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CONSTRUCTION OF CONTINUOUS TIME MARKOVIAN ARRIVAL PROCESSES*

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Abstract

Markovian arrival processes were introduced by Neuts in 1979 (Neuts 1979) and have been used extensively in the stochastic modeling of queueing, inventory, reliability, risk, and telecommunications systems. In this paper, we introduce a constructive approach to define continuous time Markovian arrival processes. The construction is based on Poisson processes, and is simple and intuitive. Such a construction makes it easy to interpret the parameters of Markovian arrival processes. The construction also makes it possible to establish rigorously basic equations, such as Kolmogorov differential equations, for Markovian arrival processes, using only elementary properties of exponential distributions and Poisson processes. In addition, the approach can be used to construct continuous time Markov chains with a finite number of states

Keywords: Markovian arrival process, Poisson process, matrix-analytic methods

1. Introduction

Markovian arrival processes are a popular tool for modeling arrival processes of stochastic systems such as queueing systems, reliability systems, telecommunications networks, inventory and supply chain systems, and risk and insurance systems. The popularity of Markovian arrival processes comes from i) its versatility in modeling stochastic systems; ii) its Markovian property that leads to Markovian structures; and iii) the maneuverability in the resulting Markov chains. In this paper, we introduce a simple, yet mathematically rigorous, approach to constructing Markovian arrival processes. As a result, learning and using Markovian arrival processes requires only basic knowledge of exponential distributions and Poisson processes.

Counting processes are important stochastic processes for science and engineering. In order to capture the characteristics of real stochastic processes, a number of counting processes have been introduced. Some well-known counting processes are Poisson processes, compound Poisson processes, Markov modulated Poisson processes, renewal processes, and semi-Markov processes (Cinlar 1969). An interesting and useful way to generate counting processes is by

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modeling the transitions of Markov chains (Rudemo 1973). By utilizing this idea in a systematic manner, Neuts introduced Markovian arrival processes as generalizations of Poisson processes, compound Poisson processes, and Markov modulated Poisson processes (Neuts 1979). Since their introduction in 1979, Markovian arrival processes have been used in the study of various queueing models in the 1980's (see Neuts (1981, 1989) and the references therein). The name of Markovian arrival processes evolved from *versatile Markovian point process* (Neuts 1979), to *Neuts process* (Ramaswami 1981), and then to *non-renewal arrival process* (Lucantoni, Meier-Hellstern & Neuts 1990), until it settled at (*batch*) *Markovian arrival processes* in Lucantoni (1991). Lucantoni (1991) also introduced a simple matrix representation for Markovian arrival processes, which made it easy to interpret parameters of Markovian arrival processes and to use Markovian arrival processes in stochastic modeling. Asmussen & Koole (1993) showed that (batch) Markovian arrival processes can approximate any stochastic arrival processes, which provided a theoretical basis for the use of Markovian arrival processes. In the 1990's, Markovian arrival processes were generalized to marked Markovian arrival processes that can be used to model arrival processes with multi-types of arrivals (He 1996, He & Neuts 1998). Such an extension led to the study of a number of queueing models with multi-types of customers (He 1996, 2001, Takine 2001), where the arrival processes of different types of customers are correlated. Meanwhile, applications of Markovian arrival processes were found in telecommunications networks, inventory-production management, risk and insurance analysis. The wide applicability of Markovian arrival processes stimulated the study of the characterization of Markovian arrival processes (e.g., Narayana & Neuts 1992, Neuts 1992, 1995, Neuts, Liu & Narayana 1992). For the development and application of Markovian arrival processes, we refer to Latouche & Ramaswami (1999) and Chakravarthy (2001).

The focus of this paper is on a constructive approach to define continuous time Markovian arrival processes. This new approach is based on Poisson processes, instead of continuous time Markov chains used in the classical definition. With the new definition, it is easy to understand the parameters of Markovian arrival processes intuitively. The new definition is simple and intuitive, yet it is mathematically rigorous. Basic results on Markovian arrival processes, such as the well-known (generalized) Kolmogorov differential equations, can be proved rigorously based on the new definition and some elementary properties of exponential distributions. In addition, the new definition leads to an easy way to simulate Markovian arrival processes. Therefore, the new definition makes it easy to introduce and to use Markovian arrival processes in stochastic modeling for both researchers and practitioners.

The remainder of the paper is organized as follow. In Section 2, exponential distributions, Poisson processes, and their properties are introduced. A constructive approach to define continuous time Markov chains with a finite number of phases is introduced in Section 3. Section 4 gives the definitions of Markovian arrival processes and batch Markovian arrival processes. Section 5 gives the definition of marked Markovian arrival processes. Some interesting examples of marked Markovian arrival processes are also presented in Section 5. Section 6 concludes the paper.

2. Preliminaries

In this section, we define exponential distributions and Poisson processes. Properties used in later sections are collected. All the properties can be proved by routine calculations. We refer to Ross (2007) for the proofs of the properties.

Definition 2.1 A nonnegative *random variable X* has an *exponential distribution* if its probability *distribution function* is given by

$$F(t) = P\{X \le t\} = 1 - \exp\{-\lambda t\}, \quad t \ge 0 \qquad (1)$$

where λ is a positive real number. We call X an exponential random variable with parameter λ.

Proposition 2.1 Properties of exponential distributions used in this paper are collected.

i) Assume that X has an exponential distribution with parameter λ. Then $P\{X>t+s|X>s\} = P\{X>t\}$ holds for $t \ge 0$ and $s \ge 0$, which is called the *memoryless property*. The memoryless property says that the distribution of the residual time $X-s$, given $X>s$, denoted by $X-s|X>s$, is independent of the time s that has elapsed.

ii) Assume that X_1, X_2, and X_3 are three independent exponential random variables with parameters λ_1, λ_2, and λ_3, respectively. We have, for small t,

$$P\{X_1 + X_2 \le t\} = 0.5\lambda_1 \lambda_2 t^2 + o(t^2) = o(t);$$

$$P\{X_1 = \min\{X_1, X_2\} \le t,\ X_1 + X_3 > t\} = \lambda_1 t + o(t) \qquad (2)$$

iii) Assume that $\{X_j,\ 1 \le j \le n\}$ are independent exponential random variables with parameters $\{\lambda_j,\ 1 \le j \le n\}$, respectively. Then $X=\min\{X_1,\ldots,X_n\}$ is exponentially distributed with parameter $\lambda_1+\ldots+\lambda_n$. (See Figure 1 for $n = 2$.)

iv) Assume that $\{X_j,\ 1 \le j \le n\}$ are independent exponential random variables with parameters $\{\lambda_j,\ 1 \le j \le n\}$, respectively. Then $P\{X_1 = \min\{X_1,\ldots, X_n\}\} = \lambda_1/(\lambda_1+\cdots+\lambda_n)$.

v) Assume that $\{X_n,\ n \ge 0\}$ are independent exponential random variables with the same parameter λ. Assume that random variable N, independent of $\{X_n, n \ge 0\}$, has a geometric

