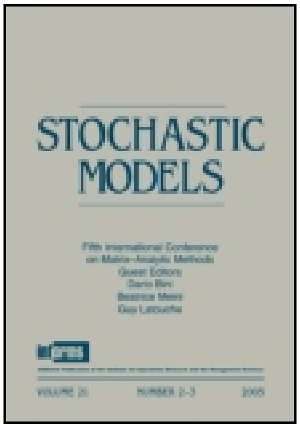

Stochastic Models
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lstm20

On Some Properties of Bivariate Exponential Distributions

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Available online: 02 May 2012

To cite this article: Qi-Ming He, Hanqin Zhang & Juan C. Vera (2012): On Some Properties of Bivariate Exponential Distributions, Stochastic Models, 28:2, 187-206

To link to this article: http://dx.doi.org/10.1080/15326349.2012.672141

Stochastic Models, 28:187–206, 2012

ISSN: 1532-6349 print/1532-4214 online
DOI: 10.1080/15326349.2012.672141

ON SOME PROPERTIES OF BIVARIATE EXPONENTIAL DISTRIBUTIONS

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□ *We show that the two bivariate exponential distributions constructed in Bladt and Nielsen*[5] *have the maximum and minimum correlation coefficients for any given order. We also generalize their constructions to the case where the matrix representations of the two (marginal) exponential distributions have different orders and show that the new constructions also have the maximum and minimum correlation coefficients. Our main tool is a majorization result for a special set of PH-generators.*

Keywords Bivariate exponential distribution; Correlation coefficient; Majorization; Multivariate distribution; Phase-type distribution.

Mathematics Subject Classification Primary 62H05; Secondary 60E10.

1. INTRODUCTION

Bivariate exponential distributions were introduced by Kibble in 1941 and later studied by a number of researchers and practitioners. Bivariate exponential distributions have been extended to multivariate exponential distributions.[3,6,10,17,18,25] Together with other types of multivariate distributions (Weibull, Gamma, phase-type, etc.), multivariate exponential distributions have been widely used in statistics, reliability, and risk analysis.[7–9,11,14,16,26]

Received May 2010; Accepted October 2011
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A number of methods have been introduced for the construction of bivariate exponential distributions. Most of them impose some restriction on the correlation coefficient. For example, Kibble[13] and Marshall and Olkin[18,19] restrict the correlation to be nonnegative. Such a restriction limits the applications of bivariate exponential distributions. In addition, some of the constructions are too complex for applications.

Recently, Bladt and Nielsen[5] introduced a class of bivariate exponential distributions through multivariate phase-type distributions.[2,15,21–23] They give a detailed analysis on two special bivariate exponential distributions and find their correlation coefficients explicitly. They have shown that the correlation coefficients of their constructed bivariate exponential distributions approach the (absolute) maximum 1 or minimum $1-\pi^2/6$, if the orders of their corresponding multivariate phase-type (*MPH*) representations go to infinity. Moreover, their construction of bivariate exponential distributions can give any correlation coefficient between $1-\pi^2/6$ and 1. They leave it open to prove that their constructions are extreme cases that give the maximum and minimum correlation coefficients for any given order of the matrix representations.

In this article, we first show that the bivariate exponential distributions constructed in Bladt and Nielsen[5] indeed give the maximum and minimum correlation coefficients within the class of bivariate exponential distributions they introduced. Then we introduce a broader class of bivariate exponential distributions by generalizing their constructions. We also find the maximum and minimum correlation coefficients corresponding to bivariate exponential distributions in the expanded class. Our results confirm that the constructions introduced in Bladt and Nielsen[5] are extreme cases, a conclusion that can be useful in the application of such multivariate phase-type distributions.

In the proofs of the main results, a majorization result related to some restricted *PH*-generators plays a key role. The majorization result provides a theoretical basis for showing that the constructions by Bladt and Nielsen[5] (Section 3) and the generalization given in this article (Section 4) are truly extreme cases, in the class of bivariate exponential distributions they belong to. A proof of the majorization result is given in this article. The majorization result is of its own interest with applications to phase-type distributions (e.g., He et al.[12]).

The remainder of the article is organized as follows. In Section 2, we briefly introduce the bivariate exponential distributions constructed in Bladt and Nielsen[5]. Section 3 shows that the constructions given in Bladt and Nielsen[5] have the maximum and minimum correlation coefficients. A technical lemma on a majorization result for *PH*-generators is proved in this section. In Section 4, we generalize the results in Section 3 to a broader class of bivariate exponential distributions. We show that the maximum/minimum correlation coefficient is a monotone function of the orders of corresponding matrix representations. Section 5 concludes the article.

2. PRELIMINARIES

The following construction of bivariate exponential distributions is given in Bladt and Nielsen[5]. This construction is based on the general theory on multivariate phase-type distributions.[2,15] Consider a continuous time Markov chain $\{X(t), t \geq 0\}$ with $2m+1$ states and an infinitesimal generator

$$\begin{pmatrix} S & D & 0 \\ 0 & T & -T\mathbf{e} \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

where S, D, and T are $m \times m$ matrices, and $\mathbf{e}$ is the column vector of ones. Matrices S and T are *PH*-generators (invertible matrix with negative diagonal elements, nonnegative all off-diagonal elements, and nonpositive row sums), D is a nonnegative matrix, S and D satisfy $S\mathbf{e} + D\mathbf{e} = 0$, and the state $2m+1$ is an absorption state.

Since both S and T are *PH*-generators, the Markov chain will eventually be absorbed into the state $2m+1$, given any initial state. Let $Y_1^{(m)}$ be the total time that the Markov chain $\{X(t), t \geq 0\}$ spent in states $\{1, 2, \ldots, m\}$ and $Y_2^{(m)}$ the total time in states $\{m+1, m+2, \ldots, 2m\}$, before the absorption into the state $2m+1$. Assume that the Markov chain $\{X(t), t \geq 0\}$ is initially in states $\{1, 2, \ldots, m\}$ with a stochastic vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, i.e., the initial distribution of the Markov chain is $(\boldsymbol{\alpha}, 0, \ldots, 0)$. We call $(\boldsymbol{\alpha}, S, D, T)$ an *MPH*-representation of the multivariate phase-type distribution $(Y_1^{(m)}, Y_2^{(m)})$.

The multidimensional Laplace–Stieltjes transform (LST) of $(Y_1^{(m)}, Y_2^{(m)})$ is given by

$$\begin{aligned} L_{st}(s_1, s_2) &= E[\exp\{-s_1 Y_1^{(m)} - s_2 Y_2^{(m)}\}] \\ &= (\boldsymbol{\alpha}, 0)\left(\begin{pmatrix} s_1 I & 0 \\ 0 & s_2 I \end{pmatrix} - \begin{pmatrix} S & D \\ 0 & T \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 \\ -T\mathbf{e} \end{pmatrix} \\ &= \boldsymbol{\alpha}(s_1 I - S)^{-1} D (s_2 I - T)^{-1}(-T)\mathbf{e}, \\ &\qquad \mathrm{Re}(s_1) > 0,\ \mathrm{Re}(s_2) > 0, \end{aligned} \tag{2.2}$$

where I is the identity matrix. By Eq. (2.2), the covariance of $(Y_1^{(m)}, Y_2^{(m)})$ can be obtained as

$$Cov(Y_1^{(m)}, Y_2^{(m)}) = \boldsymbol{\alpha}(-S)^{-1}(I - \mathbf{e}\boldsymbol{\alpha})(-S)^{-1} D (-T)^{-1}\mathbf{e}. \tag{2.3}$$

To construct a bivariate exponential distribution, the matrices S, T, and D are chosen according to the following assumption.

