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Abstract	This paper presents majorization results for *PH*-generators. Based on the majorization results, bounds on the moments and Laplace-Stieltjes transforms of phase-type distributions are found. Exponential distributions and Coxian distributions are identified to be extremal *PH*-distributions with respect to all the moments and Laplace-Stieltjes transforms for certain subsets of *PH*-distributions.

Chapter 6
Majorization and Extremal *PH* Distributions

Qi-Ming He, Hanqin Zhang, and Juan C. Vera

Introduction

Let T be an $m \times m$ invertible matrix with (1) negative diagonal elements, (2) nonnegative off-diagonal elements, and (3) nonpositive row sums, where m is a positive integer. Such a matrix T is called a *PH* generator. Let α be a substochastic vector of order m, i.e., $\alpha \geq 0$ and $\alpha\mathbf{e} \leq 1$, where $\mathbf{e}$ is the column vector of ones. Then (α, T) is called a *PH* representation of a phase-type (*PH*) random variable (distribution) X. In this chapter, we find bounds on the moments of X in terms of the elements of α and T and identify Coxian distributions to be the extremal *PH* distributions in certain subsets of *PH* distributions.

The set of *PH* distributions was introduced by Neuts [13]. Since the set of *PH* distributions is dense in the set of probability distributions on the nonnegative half-line and *PH* representations provide a Markovian structure for stochastic modeling, *PH* distributions have been used widely in the study of queueing, inventory, risk/insurance, manufacturing, and telecommunications models [9, 14]. In almost all applications of *PH* distributions, *PH* representations play a key role. Thus, the study of *PH* representations has attracted great attention from researchers (see [3, 4, 15, 17], and references therein).

Aldous and Shepp [1] find the minimum coefficient of variation of *PH* distributions with a *PH* representation of a fixed order m. They also find that the

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minimum is attained at *PH* representations of Erlang distributions. Their result is useful in determining the order of *PH* representations needed for fitting probability distributions if their coefficient of variation is known. In [17], a number of open problems related to *PH* representations are brought up and investigated. The results in [17] and in subsequent papers on the open problems (e.g., [5, 18, 19]) reveal the relationship between *PH* representations, density functions, and variances of *PH* distributions. In [17], a lower bound on the density of triangular *PH* distributions is found. In [5], it is shown that not every *PH* representation has an equivalent unicyclic *PH* representation of the same order. In [18], it is shown that, for a *PH* distribution with a *PH* representation of order 2, a minimal-norm representation can be found and the norm coincides with the minimal parameter in Maier's property [10]. While O'Cinneide [17] attempts to show *PH* distributions with a unicyclic *PH* representation as extremal *PH* distributions, this chapter aims to prove that *PH* distributions with some Coxian representations are extremal with respect to the moments of the distribution.

This chapter focuses on the relationship between *PH* representations and the moments of *PH* distributions. In the section "Two Majorization Lemmas," two majorization results are shown for the vector $-T^{-1}\mathbf{e}$. It is worth mentioning that the majorization approach [11] seems quite useful in the study of *PH* distributions and *PH* representations [7, 16]. The majorization results are used to obtain bounds on the mean (i.e., first moment) of *PH* distributions in the section "Bounds on Phase-Type Distributions." All bounds on the expectation are partially independent of the transition structure of the underlying Markov chain associated with the *PH* distribution. Results in the section "Bounds on Phase-Type Distributions" indicate that exponential/Coxian distributions are extreme cases, with respect to the mean, if the vector $-\mathbf{e}'T$ or the sum $-\mathbf{e}'T\mathbf{e}$ is fixed, where $\mathbf{e}'$ is the transpose of the vector $\mathbf{e}$. The section "Extremal Phase-Type Distributions" extends the results in the section "Bounds on Phase-Type Distributions" from the first moment to higher moments. A highlight of the results is the lower bounds on the moments of any *PH* distribution (α, T), i.e., $E[X_k] \geq k!/(-\mathbf{e}'T\mathbf{e})^k$ for all $k \geq 0$, that is independent of the order of the *PH* representation and the transitions within the underlying Markov chain. Results in the section "Extremal Phase-Type Distributions" demonstrate that exponential/Coxian distributions are extremal *PH* distributions with respect to all the moments and the Laplace–Stieltjes transform. All proofs are given in the section "Proofs." The section "Conclusion and Discussion" concludes the paper with a discussion of the potential applications of the results obtained in this chapter.

Two Majorization Lemmas

For the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, rearrange the elements of $\mathbf{x}$ in ascending order and denote the elements by $x_{[1]} \leq x_{[2]} \leq \cdots \leq x_{[m]}$, where ([1], [2], …, [$m$]) is a permutation of (1, 2, …, m). A vector $\mathbf{x}$ is *weakly submajorized* by vector $\mathbf{y}$, denoted by $\mathbf{x} \prec_w \mathbf{y}$, if $x_{[m]} + x_{[m-1]} + \cdots + x_{[k]} \le y_{[m]} + y_{[m-1]} + \cdots + y_{[k]}$ for $1 \le k \le m$. A vector $\mathbf{x}$ is *weakly supermajorized* by vector $\mathbf{y}$, denoted by $\mathbf{x} \prec^w \mathbf{y}$, if $x_{[1]} + x_{[2]} + \cdots + x_{[k]} \ge y_{[1]} + y_{[2]} + \cdots + y_{[k]}$ for $1 \le k \le m$. A vector $\mathbf{x}$ is *majorized* by $\mathbf{y}$, denoted as $\mathbf{x} \prec \mathbf{y}$, if $\mathbf{xe} = \mathbf{ye}$ and $x_{[1]} + x_{[2]} + \cdots + x_{[k]} \ge y_{[1]} + y_{[2]} + \cdots + y_{[k]}$ for $1 \le k \le m-1$, or, equivalently, $\mathbf{xe} = \mathbf{ye}$, and $x_{[m]} + x_{[m-1]} + \cdots + x_{[k]} \le y_{[m]} + y_{[m-1]} + \cdots + y_{[k]}$ for $2 \le k \le m$. It is easy to see that $\mathbf{x} \prec \mathbf{y}$ if and only if $\mathbf{x} \prec_w \mathbf{y}$ and $\mathbf{x} \prec^w \mathbf{y}$. We refer the reader to Marshall and Olkin [11] for more about majorization.