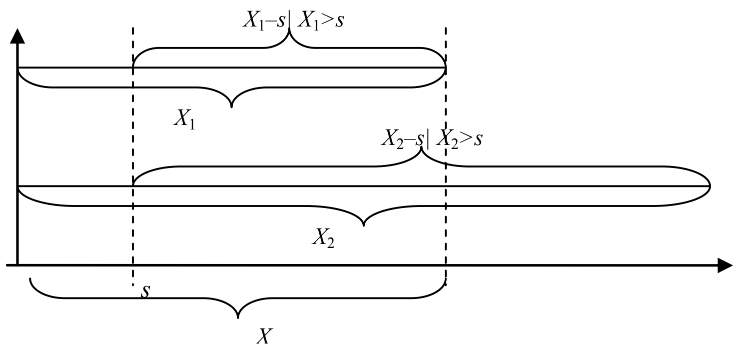

Figure 1 $X = \min\{X_1, X_2\}$ and its residual at time s

distribution with parameter p on positive integers {1, 2, …}, i.e., $P\{N=n\} = p^{n-1}(1-p)$, $n \geq 1$. Define $Y = \sum_{n=1}^{N} X_n$. Then Y has an exponential distribution with parameter $(1-p)\lambda$.

Let $\{N(t), t \geq 0\}$ be a *counting process*, i.e., $N(t)$ is the number of *events* occurring in [0, t]. An event can be a transition, the arrival of a customer, a demand, or the arrival of a group of customers. In fact, an event can be defined in any way or as anything meaningful. Poisson processes are special counting processes. The following definition is a constructive way to define Poisson processes (see Figure 2).

Definition 2.2 A counting process $\{N(t), t \geq 0\}$ is called a Poisson process if $\{N(t) \leq n\} = \{X_1 + X_2 + \cdots + X_n + X_{n+1} > t\}$, for $n \geq 0$ and $t \geq 0$, where $\{X_1, X_2, \ldots, X_n, \ldots\}$ are independent exponential random variables with parameter λ.

Proposition 2.2 For a Poisson process $\{N(t), t \geq 0\}$ with parameter λ, we have

i) $N(0) = 0$.

ii) $E(N(t)) = \lambda t$. Parameter λ is the average number of events per unit time.

iii) Let Y be the time elapsed until the first event after time t (see Figure 3). Then Y has an exponential distribution with parameter λ. This is called the memoryless property of Poisson processes.

3. Construction of CTMCs

In this section, we construct continuous time Markov chains (CTMCs) based on Poisson processes defined in Section 2.

Definition 3.1 Let $\{\alpha_i, 1 \leq i \leq m\}$ be nonnegative numbers with a unit sum (i.e., $\alpha_1 + \cdots + \alpha_m = 1$), $\{q_{i,j}, 1 \leq i \neq j \leq m\}$ nonnegative real numbers, and m a finite positive integer ($m \geq 2$). Assume $\sum_{j=1: j \neq i}^{m} q_{i,j} > 0$, for $1 \leq i \leq m$. A stochastic process $\{I(t), t \geq 0\}$ on phases $\{1, 2, \ldots, m\}$ is defined as follows.

1) Define $m(m-1)$ independent Poisson processes with parameters $\{q_{i,j}, 1 \leq i \neq j \leq m\}$. If $q_{i,j} = 0$, the corresponding Poisson process has no event at all.
2) Determine $I(0)$ by the probability distribution $\{\alpha_i, 1 \leq i \leq m\}$.
3) At time $t > 0$, if $I(t) = i$, then $I(t)$ stays in phase i until the first event occurs in the $m-1$ Poisson processes corresponding to $\{q_{i,j}, 1 \leq j \leq m, j \neq i\}$, for $1 \leq i \leq m$. If the event occurs at s ($> t$) and it comes from the Poisson process corresponding to $q_{i,j}$, the process transits from phase i to phase j at time s, i.e., $I(s-) = i$ and $I(s) = j$.

The process $\{I(t), t \geq 0\}$ is well-defined since, according to ii) in Proposition 2.1 and iii) in Proposition 2.2, the probability that two or more events from the same Poisson process or from different Poisson processes occur at the same time is zero. Figure 4 depicts a sample path of a process $\{I(t), t \geq 0\}$ with $m = 3$ (the solid line), which is generated by six Poisson processes (dashed lines). Note that for each phase i, $1 \leq i \leq 3$, there are two Poisson processes associated with it. Events of the Poisson processes are marked by "*". In Figure 4, we have $I(0) = 3$. The first event of the two Poisson processes associated with $q_{3,1}$ and $q_{3,2}$ occurs at $s = 0.5045$, and comes from the Poisson process associated with $q_{3,1}$. Then the process $\{I(t), t \geq 0\}$ transits to phase 1 at $s = 0.5045$. The rest of the sample path is interpreted similarly.

Next, we show that the process $\{I(t), t \geq 0\}$ is a CTMC.

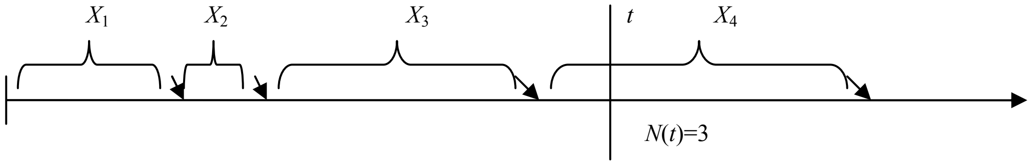

Figure 2 A sample path of a Poisson process

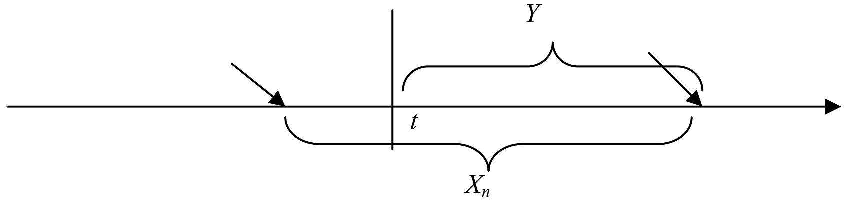

Figure 3 Memoryless property of Poisson processes

Lemma 3.1 *For the process* $\{I(t), t\geq 0\}$, *we have*

i) *The sojourn time of* $\{I(t), t\geq 0\}$ *in phase i has an exponential distribution with parameter* $q_{i,i} = -\sum_{j=1:\, j\neq i}^{m} q_{i,j}$ *, for* $1 \leq i \leq m$; *and*

ii) *The probability that the next phase is j is given by* $r_{i,j} \equiv q_{i,j}/(-q_{i,i})$, *given that the current phase is i, for* $1 \leq i \neq j \leq m$.

Proof. i) is obtained by iii) of Proposition 2.1 and iii) of Proposition 2.2. ii) is obtained by iv) of Proposition 2.1 and iii) of Proposition 2.2. This completes the proof of Lemma 3.1. ■

Theorem 3.2 *The stochastic process* $\{I(t), t\geq 0\}$ *is a CTMC with m phases. The infinitesimal generator of CTMC* $\{I(t), t\geq 0\}$ *is an* $m\times m$ *matrix* $Q = (q_{i,j})$.