Assumption A. Assume that S and T are *PH*-generators of order m, D is a nonnegative matrix of order m, $S\mathbf{e} + D\mathbf{e} = 0$, and the matrices S, D, and T satisfy

(i) $S\mathbf{e} = -\lambda\mathbf{e}$;
(ii) $\mathbf{e}'T = -\mu\mathbf{e}'$;
(iii) $-\mathbf{e}'S \geq 0$; and
(iv) $\mathbf{e}'D = \lambda\mathbf{e}'$,

where $\mathbf{e}'$ is the transpose of $\mathbf{e}$, and λ and μ are positive real numbers.

We choose $\boldsymbol{\alpha} = -\mathbf{e}'S/(m\lambda)$ in the rest of the article. By Assumption A(i) and (iii), this selection of $\boldsymbol{\alpha}$ is valid. By Eq. (2.2), it is easy to see that the LSTs of the marginal distributions $Y_1^{(m)}$ and $Y_2^{(m)}$ are given by $\lambda/(s_1 + \lambda)$ and $\mu/(s_2 + \mu)$, respectively. Thus, the marginal distributions of $(Y_1^{(m)}, Y_2^{(m)})$ are exponential with parameters λ and μ, respectively. Consequently, we call $(Y_1^{(m)}, Y_2^{(m)})$ a *bivariate exponential distribution*. Since $\boldsymbol{\alpha}$ is determined by S for the above construction, we use (S, D, T) as an *MPH*-representation for bivariate exponential distribution $(Y_1^{(m)}, Y_2^{(m)})$.

Based on Eq. (2.3), the correlation coefficient of $(Y_1^{(m)}, Y_2^{(m)})$ can be obtained as

$$Corr(Y_1^{(m)}, Y_2^{(m)}) = \frac{1}{m}\mu\mathbf{e}'(-S)^{-1}D(-T)^{-1}\mathbf{e} - 1. \tag{2.4}$$

By Assumption A and Eq. (2.4), it is straightforward to verify that the correlation coefficient is independent of the values of λ and μ, as long as they are positive. Without loss of generality, we assume $\lambda = \mu = 1$ in the rest of the article.

In Bladt and Nielsen,[5] (S, D, T) are chosen as follows:

$$S^* = \begin{pmatrix} -1 & & & & \\ 1 & -2 & & & \\ & \ddots & \ddots & & \\ & & m-2 & -(m-1) & \\ & & & m-1 & -m \end{pmatrix},$$

$$T^* = \begin{pmatrix} -m & & & & \\ m-1 & -(m-1) & & & \\ & \ddots & \ddots & & \\ & & 2 & -2 & \\ & & & 1 & -1 \end{pmatrix}, \tag{2.5}$$

$$D_{\min} = I, \quad D_{\max} = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}.$$

Denote by $Corr_{\min}(m, m)$ and $Corr_{\max}(m, m)$ the correlation coefficients of $(Y_1^{(m)}, Y_2^{(m)})$ with *MPH*-representations $(S^*, D_{\min}, T^*)$ and $(S^*, D_{\max}, T^*)$, respectively. The following results are given in Theorems 4.7 and 4.8 in Bladt and Nielsen[5]

$$\begin{aligned} Corr_{\min}(m, m) &= 1 - \sum_{i=1}^{m} \frac{1}{i^2} \overset{m\to\infty}{\longrightarrow} 1 - \frac{\pi^2}{6}; \\ Corr_{\max}(m, m) &= 1 - \frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \overset{m\to\infty}{\longrightarrow} 1. \end{aligned} \tag{2.6}$$

The correlation coefficients $Corr_{\min}(m, m)$ and $Corr_{\max}(m, m)$ are plotted in Figure 1 for $m = 1, 2, \ldots, 100$. It is clear that $Corr_{\min}(m, m)$ is a negative decreasing function of m and $Corr_{\max}(m, m)$ is a positive increasing function of m. If $D = \mathbf{e}'\mathbf{e}/m$, then the correlation coefficient of (S^*, D, T^*) is zero, which is between $Corr_{\min}(m, m)$ and $Corr_{\max}(m, m)$. In the next two sections, we show that the correlation coefficient corresponding to any (S, D, T) satisfying Assumption A is between $Corr_{\min}(m, m)$ and $Corr_{\max}(m, m)$.

As it is shown in Theorem 4.10 in Bladt and Nielsen,[5] for any $\rho \in (1 - \pi^2/6, 1)$, a bivariate exponential distribution can be constructed

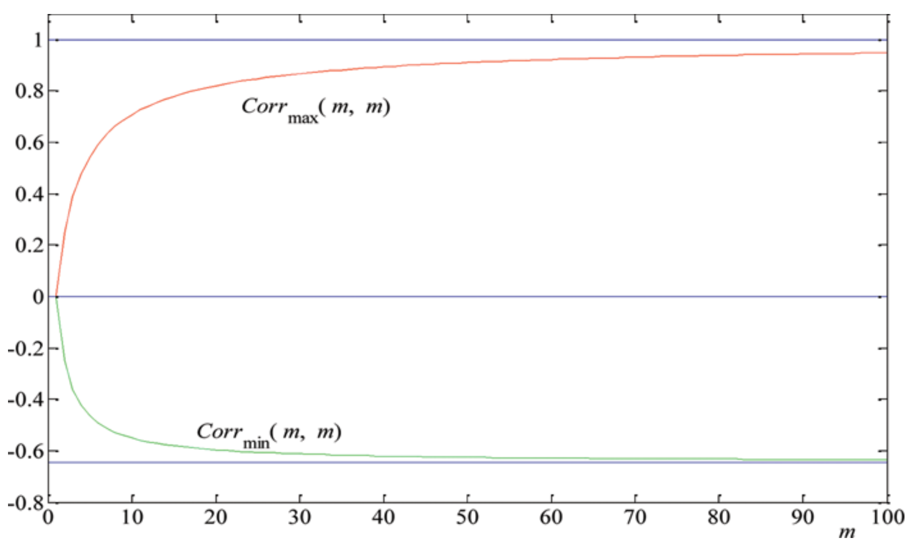

FIGURE 1 $Corr_{\min}(m, m)$ and $Corr_{\max}(m, m)$ for $m = 1, 2, \ldots, 100$ (color figure available online).

to have correlation coefficient ρ. A similar construction can be obtained by defining, for $\rho \in [Corr_{\min}(m, m), Corr_{\max}(m, m)]$,

$$D(\rho) = \left(\frac{Corr_{\max}(m, m) - \rho}{Corr_{\max}(m, m) - Corr_{\min}(m, m)}\right) D_{\min} + \left(\frac{\rho - Corr_{\min}(m, m)}{Corr_{\max}(m, m) - Corr_{\min}(m, m)}\right) D_{\max}. \tag{2.7}$$

Then the correlation coefficient corresponding to $(S^*, D(\rho), T^*)$ is ρ.

