10, 10, 5, 5, 5, 5, 5)$, and $\beta(\mathbf{x}) = (0.5, 0.175, 0.075, 0.0625, 0.0, 0.0625, 0.0875, 0.0375)$. Since it is demonstrated in Figure 3.1 that there is no bivariate Coxian representation of order 7, this bivariate Coxian representation is a minimal bivariate Coxian representation.

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