Proof. By definition, $\{I(t), t\geq 0\}$ is a CTMC if, for $t>s\geq 0$,

$$P\{I(t)=j \mid I(s)=i, I(u)=i_u, 1\leq u<s\} = P\{I(t)=j \mid I(s)=i\}, \quad 1\leq i, i_u, j\leq m \qquad (3)$$

Since all Poisson process have memoryless property, equation (3) holds. Thus, $\{I(t), t\geq 0\}$ is a CTMC.

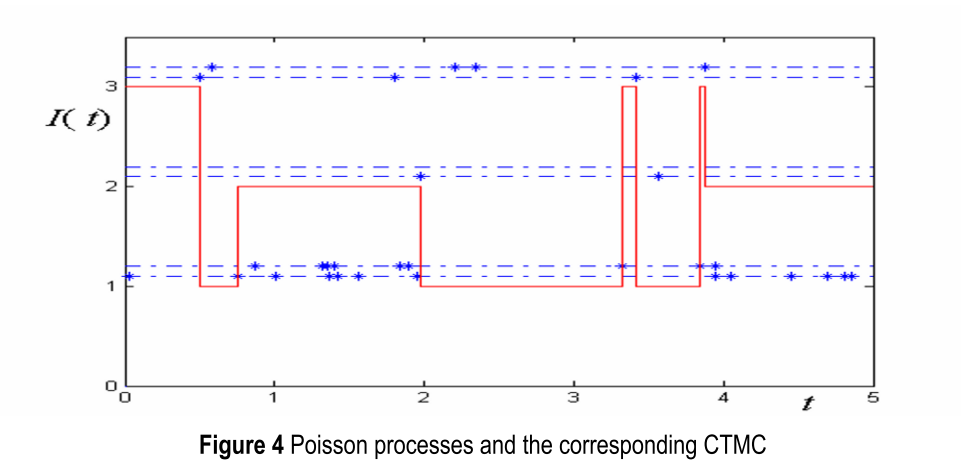

Figure 4 Poisson processes and the corresponding CTMC

CTMCs can also be defined as: 1) The sojourn time in every phase has an exponential distribution; and 2) Transitions between phases are determined by a discrete time Markov chain. By Lemma 3.1, the sojourn time in each phase is exponentially distributed with parameter $-q_{i,i}$, $1 \le i \le m$. Also by Lemma 3.1, the transitions of phases are governed by a finite Markov chain with transition probabilities $r_{i,j} = q_{i,j}/(-q_{i,i})$, for $1 \le i \ne j \le m$. Then the transition rates are given by $q_{i,j} = (-q_{i,i})r_{i,j}$, for $1 \le i \ne j \le m$. By definition, the matrix Q is the infinitesimal generator of $\{I(t), t\ge 0\}$. Once the initial state is chosen by $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, the Markov chain is determined probabilistically. This completes the proof of Theorem 3.2. ■

Theorem 3.2 indicates that any CTMC with a finite number of states can be defined by Definition 3.1. Definition 3.1 leads to intuitive understanding of the parameters of CTMCs. Parameters $\{q_{i,j}, 1 \le i \ne j \le m\}$ are the arrival rates of the $m(m-1)$ Poisson processes used to define the CTMC and they reflect the frequencies of the transitions from one phase to another. This explains why $\{q_{i,j}, 1 \le i \ne j \le m\}$ are called the transition rates and $\{-q_{i,i}, 1 \le i \le m\}$ are called the total transition rates in the literature. We call the $m(m-1)$ Poisson processes the underlying Poisson processes of the CTMC $\{I(t), t\ge 0\}$.

Definition 3.1 also leads to a simple and mathematically rigorous proof of the well-known Kolmogorov differential equations for CTMCs. Define $p_{i,j}(t) = P\{I(t)=j \mid I(0)=i\}$, for $t \ge 0$ and $1 \le i, j \le m$. Denote by $P(t) = (p_{i,j}(t))$, an $m\times m$ matrix, for $t\ge 0$.

Theorem 3.3 *For the CTMC* $\{I(t), t\ge 0\}$, *we have* $P'(t) = P(t)Q = QP(t)$, *for* $t>0$, *and* $P(0) = I$, *where I is the identity matrix.*

Proof. Let $\{\hat{X}_{i,j}, 1 \le i \ne j \le m\}$ be independent exponential random variables with parameters $q_{i,j}$, respectively. Let $\{X_{i,j}, 1 \le i \ne j \le m\}$ be the times until the next event in the Poisson processes corresponding to parameters $\{q_{i,j}, 1 \le i \ne j \le m\}$, respectively. By ii) of Proposition 2.2, $\{X_{i,j}, 1 \le i \ne j \le m\}$ are independent exponential random variables with parameter $q_{i,j}$, respectively. Assume that $\{X_{i,j}, 1 \le i \ne j \le m\}$ and $\{\hat{X}_{i,j}, 1 \le i \ne j \le m\}$ are independent. Denote by $J(t, t+\delta t)$ the number of transitions occurring in the interval $(t, t+\delta t]$. Conditioning on $I(t)$, we have the following calculations:

$$\begin{aligned}
p_{i,j}(t+\delta t) &= \sum_{k=1}^{m} P\{I(t)=k \mid I(0)=i\} \sum_{n=0}^{\infty} P\{I(t+\delta t)=j, J(t,t+\delta t)=n \mid I(t)=k, I(0)=i\} \\
&= \sum_{k=1}^{m} p_{i,k}(t) \sum_{n=0}^{\infty} P\{I(\delta t)=j, J(0,\delta t)=n \mid I(0)=k\} \\
&= \sum_{k=1}^{m} p_{i,k}(t) P\{I(\delta t)=j, J(0,\delta t)=0 \mid I(0)=k\} \\
&\quad + \sum_{k=1}^{m} p_{i,k}(t) P\{I(\delta t)=j, J(0,\delta t)=1 \mid I(0)=k\} \\
&\quad + \sum_{k=1}^{m} p_{i,k}(t) \left(\sum_{n=2}^{\infty} P\{I(\delta t)=j, J(0,\delta t)=n \mid I(0)=k\} \right)
\end{aligned} \tag{4}$$