3. MAIN RESULTS

The objective of this section is to show that the two selections of (S, T, D) constructed in Bladt and Nielsen[5] give the maximum and minimum of the correlation coefficient, among all (S, D, T)s that satisfy Assumption A.

First, we limit our attention to bivariate exponential distributions $(Y_1^{(m)}, Y_2^{(m)})$ with an *MPH*-representation (S^*, D, T^*).

Theorem 3.1. *For S^* and T^* given in Eq.* (2.5), *and any D satisfying Assumption* A, *we have $Corr_{\min}(m, m) \le Corr(Y_1^{(m)}, Y_2^{(m)}) \le Corr_{\max}(m, m)$.*

Proof. Since $S^*\mathbf{e} + D\mathbf{e} = 0$, by Assumption A(i), we obtain $D\mathbf{e} = \mathbf{e}$. Combining this result with Assumption A(iv), it is clear that D is a doubly stochastic matrix. By Birkhoff's theorem for doubly stochastic matrices,[20] we have

$$D = \sum_{\sigma \in \Omega} c_\sigma P_\sigma, \tag{3.1}$$

where Ω denotes the set of all permutations of the set $\{1, 2, \dots, m\}$, $\{c_\sigma, \sigma \in \Omega\}$ are nonnegative real numbers with unit sum, and P_σ is the permutation matrix associated with permutation σ (i.e., $P_\sigma = (\delta_{(i=\sigma(j))})\ i, j = 1, 2, \dots, m$), where $\delta_{(.)}$ is the indicator function. Let $\mathbf{a}^* = \mathbf{e}'(-S^*)^{-1}$ and $\mathbf{b}^* = (-T^*)^{-1}\mathbf{e}$. Since both S^* and T^* are *PH*-generators, matrices $-S^{*-1}$ and $-T^{*-1}$ are nonnegative, and vectors $\mathbf{a}^*$ and $\mathbf{b}^*$ are nonnegative (see Eqs. (3.3) and (3.4)). The correlation coefficient given in Eq. (2.4) becomes

$$\begin{aligned} Corr(Y_1^{(m)}, Y_2^{(m)}) &= \frac{1}{m}\mathbf{a}^* D \mathbf{b}^* - 1 \\ &= -1 + \frac{1}{m}\sum_{\sigma \in \Omega} c_\sigma \mathbf{a}^* P_\sigma \mathbf{b}^* \begin{cases} \ge -1 + \dfrac{1}{m}\min\limits_{\sigma \in \Omega}\{\mathbf{a}^* P_\sigma \mathbf{b}^*\}; \\ \le -1 + \dfrac{1}{m}\max\limits_{\sigma \in \Omega}\{\mathbf{a}^* P_\sigma \mathbf{b}^*\}. \end{cases} \end{aligned} \tag{3.2}$$

By routine calculations, we obtain

$$(-S^*)^{-1} = \begin{pmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \end{pmatrix},$$

$$(-T^*)^{-1} = \begin{pmatrix} \frac{1}{m} & & & & \\ \frac{1}{m} & \frac{1}{m-1} & & & \\ \frac{1}{m} & \frac{1}{m-1} & \frac{1}{m-2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{m} & \frac{1}{m-1} & \frac{1}{m-2} & \cdots & 1 \end{pmatrix}. \tag{3.3}$$

Consequently, we obtain

$$\mathbf{a}^* = \mathbf{e}'(-S^*)^{-1} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}, \frac{1}{2} + \cdots + \frac{1}{m}, \ldots, \frac{1}{m-1} + \frac{1}{m}, \frac{1}{m}\right);$$
$$\mathbf{b}^* = (-T^*)^{-1}\mathbf{e} = \left(\frac{1}{m}, \frac{1}{m} + \frac{1}{m-1}, \ldots, \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1\right)'. \tag{3.4}$$

Since the elements of $\mathbf{a}^*$ are in descending order, i.e., $a_1^* > a_2^* > \cdots > a_m^*$ and the elements of $\mathbf{b}^*$ are in ascending order, i.e., $b_1^* < b_2^* < \cdots < b_m^*$, it can be shown that (e.g., Lemma 2.1 in Chapter 5 in Minc[20])

$$\mathbf{a}^* D_{\min} \mathbf{b}^* = \sum_{i=1}^{m} a_i^* b_i^* \leq \mathbf{a}^* P_\sigma \mathbf{b}^* = \sum_{i=1}^{m} a_i^* b_{\sigma(i)}^* \leq \sum_{i=1}^{m} a_i^* b_{m-i+1}^* = \mathbf{a}^* D_{\max} \mathbf{b}^*, \tag{3.5}$$

for any permutation $\sigma \in \Omega$. By Eq. (3.2) and Eq. (3.5), we obtain

$$\begin{aligned} Corr(Y_1^{(m)}, Y_2^{(m)}) &\geq -1 + \frac{1}{m} \min_{\sigma\in\Omega}\{\mathbf{a}^* P_\sigma \mathbf{b}^*\} \\ &= -1 + \frac{1}{m}\mathbf{a}^* D_{\min}\mathbf{b}^* = Corr_{\min}(m, m); \\ Corr(Y_1^{(m)}, Y_2^{(m)}) &\leq -1 + \frac{1}{m} \max_{\sigma\in\Omega}\{\mathbf{a}^* P_\sigma \mathbf{b}^*\} \\ &= -1 + \frac{1}{m}\mathbf{a}^* D_{\max}\mathbf{b}^* = Corr_{\max}(m, m). \end{aligned} \tag{3.6}$$

The proof of Theorem 3.1 is completed. □

Next, we consider all (S, D, T)s satisfying Assumption A. To prove the results, we need the concept of majorization for vectors (see Marshall and Olkin[19] and Minc[20]). For vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, rearrange the elements of $\mathbf{x}$ in descending order and denote the elements as $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[m]}$. For example, $x_{[1]} = \max\{x_1, x_2, \ldots, x_m\}$ and $x_{[m]} = \min\{x_1, x_2, \ldots, x_m\}$. A vector $\mathbf{x}$ is majorized by vector $\mathbf{y}$, denoted as $\mathbf{x} \prec \mathbf{y}$, if $\mathbf{xe} = \mathbf{ye}$ and $x_{[1]} + x_{[2]} + \cdots + x_{[k]} \leq y_{[1]} + y_{[2]} + \cdots + y_{[k]}$, for $1 \leq k \leq m-1$.

Lemma 3.2. *Assume that T is an $m \times m$ matrix such that T satisfies Assumption* A *with $\mu = 1$ (i.e., T is a PH-generator satisfying $\mathbf{e}'T = -\mathbf{e}'$). Then $-T^{-1}\mathbf{e}$ is majorized by $\mathbf{b}^*$, where $\mathbf{b}^*$ is given in Eq.* (3.4).