Consider a *PH* generator T of order m. Define $\mathbf{r} = -\mathbf{e}'T = (r_1, r_2, \ldots, r_m)$. Rearrange the elements of $\mathbf{r}$ in ascending order as $r_{[1]} \le r_{[2]} \le \cdots \le r_{[m]}$. Since T is invertible and $T\mathbf{e} \le 0$, we must have $-\mathbf{e}'T\mathbf{e} = \mathbf{re} > 0$. It is possible that some of $\{r_1, r_2, \ldots, r_m\}$ are negative, but the summation $r_{[j]} + r_{[j+1]} + \cdots + r_{[m]}$ is positive for $1 \le j \le m$. For fixed $\mathbf{r}$, we shall construct two matrices $T_\downarrow^*$ and $T_\uparrow^*$ and find majorization relationships between the vectors $-T^{-1}\mathbf{e}$, $-(T_\downarrow^*)^{-1}\mathbf{e}$, and $-(T_\uparrow^*)^{-1}\mathbf{e}$. Define

$$T_\downarrow^* = \begin{pmatrix} -\sum_{j=1}^{m} r_{[j]} & & & & \\ \sum_{j=2}^{m} r_{[j]} & -\sum_{j=2}^{m} r_{[j]} & & & \\ & \ddots & \ddots & & \\ & & \sum_{j=m-1}^{m} r_{[j]} & -\sum_{j=m-1}^{m} r_{[j]} & \\ & & & r_{[m]} & -r_{[m]} \end{pmatrix}. \tag{6.1}$$

It is easy to see that the matrix $T_\downarrow^*$ is a *PH* generator. In fact, $T_\downarrow^*$ is a Coxian generator for Coxian distributions [6]. Define

$$\begin{aligned} \mathbf{b}_\downarrow^* &= -(T_\downarrow^*)^{-1}\mathbf{e} \\ &= \left(\sum_{i=m}^{m} \left(\sum_{j=m-i+1}^{m} r_{[j]} \right)^{-1}, \ldots, \sum_{i=k}^{m} \left(\sum_{j=m-i+1}^{m} r_{[j]} \right)^{-1}, \ldots, \sum_{i=1}^{m} \left(\sum_{j=m-i+1}^{m} r_{[j]} \right)^{-1} \right)'. \end{aligned} \tag{6.2}$$

It is readily seen that the elements in $\mathbf{b}_\downarrow^*$ are positive and are in ascending order.

Lemma 6.1. *Assume that T is a PH generator of order m. Then $-T^{-1}\mathbf{e}$ is weakly supermajorized by $\mathbf{b}_\downarrow^*$ defined in (6.2).*

Next, we define $T_\uparrow^*$ such that $-T^{-1}\mathbf{e}$ is weakly submajorized by $-(T_\uparrow^*)^{-1}\mathbf{e}$ under an additional condition. If $r_{[1]} = \min\{r_1, r_2, \ldots, r_m\} > 0$, then we define

$$T_{\uparrow}^{*} = \begin{pmatrix} -\sum_{j=1}^{m} r_{[j]} & & & & \\ \sum_{j=1}^{m-1} r_{[j]} & -\sum_{j=1}^{m-1} r_{[j]} & & & \\ & \ddots & \ddots & & \\ & & \sum_{j=1}^{2} r_{[j]} & -\sum_{j=1}^{2} r_{[j]} & \\ & & & r_{[1]} & -r_{[1]} \end{pmatrix}. \tag{6.3}$$

It is easy to see that the matrix $T_{\uparrow}^{*}$ is a *PH* generator. Define

$$\begin{aligned} \mathbf{b}_{\uparrow}^{*} &= -(T_{\uparrow}^{*})^{-1}\mathbf{e} \\ &= \left(\sum_{i=1}^{1} \left(\sum_{j=1}^{m-i+1} r_{[j]} \right)^{-1}, \ldots, \sum_{i=1}^{k} \left(\sum_{j=1}^{m-i+1} r_{[j]} \right)^{-1}, \ldots, \sum_{i=1}^{m} \left(\sum_{j=1}^{m-i+1} r_{[j]} \right)^{-1} \right)'. \end{aligned} \tag{6.4}$$

It is readily seen that the elements in $\mathbf{b}_{\uparrow}^{*}$ are nonnegative and are in ascending order.

Lemma 6.2. *Assume that T is a PH generator of order m, and $r_{[1]} > 0$. Then $-T^{-1}\mathbf{e}$ is weakly supermajorized by $\mathbf{b}_{\uparrow}^{*}$ defined in (6.4).*

Bounds on Phase-Type Distributions

Now we focus on a random variable X with a *PH* distribution with *PH* representation (α, T). It is well known that the expectation of *PH* distribution X is given by $E[X] = -\alpha T^{-1}\mathbf{e}$. Since $-\alpha T^{-1}\mathbf{e} = -(\alpha\mathbf{e})(\alpha/(\alpha\mathbf{e}))T^{-1}\mathbf{e}$, without loss of generality, we shall assume α normalized such that $\alpha\mathbf{e} = 1$ in the rest of the paper.