Using ii), iii), and iv) of Proposition 2.1, we obtain

$$\sum_{k=1}^{m} p_{i,k}(t)P\{I(\delta t)=j, J(0,\delta t)=0 \mid I(0)=k\} = p_{i,j}(t)P\{I(\delta t)=j, J(0,\delta t)=0 \mid I(0)=j\}$$

$$= p_{i,j}(t)P\left\{\min_{1\le k\le m, k\ne j}\{X_{j,k}\} > \delta t\right\} = p_{i,j}(t)\left(1-\left(\sum_{k=1,k\ne j}^{m} q_{k,j}\right)\delta t + o(\delta t)\right) \tag{5}$$

$$\sum_{k=1}^{m} p_{i,k}(t)P\{I(\delta t)=j, J(0,\delta t)=1 \mid I(0)=k\}$$

$$= \sum_{k=1,k\ne j}^{m} p_{i,k}(t)P\left\{X_{k,j} = \min_{1\le l\le m, l\ne k}\{X_{k,l}\} < \delta t,\ \min_{1\le l\le m, l\ne j}\{X_{k,j}+\hat{X}_{j,l}\} > \delta t\right\}$$

$$= \sum_{k=1,k\ne j}^{m} p_{i,k}(t)\left(q_{k,j}\delta t + o(\delta t)\right) \tag{6}$$

and

$$\sum_{k=1}^{m} p_{i,k}(t)\sum_{n=2}^{\infty} P\{I(\delta t)=j, J(0,\delta t)=n \mid I(0)=k\} = \sum_{k=1}^{m} p_{i,k}(t)P\left\{\min_{1\le l\le m, l\ne k, l\ne j}\{X_{k,l}+X_{l,j}\} \le \delta t\right\}$$

$$\le \sum_{k=1}^{m} p_{i,k}(t) \sum_{1\le l\le m, l\ne k, l\ne k} P\left\{ X_{k,l}+X_{l,j} \le \delta t\right\} = o(\delta t) \tag{7}$$

Combining equations (4), (5), (6), and (7) yields

$$p_{i,j}(t+\delta t) = p_{i,j}(t)\left(1+q_{j,j}\delta t\right) + \sum_{k=1,k\ne j}^{m} p_{i,k}(t)q_{k,j}\delta t + o(\delta t) \tag{8}$$

which leads to the Kolmogorov forward differential equation

$$\frac{dp_{i,j}(t)}{dt} = \sum_{k=1}^{m} p_{i,k}(t)q_{k,j} \tag{9}$$

The Kolmogorov backward differential equation can be shown similarly. This completes the proof of Theorem 3.3. ■

We remark that the advantage of Definition 3.1 for CTMCs may disappear in the proof of the Kolmogorov differential equations if there are a countable number of phases. For CTMCs with a countable number of phases, there are technical issues, such as the existence of limits, in the above proof that needs advanced techniques to resolve. We also remark that the constructive approach cannot be extended to discrete time Markov chains, since multiple events can occur simultaneously for the discrete time case.

Example 3.1 Consider a reliability system with two identical units and a repairman. If both units are functioning, then one is in work and the other one is on cold standby. If the unit in work fails, it is sent to the repairman for repair and the standby unit is put in work. If repair is completed before failure, the repaired unit is on cold standby. If failure occurs before repair completion, the failed unit has to wait for repair. A repaired unit is put in work immediately if the other unit has failed. The times to failure and repair times are exponentially distributed with parameters λ and μ, respectively.

The state of each component can be 0: in repair, 1: waiting for repair, 2: on cold standby; and 3: in work. The system has three phases (states): {(3, 2), (3, 0), (1, 0)}, since the two units are identical. Let $I(t)$ be the status of the system at time t. The process $\{I(t), t\geq 0\}$ can be defined by the following underlying Poisson processes.

i) For phase (3, 2): A Poisson process (with parameter) λ. If an event occurs, the process $\{I(t), t\geq 0\}$ transits to phase (3, 0).
ii) For phase (3, 0): A Poisson process λ and a Poisson process μ. If an event from Poisson process λ occurs first, the process transits to phase (1, 0). If an event from Poisson process μ occurs first, the process transits to phase (3, 2).
iii) For phase (1, 0): A Poisson process μ. If an event occurs, the process transits to phase (3, 0).

Then the infinitesimal generator is given by

$$\begin{array}{c} \\ Q = \begin{array}{c}(3,2)\\(3,0)\\(1,0)\end{array}\end{array}\begin{array}{c}\begin{array}{ccc}(3,2) & (3,0) & (1,0)\end{array}\\ \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{pmatrix}\end{array} \qquad (10)$$

If λ = 0.01 and μ = 0.5, the stationary distribution of the CTMC $\{I(t), t\geq 0\}$ is $\boldsymbol{\theta}$ = (0.9800, 0.0196, 0.0004), which, by definition, satisfies $\boldsymbol{\theta} Q = 0$ and $\boldsymbol{\theta}\mathbf{e} = 1$, where $\mathbf{e}$ is the column vector of ones.

4. Construction of *MAPs* and *BMAPs*

In Section 3, only transitions between the underlying Poisson processes are considered. In this section, we define some special events, called *arrivals*, and keep track of the number of arrivals. The resulting process is a Markovian arrival process (*MAP*) or a batch Markovian arrival process (*BMAP*). We begin with the construction of *MAP*s.

Definition 4.1 Let $\{\alpha_i, 1 \leq i \leq m\}$ be nonnegative numbers with a unit sum, $\{d_{0,(i,j)}, 1 \leq i \neq j \leq m\}$ and $\{d_{1,(i,j)}, 1 \leq i, j \leq m\}$ are nonnegative numbers, and m is a finite positive integer. Assume

$$d_{0,(i,i)} = -\left(\sum_{j=1:\, j\neq i}^{m} d_{0,(i,j)} + \sum_{j=1}^{m} d_{1,(i,j)}\right) < 0,$$

for $1 \leq i \leq m$. We define a stochastic process $\{(N(t), I(t)), t\geq 0\}$ as follows.

1) Define $m(2m-1)$ independent Poisson processes with parameters $\{d_{0,(i,j)}, 1\leq i\neq j \leq m\}$ and $\{d_{1,(i,j)}, 1 \leq i, j \leq m\}$. If $d_{0,(i,j)} = 0$ or $d_{1,(i,j)} = 0$, the corresponding Poisson process has no event.
2) Determine $I(0)$ by the probability distribution $\{\alpha_i, 1 \leq i \leq m\}$. Set $N(0) = 0$.
3) If $I(t) = i$, for $1 \leq i \leq m$, $I(t)$ and $N(t)$ remain the same until the first event occurs in the $2m-1$ Poisson processes corresponding to $\{d_{0,(i,j)}, 1 \leq j \leq m, j \neq i\}$ and $\{d_{1,(i,j)}, 1\leq j \leq m\}$. If the next event comes from the Poisson process corresponding to $d_{0,(i,j)}$, the variable $I(t)$ changes from phase i to phase j and $N(t)$ does not change at the epoch, for $1 \leq j \leq m$, $j \neq i$; If the next event comes from the Poisson process corresponding to $d_{1,(i,j)}$, the phase variable $I(t)$ transits from phase i to phase j and $N(t)$ is increased by one at the epoch, i.e., an arrival occurs, for $1 \leq j \leq m$.