Proof. Denote by $\mathbf{e}(i)$ the column vector with zero everywhere but one in the ith place. Since the matrix $-T$ is an M-matrix, $-T^{-1}$ is nonnegative (see Berman and Plemmons[4] and Minc[20]). Let $\mathbf{b} = -T^{-1}\mathbf{e}$. By permuting the rows and columns of T, it is always possible to obtain $\mathbf{b}$ in ascending order. To prove that $-T^{-1}\mathbf{e}$ is majorized by $\mathbf{b}^*$, it is sufficient to show that $b_1 + b_2 + \cdots + b_k \geq b_1^* + b_2^* + \cdots + b_k^* = k/m + (k-1)/(m-1) + \cdots + 1/(m-k+1)$, for $1 \leq k \leq m$. Note that $-\mathbf{e}'T^{-1}\mathbf{e} = \mathbf{e}'\mathbf{b}^* = m$.

For fixed $k \leq m$, let $\mathbf{z} = (\mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)')(-T^{-1})$. Since $\mathbf{ze} = b_1 + b_2 + \cdots + b_k$, our goal is to prove

$$\mathbf{ze} \geq k/m + (k-1)/(m-1) + \cdots + 1/(m-k+1), \quad \text{for } 1 \leq k \leq m. \tag{3.7}$$

Rearranging the elements of $\mathbf{z}$ in descending order: $z_{[1]} \geq z_{[2]} \geq \cdots \geq z_{[m]}$, where $\{[1], [2], \ldots, [m]\}$ is a permutation of $\{1, 2, \ldots, m\}$. Since $\mathbf{z}(-T) = \mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)'$, we have, for $1 \leq j \leq m$,

$$\begin{aligned} &\mathbf{z}(-T)(\mathbf{e}([j]) + \mathbf{e}([j+1]) + \cdots + \mathbf{e}([m])) \\ &\quad = (\mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)')(\mathbf{e}([j]) + \mathbf{e}([j+1]) + \cdots + \mathbf{e}([m])) \\ &\quad \geq \max\{0,\ k-j+1\}. \end{aligned} \tag{3.8}$$

By definition, we have $-T\mathbf{e} \geq 0$ and $-T_{i,j} \leq 0$, for $1 \leq i, j \leq m$, $i \neq j$. Then, for any $\{i_1, i_2, \ldots, i_n\} \subset \{1, 2, \ldots, m\}$ and $i_0 \in \{1, 2, \ldots, m\}$, we have

$$\mathbf{e}(i_0)'(-T)(\mathbf{e}(i_1) + \mathbf{e}(i_2) + \cdots + \mathbf{e}(i_n)) = -\sum_{j=1}^{n} T_{(i_0, i_j)}, \tag{3.9}$$

which is nonnegative if $i_0 \in \{i_1, i_2, \ldots, i_n\}$ and nonpositive if $i_0 \notin \{i_1, i_2, \ldots, i_n\}$. Thus, for $1 \leq i, j \leq m$, we have

$$(z_{[i]} - z_{[j]})\mathbf{e}([i])'(-T)\left(\sum_{l=j}^{m} \mathbf{e}([l])\right) \leq 0. \tag{3.10}$$

Eq. (3.10) can be proved by considering two cases: $i \leq j$ and $i > j$. For the first case, we have $z_{[i]} - z_{[j]} \geq 0$ and $\mathbf{e}([i])'(-T)(\mathbf{e}([j]) + \mathbf{e}([j+1]) + \cdots + \mathbf{e}([m])) \leq 0$. For the second case, we have $z_{[i]} - z_{[j]} \leq 0$ and $\mathbf{e}([i])'(-T)(\mathbf{e}([j]) + \mathbf{e}([j+1]) + \cdots + \mathbf{e}([m])) \geq 0$.

For $1 \leq i, j \leq m$, Eq. (3.10) leads to,

$$z_{[i]}\mathbf{e}([i])'(-T)\left(\sum_{l=j}^{m} \mathbf{e}([l])\right) \leq z_{[j]}\mathbf{e}([i])'(-T)\left(\sum_{l=j}^{m} \mathbf{e}([l])\right). \tag{3.11}$$

Summing up over $i = 1, 2, \ldots, m$, in Eq. (3.11), yields

$$\begin{aligned}
\mathbf{z}(-T)\left(\sum_{l=j}^{m} \mathbf{e}([l])\right) &= \sum_{i=1}^{m} z_{[i]}\mathbf{e}([i])'(-T)\left(\sum_{l=j}^{m} \mathbf{e}([l])\right) \\
&\leq \sum_{i=1}^{m} z_{[j]}\mathbf{e}([i])'(-T)\left(\sum_{l=j}^{m} \mathbf{e}([l])\right) \\
&= z_{[j]}\mathbf{e}'(-T)\left(\sum_{l=j}^{m} \mathbf{e}([l])\right) \\
&= z_{[j]}\mathbf{e}'\left(\sum_{l=j}^{m} \mathbf{e}([l])\right) = z_{[j]}(m - j + 1). \qquad (3.12)
\end{aligned}$$

Combining Eqs. (3.8) and (3.12), we have $z_{[j]} \geq \max\{0, k - j + 1\}/(m - j + 1)$. Adding over $j = 1, 2, \ldots, m$, Eq. (3.7) follows. This completes the proof of Lemma 3.2. □

Remark 3.1. In addition to its applications in this section and in Section 4, Lemma 3.2 can be applied in the study of phase-type distributions and M-matrices. For example, consider a PH-generator T satisfying $\mathbf{e}'T = -\mathbf{e}'$. For any probability vector $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$, define $\underline{\boldsymbol{\beta}} = (\beta_{[1]}, \beta_{[2]}, \ldots, \beta_{[m]})$ the descending rearrangement of $\boldsymbol{\beta}$, and $\bar{\boldsymbol{\beta}} = (\beta_{[m]}, \beta_{[m-1]}, \ldots, \beta_{[1]})$ the ascending rearrangement of $\boldsymbol{\beta}$. Since $-T^{-1}\mathbf{e}$ is majorized by $\mathbf{b}^* = -T^{*-1}\mathbf{e}$ given in Eq. (3.4), it is easy to obtain lower and upper bounds on the expectation of the *PH*-distribution $(\boldsymbol{\beta}, T)$:

$$\frac{1}{m} \leq -\underline{\boldsymbol{\beta}}T^{*-1}\mathbf{e} \leq -\boldsymbol{\beta}T^{-1}\mathbf{e} \leq -\bar{\boldsymbol{\beta}}T^{*-1}\mathbf{e} \leq \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1. \quad (3.13)$$

The above results are generalized to all *PH*-generators and phase-type distributions in He et al.[12].

With Lemma 3.2, we are able to extend Theorem 3.1 to all (S, D, T)s that satisfy Assumption A.