For vector $\mathbf{x}$ let $\mathbf{x}_{\uparrow} = (x_{[1]}, x_{[2]}, \ldots, x_{[m]})$ denote the ascending rearrangement of $\mathbf{x}$, and let $\mathbf{x}_{\downarrow} = (x_{[m]}, x_{[m-1]}, \ldots, x_{[1]})$ denote the descending rearrangement of $\mathbf{x}$. For stochastic vector α, the vectors $\alpha_{\uparrow}$ and $\alpha_{\downarrow}$ are defined accordingly. For any vector $\mathbf{x}$ it is easy to verify $\alpha_{\downarrow}\mathbf{x}_{\uparrow} \le \alpha\mathbf{x} \le \alpha_{\uparrow}\mathbf{x}_{\uparrow}$ [12]. For vectors $\mathbf{x}$ and $\mathbf{y}$, (1) if $\mathbf{x} \prec_{w} \mathbf{y}$, then
AQ4 we have $\alpha_{\uparrow}\mathbf{x}_{\uparrow} \le \alpha_{\uparrow}\mathbf{y}_{\uparrow}$; (2) if $\mathbf{x} \prec^{w} \mathbf{y}$, then we have $\alpha_{\downarrow}\mathbf{x}_{\uparrow} \ge \alpha_{\downarrow}\mathbf{y}_{\uparrow}$; and (3) if $\mathbf{x} \prec \mathbf{y}$, then we have $\alpha_{\downarrow}\mathbf{x}_{\uparrow} \ge \alpha_{\downarrow}\mathbf{y}_{\uparrow}$, and $\alpha_{\uparrow}\mathbf{x}_{\uparrow} \le \alpha_{\uparrow}\mathbf{y}_{\uparrow}$ [11].

AQ5 Now we are ready to state the main results.

Theorem 6.1. *Consider a PH generator T of order m. For any random variable X with a PH distribution with PH representation (α, T) we have*

$$E[X] \geq -\alpha_{\downarrow}(T_{\downarrow}^{*})^{-1}\mathbf{e} \geq -\frac{1}{\mathbf{e}'T\mathbf{e}}, \tag{6.5}$$

where $T_{\downarrow}^{*}$ is defined in (6.1). That is: the mean of the PH distribution (α, T) is greater than or equal to that of the PH distribution $(\alpha_{\downarrow}, T_{\downarrow}^{*})$.

Moreover, if all elements of $\mathbf{r} = \mathbf{e}'(-T)$ are positive, then we have

$$E[X] \leq -\alpha_{\uparrow}(T_{\uparrow}^{*})^{-1}\mathbf{e} \leq \sum_{i=1}^{m}\left(\sum_{j=1}^{m-i+1} r_{[j]}\right)^{-1}, \tag{6.6}$$

where $T_{\uparrow}^{*}$ is defined in (6.3). That is, the mean of the PH distribution (α, T) is less than or equal to that of the PH distribution $(\alpha_{\uparrow}, T_{\uparrow}^{*})$.

Note that the lower bound $-1/(\mathbf{e}'T\mathbf{e})$ in (6.5) is totally independent of the transition structure of the underlying Markov chain (i.e., the transition within T). The upper bound in (6.6) is only partially independent of the transition structure of the underlying Markov chain.

Example 6.1. Consider a *PH* generator

$$T = \begin{pmatrix} -10 & 8 \\ 2 & -2 \end{pmatrix}. \tag{6.7}$$

It is easy to find $\mathbf{e}'(-T) = (8, -6)$, $-T^{-1}\mathbf{e} = (2.5, 3)'$,

$$T_{\downarrow}^{*} = \begin{pmatrix} -2 & 0 \\ 8 & -8 \end{pmatrix}, \tag{6.8}$$

and $-(T_{\downarrow}^{*})^{-1}\mathbf{e} = (0.5, 0.625)'$. For any *PH* distribution (α, T) with $\alpha\mathbf{e} = 1$, by Theorem 6.1, we have $0.5 \leq 0.5\alpha_{[2]} + 0.625\alpha_{[1]} \leq E[X]$.

For this case, the lower bound is not sharp since $2.5 \leq E[X] \leq 3$ for all feasible α with $\alpha\mathbf{e} = 1$. Following He and Zhang [6], the *PH* generator T can be Coxianized, i.e., there is a Coxian generator

$$S = \begin{pmatrix} -6-\sqrt{32} & 0 \\ -6-\sqrt{32} & -6+\sqrt{32} \end{pmatrix} \tag{6.9}$$

such that any *PH* representation (α, T) has an equivalent Coxian representation (β, S), where β is a stochastic vector. The difference between $T_{\downarrow}^{*}$ and S explains why the lower bounds are too small for this case. This example warrants further investigation on the relationship between the matrices $T_{\downarrow}^{*}$ and $T_{\uparrow}^{*}$ and the Coxianization of T. On the other hand, finding bounds on the mean of a *PH* distribution is not the objective of this research. The results on bounds are used for characterizing *PH* distributions and for finding extremal *PH* distributions (see the section "Extremal Phase-Type Distributions").

Example 6.2. Consider a *PH* generator

$$T = \begin{pmatrix} -2 & 1 \\ x & -x \end{pmatrix}, \tag{6.10}$$

where $x > 0$. It is easy to verify $-T^{-1}\mathbf{e} = (1+1/x, 1+2/x)'$. The expectation of (α, T) with $\alpha\mathbf{e} = 1$ goes to positive infinity if x goes to zero. Note that $-\mathbf{e}'T\mathbf{e} = 1$ holds for any positive x. Thus, while there is a lower bound that is totally independent of the transition structure, there may not be such an upper bound.

For some special *PH* generators, lower and upper bounds can be obtained simultaneously.

Theorem 6.2. *Consider a PH generator T of order m and satisfying* $-\mathbf{e}'T^{-1}\mathbf{e} = -\mathbf{e}'(T_{\downarrow}^{*})^{-1}\mathbf{e}$. *For any PH distributed random variable X with PH representation* (α, T) *we have*

$$-\frac{1}{\mathbf{e}'T\mathbf{e}} \le -\alpha_{\downarrow}(T_{\downarrow}^{*})^{-1}\mathbf{e} \le E[X] \le -\alpha_{\uparrow}(T_{\downarrow}^{*})^{-1}\mathbf{e} \le \sum_{i=1}^{m}\left(\sum_{j=m-i+1}^{m} r_{[j]}\right)^{-1}. \tag{6.11}$$

Consider a PH generator T such that (i) $-\mathbf{e}'T^{-1}\mathbf{e} = -\mathbf{e}(T_{\uparrow}^{*})^{-1}\mathbf{e}$ *and (ii) all elements of* $\mathbf{r} = \mathbf{e}'(-T)$ *are positive. For any PH distributed random variable X with PH representation* (α, T) *we have*

$$-\frac{1}{\mathbf{e}'T\mathbf{e}} \le -\alpha_{\downarrow}(T_{\uparrow}^{*})^{-1}\mathbf{e} \le E[X] \le -\alpha_{\uparrow}(T_{\uparrow}^{*})^{-1}\mathbf{e} \le \sum_{i=1}^{m}\left(\sum_{j=1}^{m-i+1} r_{[j]}\right)^{-1}. \tag{6.12}$$

What follows is a special case of Theorem 6.2 that was proved in [7].