The random variable $N(t)$ records the number of arrivals associated with the Poisson processes with parameters $\{d_{1,(i,j)}, 1 \leq i, j \leq m\}$ occurring in $[0, t]$ for $t>0$. Thus, the process $\{N(t), t\geq 0\}$ is a counting process. We call the $m(2m-1)$ Poisson processes the underlying Poisson processes of the counting process. We call the process $\{I(t), t\geq 0\}$ the phase process. Figure 5 depicts the sample paths of the underlying Poisson processes, the phase process $\{I(t), t\geq 0\}$, and the counting process $\{N(t), t\geq 0\}$ for an *MAP* with $m = 3$. Poisson processes associated with $\{d_{0,(i,j)}, 1 \leq i \neq j \leq m\}$ are in blue dashed lines and Poisson processes associated with $\{d_{1,(i,j)}, 1 \leq i, j \leq m\}$ are in red dashed lines. Events in blue star represent a transition without an arrival. Events in red star represent a transition with a potential arrival.

In Figure 5, we have $I(0) = 2$. The first transition goes from phase 2 to 1 without an arrival at $t = 0.2312$. During the stay of the process $\{I(t), t\geq 0\}$ in phase 1, there are two arrivals at $t = 0.2882$ and $t = 0.4884$. At $t = 1.0311$, the process $\{I(t), t\geq 0\}$ transits to phase 3 without an arrival. At $t = 2.0843$, the process $\{I(t), t\geq 0\}$ transits to phase 2 with an arrival. The rest of Figure 5 can be interpreted similarly.

Similar to Lemma 3.1, the following results can be proved.

Lemma 4.1 *For the process* $\{(N(t), I(t)), t\geq 0\}$, *we have*

i) The sojourn time of $\{(N(t), I(t)), t\geq 0\}$ *in state* (n, i) *has an exponential distribution with parameter* $-d_{0,(i,i)}$, *for* $1 \leq i \leq m$;

ii) The probability that the next phase is j and no arrival at the transition epoch is given by $p_{0,(i,j)} \equiv d_{0,(i,j)}/(-d_{0,(i,i)})$, *given that the current state is* (n, i), *for* $1 \leq i \neq j \leq m$ *and* $n\geq 0$.

iii) The probability that the next phase is j and an arrival occurs at the transition epoch is given by $p_{1,(i,j)} \equiv d_{1,(i,j)}/(-d_{0,(i,i)})$, *given that the current state is* (n, i), *for* $1 \leq i, j \leq m$ *and* $n\geq 0$.

Let $D_0 = (d_{0,(i,j)})$, $D_1 = (d_{1,(i,j)})$, and $D=D_0+D_1$, three $m\times m$ matrices.

Theorem 4.2 *The stochastic process* $\{I(t), t\geq 0\}$ *is a continuous time Markov chain with an infinitesimal generator D.*

Proof. The sojourn time of $\{I(t), t\geq 0\}$ in phase i can be written as $Y_i = \sum_{n=1}^{N} Z_{i,n}$, where

$$Z_{i,n} = \min\left\{ \min_{1\leq j\leq m,\, j\neq i} \{X_{0,(i,j),n}\},\ \min_{1\leq j\leq m,} \{X_{1,(i,j),n}\} \right\},$$

$\{X_{0,(i,j),n}, X_{1,(i,j),n}\}$ are exponentially distributed with parameters $\{d_{0,(i,j)}, d_{1,(i,j)}\}$, respectively, all

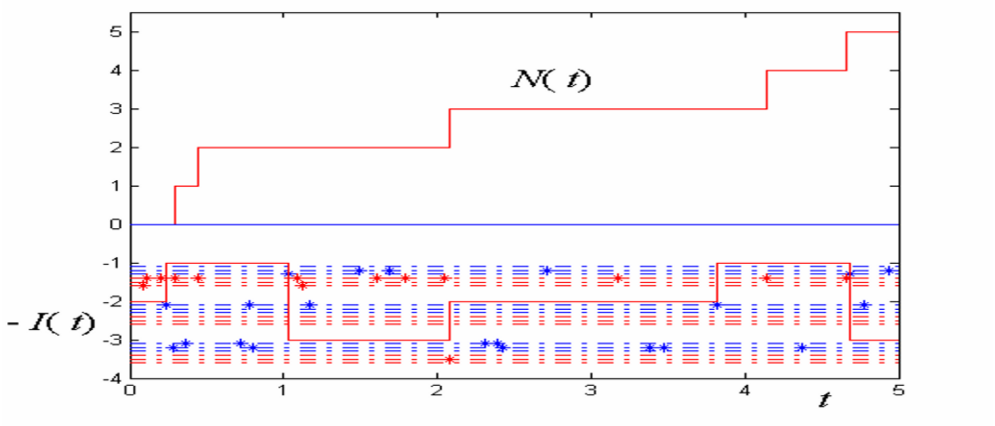

Figure 5 Sample paths of underlying Poisson processes, $I(t)$, and $N(t)$ of an *MAP*

the exponential distributions are independent, and N has a geometric distribution with parameter $p_{1,(i,i)}$. By ii) of Proposition 2.1, $Z_{i,n}$ has an exponential distribution with parameter $-d_{0,(i,i)}$. By v) of Proposition 2.1, Y_i has an exponential distribution with parameter $(-d_{0,(i,i)})(1-p_{1,(i,i)}) = -(d_{0,(i,i)} + d_{1,(i,i)})$. Thus, the process $\{I(t), t\geq 0\}$ is a CTMC. By ii) and iii) of Lemma 4.1, the probability that the process transits from phase i to phase j, given that a transition of phase was incurred, for $i \neq j$, is given by $(p_{0,(i,j)} + p_{1,(i,j)})/(1-p_{1,(i,i)})$, which implies that the transition rate from i to j is $-(d_{0,(i,i)} + d_{1,(i,i)})(p_{0,(i,j)} + p_{1,(i,j)})/(1-p_{1,(i,i)}) = -(p_{0,(i,j)} + p_{1,(i,j)})d_{0,(i,i)} = d_{0,(i,i)} + d_{1,(i,i)}$. Thus, the infinitesimal generator of the CTMC $\{I(t), t\geq 0\}$ is $D = D_0 + D_1$. This completes the proof of Theorem 4.2. ■

In the literature, $\{I(t), t\geq 0\}$ is called the underlying Markov chain of the corresponding *MAP*. In fact, in the literature, *MAP*s are defined on CTMCs, instead of Poisson processes.