Theorem 3.3. *Assume that $(Y_1^{(m)}, Y_2^{(m)})$ is a bivariate exponential distribution with an MPH-representation (S, D, T) satisfying Assumption A. Then we have $Corr_{\min}(m, m) \leq Corr(Y_1^{(m)}, Y_2^{(m)}) \leq Corr_{\max}(m, m)$.*

Proof. For any (S, D, T) satisfying Assumption A, let $\mathbf{a} = \mathbf{e}'(-S)^{-1}$ and $\mathbf{b} = (-T)^{-1}\mathbf{e}$. Since $S\mathbf{e} + D\mathbf{e} = 0$, D is a doubly stochastic matrix. Similar to Eq. (3.2), we obtain

$$\begin{aligned} Corr(Y_1^{(m)}, Y_2^{(m)}) &= \frac{1}{m}\mathbf{a}D\mathbf{b} - 1 \\ &= -1 + \frac{1}{m}\sum_{\sigma\in\Omega} c_\sigma \mathbf{a}P_\sigma\mathbf{b} \begin{cases} \geq -1 + \dfrac{1}{m}\min\limits_{\sigma\in\Omega}\{\mathbf{a}P_\sigma\mathbf{b}\}; \\ \leq -1 + \dfrac{1}{m}\max\limits_{\sigma\in\Omega}\{\mathbf{a}P_\sigma\mathbf{b}\}. \end{cases} \end{aligned} \quad (3.14)$$

By Lemma 3.2, $\mathbf{a} = \mathbf{e}'(-S)^{-1}$ is majorized by $\mathbf{a}^*$ defined in Eq. (3.4) and $\mathbf{b} = (-T)^{-1}\mathbf{e}$ is majorized by $\mathbf{b}^*$ defined in Eq. (3.4). By the Hardy–Littlewood–Polya Theorem,[20] there exists a doubly stochastic matrix P_s such that $\mathbf{a} = \mathbf{a}^*P_s$. Similarly, there exists a doubly stochastic matrix P_t such that $\mathbf{b} = P_t\mathbf{b}^*$. Then we have $\mathbf{a}P_\sigma\mathbf{b} = \mathbf{a}^*P_sP_\sigma P_t\mathbf{b}^*$. Since $P_sP_\sigma P_t$ is also a doubly stochastic matrix, we obtain

$$\begin{aligned} &\min_{\sigma\in\Omega}\{\mathbf{a}^*P_\sigma\mathbf{b}^*\} \leq \min_{\sigma\in\Omega}\{\mathbf{a}^*P_sP_\sigma P_t\mathbf{b}^*\} = \min_{\sigma\in\Omega}\{\mathbf{a}P_\sigma\mathbf{b}\}; \\ &\max_{\sigma\in\Omega}\{\mathbf{a}P_\sigma\mathbf{b}\} = \max_{\sigma\in\Omega}\{\mathbf{a}^*P_sP_\sigma P_t\mathbf{b}^*\} \leq \max_{\sigma\in\Omega}\{\mathbf{a}^*P_\sigma\mathbf{b}^*\}, \end{aligned} \quad (3.15)$$

which leads to (see Eq. (3.5))

$$\begin{aligned} \mathbf{a}^*D_{\min}\mathbf{b}^* &= \min_{\sigma\in\Omega}\{\mathbf{a}^*P_\sigma\mathbf{b}^*\} \leq \min_{\sigma\in\Omega}\{\mathbf{a}P_\sigma\mathbf{b}\} \leq \max_{\sigma\in\Omega}\{\mathbf{a}P_\sigma\mathbf{b}\} \\ &\leq \max_{\sigma\in\Omega}\{\mathbf{a}^*P_\sigma\mathbf{b}^*\} = \mathbf{a}^*D_{\max}\mathbf{b}^*. \end{aligned} \quad (3.16)$$

Combining Eqs. (3.14) and (3.16), the expected result is obtained. This completes the proof of Theorem 3.3. □

4. AN EXTENSION

In Sections 2 and 3, the orders of S and T are assumed to be the same. That condition can be relaxed. In this section, we find the maximum and minimum of the correlation coefficients for cases where S and T may have different orders.

Assumption B. Assume that S is a PH-generator of order m_1, T is a PH-generator of order m_2, D is an $m_1 \times m_2$ nonnegative matrix, $S\mathbf{e} + D\mathbf{e} = 0$, and the matrices S, D, and T satisfy

(i) $S\mathbf{e} = -\mathbf{e}$;
(ii) $\mathbf{e}'T = -\mathbf{e}'$;
(iii) $-\mathbf{e}' \geq 0$; and
(iv) $\mathbf{e}'D = m_1\mathbf{e}'/m_2$.

It is readily seen that D has unit row sums. Choose $\boldsymbol{\alpha} = -\mathbf{e}'S/m_1$. Similar to the construction given in Section 2, the triplet (S, D, T) define a bivariate exponential distribution $(Y_1^{(m_1)}, Y_2^{(m_2)})$ with correlation coefficient

$$Corr(Y_1^{(m_1)}, Y_2^{(m_2)}) = \frac{1}{m_1}\mathbf{e}'(-S)^{-1}D(-T)^{-1}\mathbf{e} - 1. \tag{4.1}$$

First, we construct two special cases $(S^*, D_{\min}^{(m_1,m_2)}, T^*)$ and $(S^*, D_{\max}^{(m_1,m_2)}, T^*)$, where S^* and T^* are given in Eq. (2.5) with orders m_1 and m_2, respectively, and the matrices $D_{\min}^{(m_1,m_2)}$ and $D_{\max}^{(m_1,m_2)}$ are constructed next. Then we prove that the two bivariate exponential distributions give the minimum and maximum correlation coefficient.

If $m_1 \geq m_2$, $D_{\min}^{(m_1,m_2)}$ and $D_{\max}^{(m_1,m_2)}$ are defined as follows:

$$D_{\min}^{(m_1,m_2)} = \begin{pmatrix} k_1\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. & & & \\ r_1 & 1-r_1 & & \\ & k_2\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. & & \\ & r_2 & \ddots & 1-r_{m_2-1} \\ & & & k_{m_2}\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. \end{pmatrix} \quad \text{and}$$

$$
D_{\max}^{(m_1,m_2)} = \begin{pmatrix}
 & & & k_1\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. \\
 & & 1-r_1 & r_1 \\
 & & k_1\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. & \\
 & & r_2 & \\
 & \vdots & & \\
1-r_{m_2-1} & & & \\
k_{m_2}\left\{\begin{matrix}1\\ \vdots\\ 1\end{matrix}\right. & & &
\end{pmatrix}, \tag{4.2}
$$

where nonnegative integers $\{k_j, 1 \le j \le m_2\}$ and real numbers $\{r_j, 1 \le j \le m_2 - 1\}$ satisfy: $0 \le r_j < 1$, for $1 \le j \le m_2 - 1$, and

$$
\begin{aligned}
r_1 + k_1 &= \frac{m_1}{m_2}; \\
1 - r_{j-1} + k_j + r_j &= \frac{m_1}{m_2}, \quad 2 \le j \le m_2 - 1; \\
1 - r_{m_2-1} + k_{m_2} &= \frac{m_1}{m_2}.
\end{aligned} \tag{4.3}
$$

It is easy to see that both matrices $D_{\min}^{(m_1,m_2)}$ and $D_{\max}^{(m_1,m_2)}$ satisfy (iv) of Assumption B and have unit row sums. The definitions are consistent with the definitions of $D_{\min}$ and $D_{\max}$ given in Section 2 in the sense that $D_{\min} = D_{\min}^{(m,m)}$ and $D_{\max} = D_{\max}^{(m,m)}$.

The case with $m_1 < m_2$ is not considered explicitly, since the correlation of coefficient function defined in Eq. (4.1) is symmetric in m_1 and m_2.