Corollary 6.1. *For any PH distribution* (α, T) *for which T satisfies* $\mathbf{e}'T = -\mu\mathbf{e}'$ *we have*

$$-\frac{1}{\mu m} \le -\alpha_{\downarrow}(T_{\downarrow}^{*})^{-1}\mathbf{e} \le E[X] \le -\alpha_{\uparrow}(T_{\downarrow}^{*})^{-1}\mathbf{e} \le \frac{1}{\mu}\sum_{i=1}^{m}\frac{1}{i}. \tag{6.13}$$

Example 6.3. Consider a *PH* generator

$$T = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}. \tag{6.14}$$

It is easy to find $\mathbf{e}'(-T) = (1,1)$, $-T^{-1}\mathbf{e} = (3/4, 5/4)'$,

$$T_{\downarrow}^{*} = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}, \tag{6.15}$$

and $-(T_{\downarrow}^{*})^{-1}\mathbf{e} = (0.5, 1.5)'$. For any *PH* distribution (α, T) with $\alpha\mathbf{e} = 1$, by Corollary 6.1, we have $0.5 \le 0.5\alpha_{[2]} + 1.5\alpha_{[1]} \le E[X] \le 0.5\alpha_{[1]} + 1.5\alpha_{[2]} \le 1.5$.

Extremal Phase-Type Distributions

Let $X_{\min}$ be the exponential random variable with parameter λ. Denote by Ω_λ the set of all *PH* distributions with a *PH* representation (α, T) satisfying $\alpha\mathbf{e} = 1$ and $\lambda = -\mathbf{e}'T\mathbf{e}$.

By Theorem 6.1, $E[X_{\min}] = \min\{E[X] : X \in \Omega_\lambda\}$, which implies that $X_{\min}$ is an extremal random variable, with respect to the first moment, in Ω_λ. Note that the result in Theorem 6.1 is independent of the order of the *PH* representation. The result can be generalized to all moments and Laplace-Stieltjes transforms (LSTs) of *PH* distributions.

Corollary 6.2. *For $\lambda > 0$ and $X \in \Omega_\lambda$ we have*

1. $E[X^k] \ge E[X_{\min}^k] = \dfrac{k!}{(-\mathbf{e}'T\mathbf{e})^k}, \quad k \ge 1;$

2. $E[e^{-sX_{\min}}] \le E[e^{-sX}]$, $s_{\text{lower}} < s \le 0$, *for some negative number* s_{lower}*; and* $E[e^{-sX_{\min}}] \ge E[e^{-sX}]$, $0 \le s < s_{\text{upper}}$, *for some positive number* s_{upper}.

Corollary 6.2 indicates that $X_{\min}$ is an extremal distribution in Ω_λ with respect to the moments and the LST. Define nonnegative random variable $Y_{\min}$ by

$$P\{Y_{\min} \le t\} = \frac{m-1}{m} + \frac{1}{m}(1 - \exp\{-\theta t\}), \quad \text{for } t \ge 0, \tag{6.16}$$

where θ is positive. Then $Y_{\min}$ equals zero, w.p. $(m-1)/m$, and an exponential random variable with parameter θ, w.p. $1/m$. Define

$$\Psi_{m,\theta} = \left\{ X : \ X \sim (\alpha, T) \ \text{of order } m, \alpha\mathbf{e} = 1, \ \theta = -\frac{\mathbf{e}'T\mathbf{e}}{m} \right\}, \tag{6.17}$$

where "$\sim$" means equivalency in probability distribution.

Corollary 6.3. *For $\theta > 0$ and $X \sim (\alpha, T) \in \Psi_{m,\theta}$ we have, for $s \ge 0$,*

$$E[e^{-sX}] \le E[e^{-sY_{\min}}] = \frac{m-1}{m} + \frac{\theta}{m(s+\theta)}. \tag{6.18}$$

We remark that, while the extremal random variable $X_{\min}$ is in Ω_λ, $Y_{\min}$ is not in $\Psi_{m,\theta}$. Yet the LST of $Y_{\min}$ provides a bound on the LSTs of all *PH* distributions in $\Psi_{m,\theta}$.

Next, let $X_{\max}$ be the exponential random variable with parameter μ. Denote by Φ_μ the set of all *PH* distributions with a *PH* representation (α, T) satisfying $\alpha\mathbf{e} = 1$ and

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{m-i+1} r_{[j]}\right)^{-1} = \frac{1}{\mu}, \tag{6.19}$$

where $\mathbf{r} = -\mathbf{e}'T > 0$ and $m = 1, 2, \ldots$. By Theorem 6.1, $E[X_{\max}] = 1/\mu = \max\{E[X] : X \in \Phi_\mu\}$, which implies that $X_{\max}$ is an extremal random variable, with respect to the first moment, in Φ_μ. The result can be generalized to all moments and LSTs of *PH* distributions.

Corollary 6.3. *For $\mu > 0$ and Φ_μ we have*

1. $E[X_{\max}^k] \geq E[X^k]$, $k \geq 1$;

2. $E[e^{-sX_{\max}}] \geq E[e^{-sX}]$, $s_{\text{lower}} < s \leq 0$, for some negative number s_{lower}; and $E[e^{-sX_{\max}}] \leq E[e^{-sX}]$, $0 \leq s < s_{\text{upper}}$, for some positive number s_{upper}.