Theorem 4.3 *The stochastic process* $\{(N(t), I(t)), t\geq 0\}$ *is a continuous time Markov chain with an infinitesimal generator*

$$Q = \begin{pmatrix} D_0 & D_1 & & & \\ & D_0 & D_1 & & \\ & & D_0 & D_1 & \\ & & & \ddots & \ddots \end{pmatrix} \qquad (11)$$

and an initial distribution $((\alpha_1,\alpha_2, \ldots,\alpha_m),0,\ldots,0)$. *The pair* (D_0, D_1) *is called a matrix representation of the MAP.*

Proof. First, the state space of $\{(N(t), I(t)), t\geq 0\}$ is $\{0, 1, 2, \ldots\}\times\{1, 2, \ldots, m\}$. The variable $N(t)$ is nondecreasing. Since all underlying Poisson processes are memoryless, the process $\{(N(t), I(t)), t\geq 0\}$ is Markovian. By i) of Lemma 4.1, the sojourn time in state (n, i) is exponentially distributed with parameter $-d_{0,(i,i)}$. By ii) of Lemma 4.1, the transition probability from state (n, i) to state (n, j) is $p_{0,(i,j)}$, which implies that the transition rate from state (n, i) to state (n, j) is $p_{0,(i,j)}(-d_{0,(i,i)}) = d_{0,(i,j)}$, for $1 \leq i \neq j \leq m$. By iii) of Lemma 4.1, the transition probability from state (n, i) to state $(n+1, j)$ is $p_{1,(i,j)}$, which implies that the transition rate from state (n, i) to state $(n+1, j)$ is $p_{1,(i,j)}(-d_{0,(i,i)}) = d_{1,(i,j)}$, for $1 \leq i, j \leq m$. Therefore, the infinitesimal generator of CTMC $\{(N(t), I(t)), t\geq 0\}$ is given in equation (11). This completes the proof of Theorem 4.3. ■

We remark that, in the literature, Lemma 4.1 and Theorem 4.3 are used to define *MAP*s. Next, we show one more way to define *MAP*s. Let τ_n be the time between the n-th and the $(n+1)$-st arrivals. Let $X_n = I((\tau_1+\cdots+\tau_{n-1})+)$. Conditioning on the phase of $\{I(t), t\geq 0\}$ at time t, we obtain, for $1 \leq i, j \leq m$,

$$P\{I(t+\delta t) = j, t < \tau_1 + \cdots + \tau_n \leq t+\delta t \mid I(0) = i\} = \sum_{k=1}^{m} p_{i,k}(t) d_{1,(k,j)}\delta t + o(\delta t), \qquad (12)$$

which leads to

$$P\{X_{n+1} = j, \tau_n \leq t \mid X_n = i\} = \left(\int_0^t \exp\{D_0 s\} D_1 ds\right)_{i,j}.$$

Then the semi-Markov chain $\{X_n, \tau_n, n\geq 0\}$ defines an *MAP* through the relationship $\{N(t) \leq n\} = \{\tau_1+\tau_2+\cdots+\tau_n \geq t\}$.

The above four definitions of *MAP*s are equivalent. Similarly, there are four definitions for *BMAP*s (Definition 4.2) and *MMAP*s (Definition 5.1) to be introduced next.

Define $p_{i,j}(n, t) = P\{N(t) = n, I(t)=j \mid I(0)=i\}$, for $1 \leq i, j \leq m$, and $n \geq 0$, the conditional distribution of the number of arrivals in $[0, t]$. Denote by $P(n, t) = (p_{i,j}(n, t))$, an $m\times m$ matrix, for t≥0 and n≥0. Similar to Theorem 3.3, based on Propositions 2.1 and 2.2, the following differential equations can be proved: $P'(0, t) = P(0, t)D_0$, $P(0, 0) = I$, $P'(n, t) = P(n, t)D_0 + P(n-1, t)D_1$, $P(n, 0) = 0$, for n≥1, which leads to a result on the distribution of the number of arrivals.

Theorem 4.4 *For an MAP* $\{(N(t), I(t)), t\geq 0\}$ *with a matrix representation* (D_0, D_1), *define* $P^*(z,t) = \sum_{n=0}^{\infty} z^n P(n,t)$, *for* z≥0. *Then* $P^*(z,t) = \exp\{(D_0 + zD_1)t\}$.

Proof. Multiplying by z^n on both sides of $P'(n, t) = P(n, t)D_0 + P(n-1, t)D_1$, and summing up over n, yields

$$\frac{\partial P^*(z,t)}{\partial t} = P^*(z,t)D_0 + zP^*(z,t)D_1 \qquad (13)$$

which leads to the expected result. This completes the proof of Theorem 4.4. ■

Based on Theorem 4.4, the average number of arrivals per unit time, called the (stationary) *arrival rate*, can be found as $\lambda = \boldsymbol{\theta} D_1 \mathbf{e}$, where $\boldsymbol{\theta}$ is the stationary distribution of D (assuming that D is irreducible), i.e., $\boldsymbol{\theta} D = 0$ and $\boldsymbol{\theta}\mathbf{e} = 1$.

Example 4.1 (Example 3.1 continued) For the model introduced in Example 3.1, we are interested in the number of repairs in [0, t]. We define the events of Poisson processes associated with repairs as arrivals (corresponding to D_1). The rest of the Poisson processes correspond to D_0. Let $N(t)$ be the number of repairs completed in [0, t]. Then it is easy to see that $\{(N(t), I(t)), t\geq 0\}$ is an *MAP* with a matrix representation

$$D_0 = \begin{matrix} (3,2) \\ (3,0) \\ (1,0) \end{matrix} \overset{\begin{matrix} (3,2) & (3,0) & (1,0) \end{matrix}}{\begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda-\mu & \lambda \\ 0 & 0 & -\mu \end{pmatrix}},$$

$$D_1 = \begin{matrix} (3,2) \\ (3,0) \\ (1,0) \end{matrix} \overset{\begin{matrix} (3,2) & (3,0) & (1,0) \end{matrix}}{\begin{pmatrix} 0 & 0 & 0 \\ \mu & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}} \qquad (14)$$

If $\lambda = 0.01$ and $\mu = 0.5$, the stationary distribution of $\{I(t), t\geq 0\}$ is $\boldsymbol{\theta}$ = (0.9800, 0.0196, 0.0004) and the arrival rate $\boldsymbol{\theta} D_1 \mathbf{e}$ = 0.009996, i.e., repair completion occurs 0.009996 times per unit time.

In Definition 3.1, events of underlying Poisson processes trigger only one thing: a transition of phase for the CTMC $\{I(t), t\geq 0\}$. In Definition 4.1, events of underlying Poisson processes are categorized in two types: One type of events triggers only a transition of phase and the other type of events triggers a transition of phase and an arrival. To extend Definition 4.1, we consider events that trigger a batch of arrivals.

Definition 4.2 Let $\{\alpha_i, 1 \leq i \leq m\}$ be nonnegative numbers with a unit sum, $\{d_{0,(i,j)}, 1 \leq i \neq j \leq m\}$, $\{d_{n,(i,j)}, 1 \leq i, j \leq m\}$ are nonnegative numbers, for $1 \leq n \leq N < \infty$, and m is a finite positive integer. Assume

$$d_{0,(i,i)} = -\left(\sum_{j=1:\, j\neq i}^{m} d_{0,(i,j)} + \sum_{n=1}^{N}\sum_{j=1}^{m} d_{n,(i,j)}\right)$$

< 0, for $1 \leq i \leq m$. We define a stochastic process $\{(N(t), I(t)), t\geq 0\}$ as follows.