Example 4.1. For $m_1 = 5$ and $m_2 = 3$, we have $k_1 = 1$, $r_1 = 2/3$, $k_2 = 1$, $r_2 = 1/3$, $k_3 = 1$,

$$
D_{\min}^{(m_1,m_2)} = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 2/3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D_{\max}^{(m_1,m_2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1/3 & 2/3 \\ 0 & 1 & 0 \\ 2/3 & 1/3 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.4}
$$

For the above construction, we have, for $1 \le i \le m_1$ and $1 \le j \le m_2$,

$$(\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D_{\min}^{(m_1,m_2)} (\mathbf{e}(1) + \cdots + \mathbf{e}(j)) = \min \left\{ i, j\frac{m_1}{m_2} \right\};$$
$$(\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D_{\max}^{(m_1,m_2)} (\mathbf{e}(j) + \cdots + \mathbf{e}(m_2)) = \min \left\{ i, (m_2 - j + 1)\frac{m_1}{m_2} \right\}. \tag{4.5}$$

These observations and the following lemma are used in the proofs of the main results.

Lemma 4.1. *Assume that D is a nonnegative matrix with unit row sums and satisfies* (iv) *of Assumption* B. *For nonnegative row vector* $\mathbf{a}$ *whose elements are in descending order and nonnegative column vector* $\mathbf{b}$ *whose elements are in ascending order, we have* $\mathbf{a}D_{\min}^{(m_1,m_2)}\mathbf{b} \le \mathbf{a}D\mathbf{b} \le \mathbf{a}D_{\max}^{(m_1,m_2)}\mathbf{b}$.

Proof. For the matrix D, we have, for $1 \le i \le m_1$ and $1 \le j \le m_2$,

$$(\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D (\mathbf{e}(1) + \cdots + \mathbf{e}(j)) \le (\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D\mathbf{e} = i;$$
$$(\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D (\mathbf{e}(j) + \cdots + \mathbf{e}(m_2)) \le \mathbf{e}' D (\mathbf{e}(1) + \cdots + \mathbf{e}(j)) = j\frac{m_1}{m_2}. \tag{4.6}$$

Combining Eqs. (4.5) and (4.6), we obtain, for $1 \le i \le m_1$ and $1 \le j \le m_2$,

$$(\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D_{\min}^{(m_1,m_2)} (\mathbf{e}(1) + \cdots + \mathbf{e}(j)) \ge (\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D (\mathbf{e}(1) + \cdots + \mathbf{e}(j)), \tag{4.7}$$

which leads to

$$\begin{aligned}
&(\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D_{\min}^{(m_1,m_2)} (\mathbf{e}(j) + \cdots + \mathbf{e}(m_2)) \\
&\quad = (\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D_{\min}^{(m_1,m_2)} [\mathbf{e} - (\mathbf{e}(1) + \cdots + \mathbf{e}(j-1))] \\
&\quad \le (\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D [\mathbf{e} - (\mathbf{e}(1) + \cdots + \mathbf{e}(j-1))] \\
&\quad = (\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D (\mathbf{e}(j) + \cdots + \mathbf{e}(m_2)).
\end{aligned} \tag{4.8}$$

Since elements in $\mathbf{a}$ are in descending order, the vector $\mathbf{a}$ can be written as $\mathbf{a} = \sum_{i=1}^{m_1} \lambda_i (\mathbf{e}(1) + \cdots + \mathbf{e}(i))'$, where $\{\lambda_{i,} 1 \le i \le m_1\}$ are nonnegative. Similarly, the vector $\mathbf{b}$ can be written as $\mathbf{b} = \sum_{j=1}^{m_2} \mu_j (\mathbf{e}(j) + \cdots + \mathbf{e}(m_2))$, where $\{\mu_{i,} 1 \le i \le m_1\}$ are nonnegative. Then we have

$$\mathbf{a}D_{\min}^{(m_1,m_2)}\mathbf{b} = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \lambda_i \mu_j (\mathbf{e}(1) + \cdots + \mathbf{e}(i))' D_{\min}^{(m_1,m_2)} (\mathbf{e}(j) + \cdots + \mathbf{e}(m_2))$$

$$\leq \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\lambda_i\mu_j(\mathbf{e}(1)+\cdots+\mathbf{e}(i))'D(\mathbf{e}(j)+\cdots+\mathbf{e}(m_2))$$
$$= \mathbf{a}D\mathbf{b}, \tag{4.9}$$

which proves the first part of the lemma.

To prove $\mathbf{a}D\mathbf{b} \leq \mathbf{a}D_{\max}^{(m_1,m_2)}\mathbf{b}$, we note that, for $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$,

$$\begin{aligned}&(\mathbf{e}(1)+\cdots+\mathbf{e}(i))'D_{\max}^{(m_1,m_2)}(\mathbf{e}(j)+\cdots+\mathbf{e}(m_2))\\ &\quad= \min\{i, \quad (m_2-j+1)m_1/m_2\}\\ &\quad\geq (\mathbf{e}(1)+\cdots+\mathbf{e}(i))'D(\mathbf{e}(j)+\cdots+\mathbf{e}(m_2)).\end{aligned} \tag{4.10}$$

The rest of the proof is similar. This completes the proof of Lemma 4.1. □

Lemma 4.1 generalizes results in Eq. (3.5).

For S^* of order m_1 and T^* of order m_2 (see Eq. (2.5) for definition), the correlation coefficients of *MPH*-representations $(S^*, D_{\min}^{(m_1,m_2)}, T^*)$ and $(S^*, D_{\max}^{(m_1,m_2)}, T^*)$ are given, respectively, as follows:

$$\begin{aligned}Corr_{\min}(m_1,m_2) &= \frac{1}{m_1}\mathbf{e}'(-S^*)^{-1}D_{\min}^{(m_1,m_2)}(-T^*)^{-1}\mathbf{e}-1\\ &= \frac{1}{m_1}\mathbf{a}^*D_{\min}^{(m_1,m_2)}\mathbf{b}^*-1;\\ Corr_{\max}(m_1,m_2) &= \frac{1}{m_1}\mathbf{e}'(-S^*)^{-1}D_{\max}^{(m_1,m_2)}(-T^*)^{-1}\mathbf{e}-1\\ &= \frac{1}{m_1}\mathbf{a}^*D_{\max}^{(m_1,m_2)}\mathbf{b}^*-1.\end{aligned} \tag{4.11}$$

Similar to Section 3, we first limit our attention to bivariate exponential distributions with an *MPH*-representation (S^*, D, T^*).

Theorem 4.2. *For S^* of order m_1 and T^* of order m_2 defined in Eq.* (2.5)*, and any D satisfying Assumption* B*, we have $Corr_{\min}(m_1,m_2) \leq Corr(Y_1^{(m_1)}, Y_2^{(m_2)}) \leq Corr_{\max}(m_1,m_2)$, for $(Y_1^{(m_1)}, Y_2^{(m_2)})$ with MPH-representation (S^*, D, T^*).*

Proof. By Eq. (4.1), it is easy to obtain $Corr(Y_1^{(m_1)}, Y_2^{(m_2)}) = m_1^{-1}\mathbf{a}^*D\mathbf{b}^*-1$. Since the elements in $\mathbf{a}^*$ are in descending order and elements in $\mathbf{b}^*$ are in ascending order, the theorem is proved by applying Lemma 4.1. □

For the general case, similar to Theorem 3.3, we have the following result.