Define

$$\Theta_m = \{X :\ X \sim (\alpha, T) \text{ of order } m,\ \alpha\mathbf{e} = 1,\ -\mathbf{e}'T > 0\}. \tag{6.20}$$

Corollary 6.4. *For $X \sim (\alpha, T) \in \Theta_m$ we have, for $s \geq 0$,*

$$E[e^{-sX}] \geq 1 - \sum_{i=1}^{m} \frac{s}{i(s+\delta_i)}, \tag{6.21}$$

where $\delta_i = r_{[1]} + \cdots + r_{[i]}$, $i = 1, 2, \ldots, m$, and $\mathbf{r} = -\mathbf{e}'T$.

Note that $\mathbf{e}'(sI - T) > 0$ for sufficiently large s. For any *PH* distribution, (6.21) holds if s is sufficiently large.

Example 6.1. Consider *PH* generator T defined as

$$T = \begin{pmatrix} -5 & 1 & 1 & 0 & 1 \\ 2 & -15 & 0 & 1 & 5 \\ 0 & 1 & -3 & 1 & 0 \\ 1 & 0 & 0 & -5 & 1 \\ 1 & 1 & 1 & 0 & -8 \end{pmatrix}. \tag{6.22}$$

Note that $-\mathbf{e}'T = (1, 12, 1, 3, 1)$ is positive.

For $X \sim (\alpha, T)$ with $\alpha = (0.2, 0.5, 0.1, 0.1, 0.1)$ we have $E[X_{\min}^k] \leq E[X^k] = k!\alpha(-T^{-1})^k\mathbf{e} \leq E[X_{\max}^k]$ for $k \geq 1$. As shown in Fig. 6.1, the two logarithmic ratios are less than zero for all k, which confirms the inequalities numerically. We further obtain $E[e^{-sX_{\min}}] \leq E[e^{-sX}] \leq E[e^{-sX_{\max}}]$ for $s \leq 0$, for which the expectations exist.

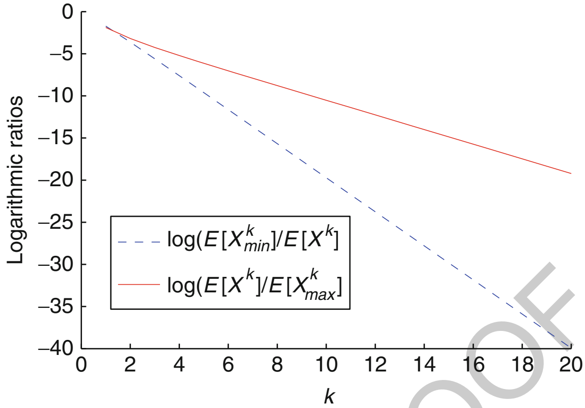

Fig. 6.1 Logarithmic ratios $\log(E[X_{\min}^k]/E[X^k])$ and $\log(E[X^k]/E[X_{\max}^k])$

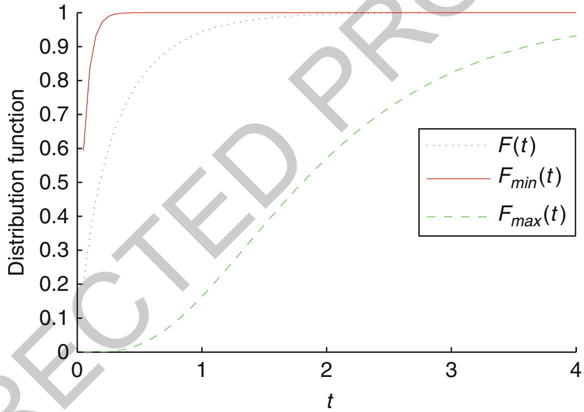

Fig. 6.2 Distribution functions $F_{\max}(t)$, $F(t)$, and $F_{\min}(t)$

For Example 6.1, further numerical results indicate that $(-T^{-1})^k\mathbf{e} \prec^w (-(T_\downarrow^*)^{-1})^k\mathbf{e}$ and $(-T^{-1})^k\mathbf{e} \prec_w (-(T_\uparrow^*)^{-1})^k\mathbf{e}$ for $k \geq 1$. Such results are stronger than those in Corollaries 6.2 and 6.3. If the results are true, then the moments of (α, T) are upper bounded by that of the Coxian distribution $((0,\ldots,0,1), T_\uparrow^*)$, which is different from the distribution function of $X_{\max}$ (which is actually an exponential random variable). Denote by $F_{\min}(t)$, $F(t)$, and $F_{\max}(t)$ the probability distribution functions of the *PH* distributions $((1,0,\ldots,0), T_\downarrow^*)$, (α, T), and $((0,\ldots,0,1), T_\uparrow^*)$, respectively. Numerical results also indicate that $F_{\max}(t) \leq F(t) \leq F_{\min}(t)$ for $t \geq 0$ (Fig. 6.2), which implies that the three probability distributions are stochastically ordered. The result is interesting since $F_{\max}(t)$ is a Coxian (not an exponential) distribution in general. Extensive numerical tests demonstrate that those results may hold for all *PH* distributions with *PH* generators satisfying $-\mathbf{e}'T > 0$.

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Proofs

Proof of Lemma 6.1. Denote by $\mathbf{e}(i)$ the column vector with zero everywhere but one in the ith place. Since the matrix $-T$ is an M-matrix, $-T^{-1}$ is nonnegative (Theorem 4.5 [12]). Let $\mathbf{b} = -T^{-1}\mathbf{e}$. Without loss of generality, we assume that elements of $\mathbf{b}$ are in ascending order, i.e., $b_1 \le b_2 \le \cdots \le b_m$, which can be done by permuting the rows and columns of matrix T. To prove that $-T^{-1}\mathbf{e}$ is weakly supermajorized by $\mathbf{b}^*_{\downarrow}$, by definition, it is sufficient to show that $b_1 + b_2 + \cdots + b_k \ge (\mathbf{b}^*_{\downarrow})_1 + (\mathbf{b}^*_{\downarrow})_2 + \cdots + (\mathbf{b}^*_{\downarrow})_k$, for $1 \le k \le m$.