1) Define $m((N+1)m-1)$ independent Poisson processes with parameters $\{d_{0,(i,j)}, 1 \leq i \neq j \leq m\}$ and $\{d_{n,(i,j)}, 1 \leq i, j \leq m, 1 \leq n \leq N\}$. If $d_{n,(i,j)} = 0$, the corresponding Poisson process has no event.
2) Determine $I(0)$ by the probability distribution $\{\alpha_i, 1 \leq i \leq m\}$. Set $N(0) = 0$.
3) If $I(t) = i$, for $1 \leq i \leq m$, $I(t)$ and $N(t)$ remain the same until the first event occurs in the

$(N+1)m-1$ Poisson processes corresponding to $\{d_{0,(i,j)}, 1 \le j \le m, j \ne i\}$ and $\{d_{n,(i,j)}, 1 \le j \le m, 1 \le n \le N\}$. If the next event comes from the Poisson process corresponding to $d_{0,(i,j)}$, the variable $I(t)$ changes from phase i to phase j and $N(t)$ does not change at the epoch, for $1 \le j \le m, j \ne i$; If the next event comes from the Poisson process corresponding to $d_{n,(i,j)}$, the phase variable $I(t)$ transits from phase i to phase j and $N(t)$ is increased by n at the epoch, i.e., that a batch of n arrivals is associated with the event, for $1 \le j \le m$ and $1 \le n \le N$.

An example of *BMAP* is shown in Figure 6. In Figure 6, a batch of size 2 arrives at t = 1.8408 when the phase goes from 1 to 3.

Similar to Theorems 4.2 to 4.4, it can be shown that $\{I(t), t \ge 0\}$ and $\{(N(t), I(t)), t \ge 0\}$ are CTMCs. That $\{(N(t), I(t)), t \ge 0\}$ is a *BMAP* with matrix representation $(D_0, D_1, \ldots, D_N)$, where $D_n = (d_{n,(i,j)})$, for $0 \le n \le N$. The infinitesimal generator of $\{I(t), t \ge 0\}$ is $D = D_0 + \cdots + D_N$. The conditional distributions of the number of arrivals in $[0, t]$ can be obtained from $P^*(z,t) = \exp\left\{\left(\sum_{n=0}^{N} z^n D_n\right)t\right\}$. The arrival rate is $\lambda = \boldsymbol{\theta}(D_1 + 2D_2 + \cdots + ND_N)\mathbf{e}$, where $\boldsymbol{\theta}$ is the stationary distribution of D (assuming that D is irreducible).

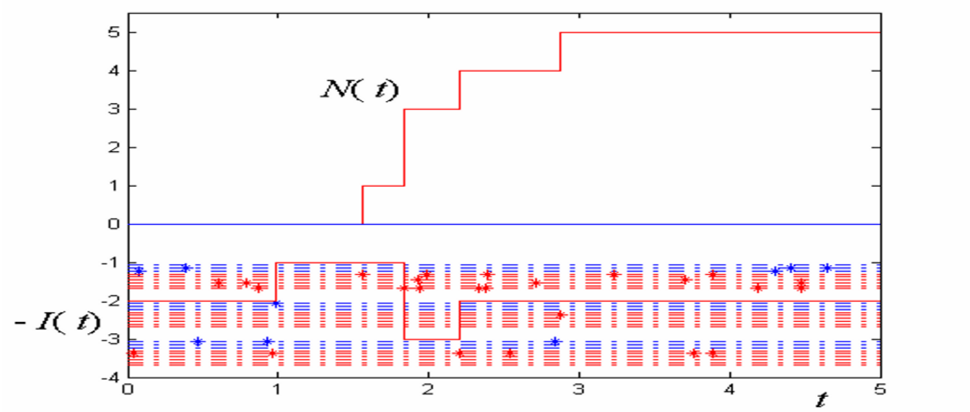

Figure 6 Sample paths of underlying Poisson processes, $I(t)$, and $N(t)$ of a *BMAP*

5. Construction of *MMAP*s and Examples

Definitions 3.1, 4.1, and 4.2 demonstrate that, by assigning different meanings to events of different Poisson processes, more versatile arrival processes can be introduced. In this section, we further extend those definitions to construct Markovian arrival processes with different types of arrivals, called Marked Markovian arrival processes (*MMAP*s).

Let C^0 be a finite set of indices. Examples of C^0 are: i) C^0 = {man, woman, {man, woman, child}}; ii) C^0 = {1, 2, 11, 12, 21, 22, 122, 212}; iii) C^0 = {1, 11, 111, …, 1…1}, and iv) C^0 = {failure, repair}.

Definition 5.1 Let $\{\alpha_i, 1 \le i \le m\}$ be nonnegative numbers with a unit sum, $\{d_{0,(i,j)}, 1 \le i \ne j \le m\}$, $\{d_{h,(i,j)}, 1 \le i, j \le m, h \in C^0\}$ are nonnegative numbers, and m is a finite positive integer. Assume

$d_{0,(i,i)} \equiv -\left(\sum_{j=1,j\neq i}^{m} d_{0,(i,j)} + \sum_{h\in C^o}\sum_{j=1}^{m} d_{h,(i,j)}\right)$ <0, for $1 \le i \le m$. We define a stochastic process $\{(N_h(t), h \in C^0, I(t)), t\ge 0\}$ as follows.

1) Define independent Poisson processes with parameters $\{d_{0,(i,j)}, 1 \le i \ne j \le m\}$ and $\{d_{h,(i,j)}, 1 \le i, j \le m, h \in C^0\}$. If $d_{0,(i,j)} = 0$ or $d_{h,(i,j)}=0$, the corresponding Poisson process has no event.
2) Determine $I(0)$ by the probability distribution $\{\alpha_i, 1 \le i \le m\}$. Set $N_h(0) = 0$, for $h \in C^0$.
3) If $I(t)=i$, for $1 \le i \le m$, $I(t)$ and $\{N_h(t), h\in C^0\}$ remain the same until the first event occurs in the Poisson processes corresponding to $\{d_{0,(i,j)}, 1 \le j \le m, j \ne i\}$ and $\{d_{h,(i,j)}, 1 \le j \le m, h \in C^0\}$. If the next event comes from the Poisson process corresponding to $d_{0,(i,j)}$, the variable $I(t)$ changes from phase i to phase j and $\{N_h(t), h \in C^0\}$ do not change at the epoch, for $1 \le j \le m, j \ne i$; If the next event comes from the Poisson process corresponding to $d_{h,(i,j)}$, the phase variable $I(t)$ changes from phase i to phase j, $N_h(t)$ is increased by one (or by a pre-specified number, such as the batch size) at the epoch, and $N_l(t)$ remains the same for $l\neq h$ and $l\in C^0$, for $1 \le i, j \le m, h \in C^0$.