Theorem 4.3. *Assume that* $(Y_1^{(m_1)}, Y_2^{(m_2)})$ *is a bivariate exponential distribution with MPH-representation* (S, D, T) *satisfying Assumption* B. *Then* $Corr_{\min}(m_1, m_2) \le Corr(Y_1^{(m_1)}, Y_2^{(m_2)}) \le Corr_{\max}(m_1, m_2)$, *where* $Corr_{min}(m_1, m_2)$ *and* $Corr_{max}(m_1, m_2)$ *are given in Eq.* (4.11).

Proof. By Lemma 3.2, $\mathbf{a} = \mathbf{e}'(-S)^{-1}$ is majorized by $\mathbf{a}^*$ and $\mathbf{b} = (-T)^{-1}\mathbf{e}$ is majorized by $\mathbf{b}^*$. Similar to the proof of Theorem 3.3, there exist a doubly stochastic matrix P_s such that $\mathbf{a} = \mathbf{a}^* P_s$, and a doubly stochastic matrix P_t such that $\mathbf{b} = P_t \mathbf{b}^*$. Then we have $\mathbf{a}D\mathbf{b} = \mathbf{a}^* P_s D P_t \mathbf{b}^*$. Let $\widehat{D} = P_s D P_t$. It is easy to verify that $\widehat{D}\mathbf{e} = \mathbf{e}$ and $\mathbf{e}'\widehat{D} = m_1\mathbf{e}'/m_2$. The results are obtained by applying Theorem 4.2 to $(S^*, \widehat{D}, T^*)$. This completes the proof Theorem 4.3. □

The following result show that the maximum and minimum of correlation coefficients $Corr_{\min}(m_1, m_2)$ and $Corr_{\max}(m_1, m_2)$ are increasing and decreasing in m_1 and m_2, respectively, which generalizes the expressions in Eq. (2.6).

Theorem 4.4. *For positive integers* m_1 *and* m_2, *we have*

$$\begin{aligned} Corr_{\min}(m_1, m_2) &= \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} \max\left\{0, \ \frac{1}{jm_1} + \frac{1}{im_2} - \frac{1}{ij}\right\} - 1; \\ Corr_{\max}(m_1, m_2) &= \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} \min\left\{\frac{1}{jm_1}, \ \frac{1}{im_2}\right\} - 1. \end{aligned} \tag{4.12}$$

Both functions $Corr_{\min}(m_1, m_2)$ *and* $Corr_{\max}(m_1, m_2)$ *are symmetric. The function* $Corr_{\min}(m_1, m_2)$ *is non-increasing in* m_1 *and* m_2, *and the function* $Corr_{\max}(m_1, m_2)$ *is non-decreasing in* m_1 *and* m_2. *In addition, we have* $Corr_{\min}(m, km) = Corr_{\min}(m, km+1)$ *and* $Corr_{\max}(m, km) = Corr_{\max}(m, km+1)$ *for all positive* m *and* k.

Proof. We prove the result for $Corr_{\min}(m_1, m_2)$, as the proof for $Corr_{\max}(m_1, m_2)$ follows in a similar way. First note that

$$\begin{aligned} \mathbf{b}^* &= \sum_{j=1}^{m_2} \frac{1}{j}(\mathbf{e}(m_2 - j + 1) + \cdots + \mathbf{e}(m_2)) \\ &= \sum_{j=1}^{m_2} \frac{1}{j}[\mathbf{e} - (\mathbf{e}(1) + \cdots + \mathbf{e}(m_2 - j))]. \end{aligned} \tag{4.13}$$

Using the first equation in (4.5), we have, for $1 \leq i \leq m_1$,

$$\begin{aligned} F_i^{(m_1,m_2)} &\equiv (\mathbf{e}(1)+\cdots+\mathbf{e}(i))' D_{\min}^{(m_1,m_2)} \mathbf{b}^* \\ &= \sum_{j=1}^{m_2} \frac{1}{j} (\mathbf{e}(1)+\cdots+\mathbf{e}(i))' D_{\min}^{(m_1,m_2)} [\mathbf{e} - (\mathbf{e}(1)+\cdots+\mathbf{e}(m_2-j))] \\ &= \sum_{j=1}^{m_2} \frac{1}{j} (\mathbf{e}(1)+\cdots+\mathbf{e}(i))' \left[\mathbf{e} - D_{\min}^{(m_1,m_2)} (\mathbf{e}(1)+\cdots+\mathbf{e}(m_2-j))\right] \\ &= \sum_{j=1}^{m_2} \frac{1}{j} \left(i - \min\left\{ i, (m_2-j)\frac{m_1}{m_2} \right\} \right) \\ &= \sum_{j=1}^{m_2} \max\left\{ 0, \frac{m_1}{m_2} + \frac{i}{j} - \frac{m_1}{j} \right\}. \end{aligned} \tag{4.14}$$

Using $\mathbf{a}^* = \sum_{i=1}^{m_1} \frac{1}{i} (\mathbf{e}(1)+\cdots+\mathbf{e}(i))'$, we obtain

$$\begin{aligned} Corr_{\min}(m_1, m_2) &= \frac{1}{m_1} \mathbf{a}^* D_{\min}^{(m_1,m_2)} \mathbf{b}^* - 1 = \frac{1}{m_1} \sum_{i=1}^{m_1} \frac{1}{i} F_i^{(m_1,m_2)} - 1 \\ &= \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \max\left\{ 0, \frac{1}{jm_1} + \frac{1}{im_2} - \frac{1}{ij} \right\} - 1. \end{aligned} \tag{4.15}$$

To show the second part, we focus on the monotonicity of the function $Corr_{\min}(m_1, m_2)$. Since the function $Corr_{\min}(m_1, m_2)$ is symmetric, it is sufficient to show $Corr_{\min}(m_1, m_2+1) \leq Corr_{\min}(m_1, m_2)$. To that end, by Eq. (4.15), it is sufficient to show $F_i^{(m_1,m_2+1)} \leq F_i^{(m_1,m_2)}$ for $1 \leq i \leq m_1$. For $j \geq 1$, fixing i, we have

$$\max\left\{ 0, \frac{m_1}{m_2} - \frac{m_1 - i}{j} \right\} = \begin{cases} 0, & \text{if } j \leq (1 - i/m_1) m_2; \\ \dfrac{m_1}{m_2} - \dfrac{m_1 - i}{j}, & \text{if } j \geq (1 - i/m_1) m_2. \end{cases} \tag{4.16}$$

If there exists an integer j^* such that $(1 - i/m_1)m_2 \leq j^* < (1 - i/m_1)(m_2 + 1)$, we have

$$\max\left\{ 0, \frac{m_1}{m_2} - \frac{m_1 - i}{j} \right\} - \max\left\{ 0, \frac{m_1}{m_2+1} - \frac{m_1 - i}{j} \right\}$$

$$= \begin{cases} 0, & \text{if } 1 \le j < (1 - i/m_1)m_2; \\ \dfrac{m_1}{m_2} - \dfrac{m_1 - i}{j^*}, & \text{if } j = j^*; \\ \dfrac{m_1}{m_2(m_2+1)}, & \text{if } (1 - i/m_1)(m_2+1) \le j \le m_2 + 1. \end{cases} \tag{4.17}$$