For fixed $k \le m$ let $\mathbf{z} = (\mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)')(-T^{-1})$. Let $z_{n_1} \ge z_{n_2} \ge \cdots \ge z_{n_m}$ be the elements of $\mathbf{z}$ in descending order. Since $\mathbf{ze} = b_1 + b_2 + \cdots + b_k$, our goal is to prove, for $1 \le k \le m$,

$$\mathbf{ze} \ge k\left(\sum_{j=1}^{m} r_{[j]}\right)^{-1} + (k-1)\left(\sum_{j=2}^{m} r_{[j]}\right)^{-1} + \cdots + \left(\sum_{j=k}^{m} r_{[j]}\right)^{-1}. \tag{6.23}$$

Since $\mathbf{z}(-T) = \mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)'$, we have, for $1 \le j \le m$,

$$\begin{aligned} &\mathbf{z}(-T)(\mathbf{e}(n_j) + \mathbf{e}(n_{j+1}) + \cdots + \mathbf{e}(n_m)) \\ &\quad = (\mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)')(\mathbf{e}(n_j) + \mathbf{e}(n_{j+1}) + \cdots + \mathbf{e}(n_m)) \\ &\quad \ge \max\{0,\ k - j + 1\}. \end{aligned} \tag{6.24}$$

By definition, we have $T\mathbf{e} \le 0$ and $T_{i,j} \ge 0$ for $1 \le i \le j \le m$. Then, for any $\{i_1, i_2, \ldots, i_n\} \subset \{1, 2, \ldots, m\}$ and $i_0 \in \{1, 2, \ldots, m\}$, note that

$$\mathbf{e}(i_0)'(-T)(\mathbf{e}(i_1) + \mathbf{e}(i_2) + \cdots + \mathbf{e}(i_n)) = -\sum_{j=1}^{n} T_{(i_0, i_j)}, \tag{6.25}$$

which is nonnegative if $i_0 \in \{i_1, i_2, \ldots, i_n\}$ and nonpositive if $i_0 \notin \{i_1, i_2, \ldots, i_n\}$. For $i < j$ we have $z_{n_i} - z_{n_j} \ge 0$ and $\mathbf{e}(n_i)'(-T)(\mathbf{e}(n_j) + \mathbf{e}(n_{j+1}) + \cdots + \mathbf{e}(n_m)) \le 0$. For $i \ge j$, we have $z_{n_i} - z_{n_j} \le 0$ and $\mathbf{e}(n_i)'(-T)(\mathbf{e}(n_j) + \mathbf{e}(n_{j+1}) + \cdots + \mathbf{e}(n_m)) \ge 0$. Combining the two cases, for $1 \le i, j \le m$ we obtain

$$(z_{n_i} - z_{n_j})\mathbf{e}(n_i)'(-T)(\mathbf{e}(n_j) + \mathbf{e}(n_{j+1}) + \cdots + \mathbf{e}(n_m)) \le 0. \tag{6.26}$$

Equation (6.26) leads to

$$z_{n_i}\mathbf{e}(n_i)'(-T)\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) \le z_{n_j}\mathbf{e}(n_i)'(-T)\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right). \tag{6.27}$$

AQ8 Summing up over $i = 1, 2, \ldots, m$, in (6.27), yields

$$\begin{aligned}
\mathbf{z}(-T)\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) &= \sum_{i=1}^{m} z_{n_i}\mathbf{e}(n_i)'(-T)\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) \\
&\le \sum_{i=1}^{m} z_{n_j}\mathbf{e}(n_i)'(-T)\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) \\
&= z_{n_j}\mathbf{e}'(-T)\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) \\
&= z_{n_j}\mathbf{r}\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right). \qquad (6.28)
\end{aligned}$$

We also have

$$\mathbf{r}\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) = \sum_{h=j}^{m} r_{n_h} \le \sum_{h=j}^{m} r_{[h]}. \qquad (6.29)$$

Combining (6.24), (6.28), and (6.29) we obtain

$$z_{n_j} \ge \max\{0, k-j+1\}\left(\sum_{h=j}^{m} r_{[h]}\right)^{-1}. \qquad (6.30)$$

Adding over $j = 1, 2, \ldots, m$, (6.23) follows. This completes the proof of Lemma 6.1.

Proof of Lemma 6.2. This proof is similar to that of Lemma 6.1, but some details are different. Let $\mathbf{b} = -T^{-1}\mathbf{e}$. Without loss of generality, we assume that elements of $\mathbf{b}$ are in descending order, i.e., $b_1 \ge b_2 \ge \cdots \ge b_m$. To prove that $-T^{-1}\mathbf{e}$ is weakly submajorized by $\mathbf{b}^*_{\uparrow}$, it is sufficient to show that $b_1 + b_2 + \cdots + b_k \le (\mathbf{b}^*_{\uparrow})_m + (\mathbf{b}^*_{\uparrow})_{(m-1)} + \cdots + (\mathbf{b}^*_{\uparrow})_{(m-k+1)}$, for $1 \le k \le m$.

For fixed $k \le m$ let $\mathbf{z} = (\mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)')(-T^{-1})$. Let $z_{n_1} \le z_{n_2} \le \cdots \le z_{n_m}$ be the elements of $\mathbf{z}$ in ascending order, where $(n_1, n_2, \ldots, n_m)$ is a permutation of $(1, 2, \ldots, m)$. Since $\mathbf{ze} = b_1 + b_2 + \cdots + b_k$, our goal is to prove

$$\mathbf{ze} \le \sum_{i=1}^{m} \min\{k, m-i+1\}\left(\sum_{j=1}^{m-i+1} r_{[j]}\right)^{-1}. \qquad (6.31)$$

Since $\mathbf{z}(-T) = \mathbf{e}(1)' + \mathbf{e}(2)' + \cdots + \mathbf{e}(k)'$, we have, for $1 \le j \le m$,

$$\mathbf{z}(-T)(\mathbf{e}(n_j) + \mathbf{e}(n_{j+1}) + \cdots + \mathbf{e}(n_m)) \le \min\{k, m-j+1\}. \qquad (6.32)$$