Similar to Theorems 4.2 to 4.4, it can be shown that $\{I(t), t\ge 0\}$ and $\{(N_h(t), h \in C^0, I(t)), t\ge 0\}$ are CTMCs. The infinitesimal generator of $\{I(t), t\ge 0\}$ is $D = D_0 + \sum_{h\in C^0} D_h$. The transition rates for $\{(N_h(t), h \in C^0, I(t)), t\ge 0\}$ are D_0 for transitions from $(\mathbf{n}, i)$ to $(\mathbf{n}, j)$, D_h from $(\mathbf{n}, i)$ to $(n_l, l \in C^0, n_h+1, j)$, for $h \in C^0$, where $\mathbf{n} = (n_k, k \in C^0)$. That $\{(N_h(t), h \in C^0, I(t)), t\ge 0\}$ is called an *MMAP* with a matrix representation $(D_0, D_h, h \in C^0)$, where $D_h = (d_{h,(i,j)})$, for $h\in C^0$. The conditional joint distributions of the numbers of arrivals in [0, t] can be obtained from

$$P^*(z_h, h\in C^o, t) = \exp\left\{\left(D_0 + \sum_{h\in C^o} z_h D_h\right)t\right\} .$$

The arrival rate of type h arrivals is $\lambda_h = \boldsymbol{\theta} D_h \mathrm{e}$, where $\boldsymbol{\theta}$ is the stationary distribution of D (Assuming that D is irreducible).

Example 5.1 (Examples 3.1 and 4.1 continued) For the model introduced in Example 3.1, we are interested in the number of repairs and the number of failures in [0, t]. We define the events of Poisson processes associated with repairs as type r arrivals (corresponding to D_r) and the events of Poisson processes associated with failures as type f arrivals (corresponding to D_f). Set C^0 = {failure, repair}. Let $N_r(t)$ be the number of repairs completed in [0, t] and $N_f(t)$ the number of failures occurring in [0, t]. Then it is easy to see that $\{(N_r(t), N_f(t), I(t)), t\ge 0\}$ is an *MMAP* with a matrix representation

$$D_0 = \begin{matrix} & (3,2) & (3,0) & (1,0) \\ (3,2) & -\lambda & 0 & 0 \\ (3,0) & 0 & -\lambda-\mu & 0 \\ (1,0) & 0 & 0 & -\mu \end{matrix},$$

$$D_r = \begin{matrix} & (3,2) & (3,0) & (1,0) \\ (3,2) & 0 & 0 & 0 \\ (3,0) & \mu & 0 & 0 \\ (1,0) & 0 & \mu & 0 \end{matrix}, \quad D_f = \begin{matrix} & (3,2) & (3,0) & (1,0) \\ (3,2) & 0 & \lambda & 0 \\ (3,0) & 0 & 0 & \lambda \\ (1,0) & 0 & 0 & 0 \end{matrix} \qquad (15)$$

If λ = 0.01 and μ = 0.5, the stationary distribution of $\{I(t), t\ge 0\}$ is $\boldsymbol{\theta}$ = (0.9800, 0.0196, 0.0004), the arrival rate of failures is $\boldsymbol{\theta} D_f \mathbf{e}$ = 0.009996, and the arrival rate of repairs is $\boldsymbol{\theta} D_r \mathbf{e}$ = 0.009996.

The model considered in Examples 3.1, 4.1, and 5.1 can be extended to the case in which the two units have different distributions for their times to failure and their repair times. The extended model can be analyzed in a similar way.

Example 5.2 (Examples 3.1, 4.1, and 5.1 continued) For the model introduced in Example 3.1, assume that the times to failure have a common *PH*-distribution with matrix representation (α, T) (see Neuts (1981)) and the repair times have a common *PH*-distribution with matrix representation (β, S). Then an *MMAP* $\{(N_r(t), N_f(t), I(t)), t\geq 0\}$ can be constructed for the numbers of failures and repairs, which has a matrix representation

$$D_0 = \begin{array}{c} \\ (3,2) \\ (3,0) \\ (1,0) \end{array} \begin{array}{c} \begin{array}{ccc} (3,2) & (3,0) & (1,0) \end{array} \\ \begin{pmatrix} T & 0 & 0 \\ 0 & T\otimes I + I\otimes S & 0 \\ 0 & 0 & S \end{pmatrix} \end{array},$$

$$D_r = \begin{array}{c} \\ (3,2) \\ (3,0) \\ (1,0) \end{array} \begin{array}{c} \begin{array}{ccc} (3,2) & (3,0) & (1,0) \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ I\otimes \mathrm{S}^0 & 0 & 0 \\ 0 & \boldsymbol{\alpha}\otimes(\mathrm{S}^0\boldsymbol{\beta}) & 0 \end{pmatrix} \end{array},$$

$$D_f = \begin{array}{c} \\ (3,2) \\ (3,0) \\ (1,0) \end{array} \begin{array}{c} \begin{array}{ccc} (3,2) & (3,0) & (1,0) \end{array} \\ \begin{pmatrix} 0 & (\mathrm{T}^0\boldsymbol{\alpha})\otimes\boldsymbol{\beta} & 0 \\ 0 & 0 & \mathrm{T}^0\otimes I \\ 0 & 0 & 0 \end{pmatrix} \end{array} \tag{16}$$

where "$\otimes$" is for Kronecker product of matrices, $\mathbf{T}^0 = -T\mathbf{e}$ and $\mathbf{S}^0 = -S\mathbf{e}$.

The set of *MMAP*s is versatile. *MAPs* and *BMAPs* are special *MMAPs*. Well-known arrival processes such as Poisson processes, Markov-modulated Poisson processes, and *PH*-renewal processes are also special *MMAP*s. *MMAP*s can be used to model arrival processes with special features. Here are some simple examples.

Example 5.3

i) Cyclic arrival: $D_0 = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$, $D_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, and $D_2 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$. Type 1 and type 2 customers arrive cyclically.

ii) Bursty vs smooth: $D_0 = \begin{pmatrix} -1 & 0 \\ 0 & -100 \end{pmatrix}$, $D_1 = \begin{pmatrix} 0 & 0 \\ 1 & 99 \end{pmatrix}$, $D_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Type 1 process is bursty, while type 2 is smooth.

iii) Individual vs group: $D_0 = \begin{pmatrix} -5 & 0 \\ 0 & -10 \end{pmatrix}$, $D_1 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$, $D_{2,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Every type 2 arrival is accompanied by a type 1 arrival.

iv) Type 2 always follows type 1: $D_0 = \begin{pmatrix} -5 & 0 \\ 0 & -10 \end{pmatrix}$, $D_1 = \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$, $D_2 = \begin{pmatrix} 0 & 0 \\ 10 & 0 \end{pmatrix}$.

v) Orders in individual batches: $D_0 = \begin{pmatrix} -5 & 0 \\ 0 & -10 \end{pmatrix}$, $D_{3,1,1,2,2,1} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$, $D_{1,2,1,3,,1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, if the orders within batches do matter.

We remark that, for all above examples, the interarrival times between the same type of arrivals and different types of arrivals can be more generally distributed by using phase-type distributions and by introducing more phases in the *MMAPs* properly.

6. Conclusion

This paper presented an elementary approach to introduce *MAP*s. The alternative approach is based on Poisson processes, makes it easy to understand parameters of *MAP*s intuitively, and provides more insight into how *MAP*s are generated. As shown by examples, the approach also makes it easy to use *MAP*s in stochastic modeling.

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