Then we obtain, for $1 \le i \le m_1$,

$$\begin{aligned} &F_i^{(m_1,m_2)} - F_i^{(m_1,m_2+1)} \\ &= \sum_{j=1}^{m_2} \left(\max\left\{0, \frac{m_1}{m_2} - \frac{m_1 - i}{j}\right\} - \max\left\{0, \frac{m_1}{m_2+1} - \frac{m_1 - i}{j}\right\}\right) - \frac{i}{m_2+1} \\ &= \frac{m_1}{m_2} - \frac{m_1 - i}{j^*} + (m_2 - j^*)\frac{m_1}{m_2(m_2+1)} - \frac{i}{m_2+1} \\ &= \left(j^* - m_2\left(1 - \frac{i}{m_1}\right)\right)\frac{m_1}{m_2}\left(\frac{1}{j^*} - \frac{1}{m_2+1}\right) \ge 0. \end{aligned} \tag{4.18}$$

If there is no integer in the interval $[(1 - i/m_1)m_2, (1 - i/m_1)(m_2 + 1))$, we choose j^* to be the smallest integer that is greater than or equal to $(1 - i/m_1)(m_2 + 1)$. That implies $(1 - i/m_1)m_2 < j^* < 1 + (1 - i/m_1)m_2$. Then the calculations in Eq. (4.17) become, for $1 \le i \le m_1$,

$$\begin{aligned} &\max\left\{0, \frac{m_1}{m_2} - \frac{m_1 - i}{j}\right\} - \max\left\{0, \frac{m_1}{m_2+1} - \frac{m_1 - i}{j}\right\} \\ &\quad = \begin{cases} 0, & \text{if } 1 \le j < (1 - i/m_1)m_2; \\ \dfrac{m_1}{m_2(m_2+1)}, & \text{if } (1 - i/m_1)(m_2+1) \le j \le m_2 + 1. \end{cases} \end{aligned} \tag{4.19}$$

The calculations in Eq. (4.18) become

$$\begin{aligned} F_i^{(m_1,m_2)} - F_i^{(m_1,m_2+1)} &= (m_2 - j^* + 1)\frac{m_1}{m_2(m_2+1)} - \frac{i}{m_2+1} \\ &= \left(m_2\left(1 - \frac{i}{m_1}\right) + 1 - j^*\right)\frac{m_1}{m_2(m_2+1)} \ge 0. \end{aligned} \tag{4.20}$$

Thus, the function $Corr_{\min}(m_1, m_2)$ is non-increasing in m_1 and m_2.

To prove the last part of the theorem, note that if $m_2 = km_1$, then $j^* = k(m_1 - i)$, which leads to $F_i^{(m_1,m_2)} - F_i^{(m_1,m_2+1)} = 0$ in Eq. (4.18), for $1 \le i \le m_1$. Consequently, $Corr_{\min}(m, km) = Corr_{\min}(m, km + 1)$ holds for all positive m and k. This completes the proof of Theorem 4.4. □

Example 4.2. Some values of $Corr_{\min}(m_1, m_2)$ are given in Table 1.

TABLE 1 The minimum correlation coefficient $Corr_{\min}(m_1, m_2)$

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	−0.2500	−0.2500	−0.2917	−0.2917	−0.3083	−0.3083	−0.3173
3	0	−0.2500	−0.3611	−0.3611	−0.3806	−0.4028	−0.4028	−0.4093
4	0	−0.2917	−0.3611	−0.4236	−0.4236	−0.4389	−0.4476	−0.4610
5	0	−0.2917	−0.3806	−0.4236	−0.4636	−0.4636	−0.4735	−0.4814
6	0	−0.3083	−0.4028	−0.4389	−0.4636	−0.4914	−0.4914	−0.4992
7	0	−0.3083	−0.4028	−0.4476	−0.4735	−0.4914	−0.5118	−0.5118
8	0	−0.3173	−0.4093	−0.4610	−0.4814	−0.4992	−0.5118	−0.5274

Some values of $Corr_{\max}(m_1, m_2)$ are given in Table 2.

Tables 1 and 2 indicate that, for fixed m, the function $Corr_{\min}(k, m-k)$ is minimized at $k = \lfloor m/2 \rfloor$ (i.e., the greatest integer that is less than or equal to $m/2$) or $k = \lceil m/2 \rceil$ (i.e., the smallest integer that is greater than or equal to $m/2$), and the function $Corr_{\max}(k, m-k)$ is maximized at $k = \lfloor m/2 \rfloor$ or $k = \lceil m/2 \rceil$. Alternatively, Tables 1 and 2 suggests $[Corr_{\min}(k, m-k),\ Corr_{\max}(k, m-k)] \subset [Corr_{\min}(\lfloor m/2 \rfloor, \lfloor m/2 \rfloor),\ Corr_{\max}(\lfloor m/2 \rfloor, \lfloor m/2 \rfloor)]$ for $1 \le k \le \lfloor m/2 \rfloor$. In general, if $m_1 + m_2 = 2m$, then $[Corr_{\min}(m_1, m_2), Corr_{\max}(m_1, m_2)] \subset [Corr_{\min}(m, m), Corr_{\max}(m, m)]$ may hold. If $m_1 + m_2 = 2m + 1$, by Theorem 4.4, then $[Corr_{\min}(m_1, m_2), Corr_{\max}(m_1, m_2)] \subset [Corr_{\min}(m + 1, m), Corr_{\max}(m + 1, m)] = [Corr_{\min}(m, m), Corr_{\max}(m, m)]$ may hold. Consequently, Tables 1 and 2, together with Theorem 4.4, suggest that the smallest orders, for which a bivariate exponential distribution has correlation coefficient $\rho \in (1 - \pi^2/6, 1)$, can be achieved at the smallest m such that $\rho \in [Corr_{\min}(m, m), Corr_{\max}(m, m)]$.

TABLE 2 The maximum correlation coefficient $Corr_{\max}(m_1, m_2)$

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0.2500	0.2500	0.2917	0.2917	0.3083	0.3083	0.3173
3	0	0.2500	0.3889	0.3889	0.4111	0.4389	0.4389	0.4466
4	0	0.2917	0.3889	0.4792	0.4792	0.4972	0.5107	0.5301
5	0	0.2917	0.4111	0.4792	0.5433	0.5433	0.5557	0.5676
6	0	0.3083	0.4389	0.4972	0.5433	0.5917	0.5917	0.6017
7	0	0.3083	0.4389	0.5107	0.5557	0.5917	0.6296	0.6296
8	0	0.3173	0.4466	0.5301	0.5676	0.6017	0.6296	0.6603

5. DISCUSSION

An interesting issue for further research is to find the maximum and minimum correlation coefficient(s) for multivariate *PH*-distribution with a given order of matrix representation. Based on Aldous and Shepp[1] and O'Cinneide[24] for univariate *PH*-distributions, the maximum and minimum of the correlation coefficients of bivariate *PH*-distributions with any given order of representation cannot reach 1 or −1. Thus, finding the maximum and minimum can be useful for approximating bivariate distributions with bivariate exponential distributions.

ACKNOWLEDGMENT

We thank a referee and an editor for their valuable comments and suggestions.

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