Similar to (6.26), we can show, for $1 \le i, j \le m$,

$$(z_{n_i} - z_{n_j})\mathbf{e}(n_i)'(-T)(\mathbf{e}(n_j) + \mathbf{e}(n_{j+1}) + \cdots + \mathbf{e}(n_m)) \ge 0, \tag{6.33}$$

which leads to

$$\begin{aligned} \mathbf{z}(-T)\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) &\ge z_{n_j}\mathbf{r}\left(\sum_{h=j}^{m} \mathbf{e}(n_h)\right) \\ &= z_{n_j}\left(\sum_{h=j}^{m} r_{n_h}\right) \\ &\ge z_{n_j}\left(\sum_{h=1}^{m-j+1} r_{[h]}\right). \end{aligned} \tag{6.34}$$

Combining (6.32) and (6.34), since $\sum_{h=1}^{m-j+1} r_{[h]} > 0$, we obtain

$$z_{n_j} \le \min\{k, m-j+1\}\left(\sum_{h=1}^{m-j+1} r_{[h]}\right)^{-1}. \tag{6.35}$$

Adding over $j = 1, 2, \ldots, m$, (6.31) follows. This completes the proof of Lemma 6.2.

Proof of Theorem 6.1. By Lemma 6.1, we have $-T^{-1}\mathbf{e} \prec^{w} -(T_{\downarrow}^{*})^{-1}\mathbf{e}$, or, equivalently, $(-T^{-1}\mathbf{e})_{\uparrow} \prec^{w} -(T_{\downarrow}^{*})^{-1}\mathbf{e}$. Since the elements in $\alpha_{\downarrow}$ are in descending order, we obtain $\alpha_{\downarrow}(-T^{-1}\mathbf{e})_{\uparrow} \ge -\alpha_{\downarrow}(T_{\downarrow}^{*})^{-1}\mathbf{e}$, which leads to

$$E[X] = -\alpha T^{-1}\mathbf{e} \ge \alpha_{\downarrow}(-T^{-1}\mathbf{e})_{\uparrow} \ge -\alpha_{\downarrow}(T_{\downarrow}^{*})^{-1}\mathbf{e}. \tag{6.36}$$

AQ9 Since the vector $-(T_{\downarrow}^{*})^{-1}\mathbf{e}$ are in ascending order, $-\alpha_{\downarrow}(T_{\downarrow}^{*})^{-1}\mathbf{e} \ge (-(T_{\downarrow}^{*})^{-1}\mathbf{e})_1 = (r_1 + r_2 + \cdots + r_m)^{-1} = -1/(\mathbf{e}'T\mathbf{e})$. This proves the first part of the theorem.

By Lemma 6.2, we have $-T^{-1}\mathbf{e} \prec_{w} -(T_{\uparrow}^{*})^{-1}\mathbf{e}$, or, equivalently, $(-T^{-1}\mathbf{e})_{\uparrow} \prec_{w} -(T_{\uparrow}^{*})^{-1}\mathbf{e}$. Since the elements of $\alpha_{\uparrow}$ are in ascending order, we obtain $\alpha_{\uparrow}(-T^{-1}\mathbf{e})_{\uparrow} \le -\alpha_{\uparrow}(T_{\uparrow}^{*})^{-1}\mathbf{e}$, which leads to

$$E[X] = -\alpha T^{-1}\mathbf{e} \le \alpha_{\uparrow}(-T^{-1}\mathbf{e})_{\uparrow} \le -\alpha_{\uparrow}(T_{\uparrow}^{*})^{-1}\mathbf{e}. \tag{6.37}$$

Since the elements of the vector $-(T_{\uparrow}^{*})^{-1}\mathbf{e}$ are in ascending order, $-\alpha_{\uparrow}(T_{\uparrow}^{*})^{-1}\mathbf{e} \le (-(T_{\uparrow}^{*})^{-1}\mathbf{e})_m = \sum_{i=1}^{m}(\sum_{j=1}^{m-i+1} r_{[j]})^{-1}$. This proves the second part and concludes the proof of Theorem 6.1.

AQ10 **Proof of Theorem 6.2.** Under the conditions, we have $-T^{-1}\mathbf{e} \prec -(T_{\downarrow}^{*})^{-1}\mathbf{e}$ and $-T^{-1}\mathbf{e} \prec -(T_{\uparrow}^{*})^{-1}\mathbf{e}$, respectively. The rest of the proof is similar to that of
AQ11 Theorem 6.1. This completes the proof of Theorem 6.2. xxxxx

AQ12 **Proof of Corollary 6.1.** By Theorem 6.1, part 1) holds for $k = 1$, i.e., $E[X] \geq 1/\lambda$. We prove the result for $k > 1$ by induction. Consider the stochastic vector $\gamma = \alpha(-T^{-1})^k/(\alpha(-T^{-1})^k\mathbf{e})$. Note that $E[X^k] = k!\alpha(-T^{-1})^k\mathbf{e}$ and $E[X_{\min}^k] = k!/\lambda^k$ for $k \geq 1$. Applying Theorem 6.1 to $(\gamma, T) \in \Omega_\lambda$ we obtain

$$\frac{\alpha(-T^{-1})^{k+1}\mathbf{e}}{\alpha(-T^{-1})^k\mathbf{e}} = \gamma(-T^{-1})\mathbf{e} \geq \frac{1}{(-\mathbf{e}'T\mathbf{e})} = \frac{1}{\lambda}, \tag{6.38}$$

which leads to $E[X^{k+1}] \geq (k+1)E[X^k]/\lambda$. By induction, we obtain $E[X^{k+1}] \geq$
AQ13 $(k+1)!/\lambda^{k+1} = E[X_{\min}^{k+1}]$. This proves part 1) of Corollary 6.1. To prove part 2), we first note that, by definition,

$$E[\mathrm{e}^{-sX}] = \sum_{n=0}^{\infty} \frac{(-s)^n E[X^n]}{n!} \tag{6.39}$$

if the summation exists. Then $E[\mathrm{e}^{-sX_{\min}}] \leq E[\mathrm{e}^{-sX}]$ for $s_{\text{lower}} < s \leq 0$, is obtained from part 1), for some negative number s_{lower}. Since both functions $E[\mathrm{e}^{-sX_{\min}}]$ and $E[\mathrm{e}^{sX}]$ equal one at $s = 0$, by continuous extension at $s = 0$, we obtain $E[\mathrm{e}^{-sX_{\min}}] \geq E[\mathrm{e}^{-sX}]$, for $0 \leq s < s_{\text{upper}}$ and some positive number s_{upper}. This completes the proof of Corollary 6.1.

Proof of Corollary 6.2. We consider the *PH* generator $-sI + T$ for $s \geq 0$. Lemma 6.1 indicates that $(sI - T)^{-1}\mathbf{e}$ is weakly supermajorized by $(sJ - T_{\downarrow}^*)^{-1}\mathbf{e}$, where

$$J = \begin{pmatrix} m & & & & \\ -(m-1) & m-1 & & & \\ & \ddots & \ddots & & \\ & & -2 & 2 & \\ & & & -1 & 1 \end{pmatrix}, \tag{6.40}$$

and $T_{\downarrow}^*$ was defined in (6.1). Applying Theorem 6.1 to $(\alpha, -sI + T)$ we obtain that $\alpha(sI - T)^{-1}\mathbf{e}$ is greater than or equal to the first element in the column vector $(sJ - T_{\downarrow}^*)^{-1}\mathbf{e}$. Since $s \geq 0$, $s\alpha(sI - T)^{-1}\mathbf{e}$ is greater than or equal to the first element in the column vector $s(sJ - T_{\downarrow}^*)^{-1}\mathbf{e}$, which is given by

$$\frac{1}{m} - \frac{\theta}{m(s+\theta)}. \tag{6.41}$$

Note that $1 - s\alpha(sI - T)^{-1}\mathbf{e} = \alpha(sI - T)^{-1}(-T)\mathbf{e} = E[\mathrm{e}^{-sX}]$. Then (6.18) is obtained from (6.41). This completes the proof of Corollary 6.2.

Proof of Corollary 6.3. The proof is similar to that of Corollary 6.1. Details are omitted.

Proof of Corollary 6.4. The proof is similar to that of Corollary 6.2. Details are omitted.

Conclusion and Discussion

For some subsets of *PH* distributions, in this chapter, it is found that the exponential distributions and Coxian distributions are extremal distributions with respect to all the moments and the LSTs of *PH* distributions. The results have potential applications in several areas.

- The results can be useful in parameter estimation of *PH* distributions. For instance, the relationship $E[X^k] \geq k!/(-\mathbf{e}'T\mathbf{e})^k$, for $k \geq 1$, provides constraints on the parameters in T if the sample moments of the *PH* distribution X can be found (through other methods). The constraints can be used in nonlinear programs (e.g., EM algorithm) for parameter estimation of *PH* distributions [2, 8]. The potential of the results in this area is yet to be explored.
- The results can be used in optimization. Consider the case $\mathbf{e}'T = -\mu\mathbf{e}$, where $\mu > 0$. Without loss of generality, we assume $\mu = 1$. Then we obtain $\mathbf{e}'(-T)^{-1} = \mathbf{e}'$. Denote by $\mathbf{a}_1, \mathbf{a}_2, \ldots,$ and $\mathbf{a}_m$ the column vectors of $-T^{-1}$, which is nonnegative. Then the vector $\mathbf{e}'/m$ is in the polytope generated by $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$. Then Corollary 6.1 gives the optimal solution(s) to the following optimization problem:

$$\begin{aligned}
&\max / \min_{\{\alpha_i, \mathbf{a}_i,\ 1 \leq i \leq m\}} \left(\sum_{i=1}^{m} \alpha_i \mathbf{a}_i \right) \mathbf{e} \\
&s.t.\ \ \alpha_i \geq 0,\ \sum_{i=1}^{m} \alpha_i = 1; \\
&(\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots, \mathbf{a}_m)T = I; \\
&\mathbf{e}'T = -\mathbf{e}'; \\
&T \text{ is a } PH \text{ generator.}
\end{aligned} \tag{6.42}$$

 Geometrically, optimization problem (6.42) is to find a point in the polytope generated by extreme points $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ such that the objective function is either minimized or maximized.
- Because the bounds obtained in the sections "Bounds on Phase-Type Distributions" and "Extremal Phase-Type Distributions" are either partially or completely independent of the transition structure within T, they have the potential to be used in resource allocation if the transitions are affected by resources allocated to different phases.

Naturally, the preceding applications are interesting topics for future research. In addition, the issues on the distribution functions of *PH* distributions and extremal *PH* distributions raised at the end of the section "Extremal Phase-Type Distributions" are of theoretical interest for further investigation.

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AUTHOR QUERIES

AQ1. Is it necessary for "PH" to be italicized at every instance? It is generally not italicized. Please consider removing italics after the first occurrence.

AQ2. To be consistent with the other contributions in this book, the hyphen has been removed from "PH".

AQ3. First author has been considered as the corresponding author. Please check.

AQ4. Please verify "i)" here after "(3)." Can it be deleted?

AQ5. "(i)" given in sentence "For any vector **x**, it is easy to verify..." has been deleted. Please check if this is appropriate.

AQ6. Do you mean "bound is" or "bounds are"? Please revise.

AQ7. Changed "that" to "those" here. Please confirm.

AQ8. Please make sure final period is placed correctly at the end of the following equation.

AQ9. This sentence needs revising. It says "Since the vector" (where "vector" is singular) "are in ascending order," where the verb is plural. Do you mean "since the elements of the vector $-(T^*_{\downarrow})^{-1}\mathbf{e}$ are..."?

AQ10. Under what conditions? Please specify.

AQ11. Why "respectively"? Please verify and revise if necessary.

AQ12. Part 1 of what? It is not clear.

AQ13. Added "of Corollary 6.1" here. Please confirm.

AQ14. This could be revised to read better. Perhaps "The objective of optimization problem (6.42) is to find..." or "optimization problem (6.42) can be used to find...". Please consider revising.

AQ15. Please provide complete details for reference [3].

AQ16. Please update reference [7].