Methodol Comput Appl Probab
DOI 10.1007/s11009-014-9420-9

Analysis of a Multivariate Claim Process

Qi-Ming He · Jiandong Ren

Received: 8 September 2012 / Revised: 4 June 2014 / Accepted: 11 July 2014

Abstract The first part of this paper introduces a class of discrete multivariate phase-type (*MPH*) distributions. Recursive formulas are found for joint probabilities. Explicit expressions are obtained for means, variances and co-variances. The discrete *MPH*-distributions are used in the second part of the paper to study multivariate insurance claim processes in risk analysis, where claims may arrive in batches, the arrivals of different types of batches may be correlated and the amounts of different types of claims in a batch may be dependent. Under certain conditions, it is shown that the total amounts of claims accumulated in some random time horizon are discrete *MPH* random vectors. Matrix-representations of the discrete *MPH*-distributions are constructed explicitly. Efficient computational methods are developed for computing risk measures of the total claims of different types of claim batches and individual types of claims (e.g., joint distribution, mean, variance, correlation and value at risk.)

Keywords Matrix-analytic methods · Risk analysis · Markovian arrival process · Multivariate phase-type distribution.

Mathematics Subject Classification 60K

1 Introduction

Multivariate Phase-Type Distributions are introduced in Assaf et al. (1984) and Kulkarni (1989). They have found applications in many areas. For example, in insurance risk modeling, Cai and Li (2005) use it to develop a multivariate risk model in which different

Q.-M. He (✉)
Department of Management Sciences, University of Waterloo, Waterloo, ON, N2L 3G1 Canada
e-mail: q7he@uwaterloo.ca

J. Ren
Department of Statistical and Actuarial Sciences, University of Western Ontario,
London, ON, N6A 5B7 Canada

Published online: 26 July 2014

types of claims are stochastically dependent. In credit risk analysis, Herbertsson (2011) uses multivariate phase-type distributions to model default contagion in credit risk. Some other developments can be found in Rabehasaina (2009) and Zaphiropoulos and Zazanis (2010).

In the first part of the paper, we present a class of discrete multivariate phase-type (*MPH*) distributions, which generalizes the discrete univariate phase-type distributions (Neuts 1975, 1981) and is a discrete version of the continuous multivariate phase-type distributions (Assaf et al. 1984; Kulkarni 1989). Taking advantages of Markovian properties provided by an auxiliary process, we develop efficient computational methods for the joint distribution functions and the joint moments of the *MPH* distributions. According to Assaf et al. (1984), the set of continuous *MPH* is dense in the set of nonnegative multivariate distributions. The same conclusion holds for discrete *MPH*, which implies a wide variety of applications.

In the second part of this paper, we introduce a novel multivariate insurance claim process that considers $K>1$ types of claims and show that the joint distribution of the various types of losses follow the *MPH* distribution introduced in the first part of the paper. As a result, the joint distribution as well as the joint moments of the losses of different types can be efficiently evaluated.

The remainder of the paper is organized as follows. In Section 2, a class of discrete multivariate phase-type distributions is introduced. Analyses on the distributions at the batch level and individual item level are carried out in Sections 2.1 and 2.2, respectively. A multivariate insurance claim process is introduced and studied in Section 3. Efficient algorithms are developed for computing performance measures such as the value at risk. Section 4 gives a brief discussion on some extensions.

2 Discrete Multivariate Phase-Type Distributions

Similar to the continuous multivariate phase-type distributions (e.g., Kulkarni 1989), we introduce discrete multivariate phase-type distributions (*MPH*-distributions). Like the continuous case, the discrete *MPH*s are defined as the numbers of different types of batches arrived before the absorption of a discrete time Markov chain.

Assume that there are K types of items of interest, where K is a positive integer. The set of batches of interest is denoted as C^0, where a batch in C^0 is represented by a string of integers between 1 and K. For example, for $K=2$, C^0 can be {{1}} , {2}, {11}, {12}, {111}, {112}. In batch $\mathbf{h}=\{112\}$, there are two type 1 items and one type 2 item. In general, a batch in C^0 can be denoted as $\mathbf{h}=\{k_1k_2\ldots k_{|\mathbf{h}|}\}$, where $1 \le k_1 \le k_2 \le \ldots \le k_{|\mathbf{h}|} \le K$, $|\mathbf{h}|$is the number of integers in the string $\mathbf{h}$ (counting multiplicities). In other words, $|\mathbf{h}|$is the number of items in the batch $\mathbf{h}$. Note that we shall use {.} to represent a batch. For example, "{1} " represent a batch with a single item of type 1 and "1" represents a type 1 item.

Definition 2.1 Let $\{B_0,\ B_{\mathbf{h}},\ \mathbf{h} \in C^0\}$ be nonnegative matrices of order m, where m is a positive integer. Assume that $B = B_0 + \sum_{\mathbf{h}\in C^0} B_{\mathbf{h}}$ is a substochastic matrix (i.e., $B\mathbf{e} \le \mathbf{e}$). A multivariate phase-type random vector $\{ X_{\mathbf{h}},\ \mathbf{h} \in C^0\}$ can be defined as follows.

i) Define a discrete time Markov chain $\{ I_n, n \ge 0\}$ with $m+1$ phases $\{ 1, 2, \ldots, m, m+1\}$ and transition probability matrix $\begin{pmatrix} B & \mathbf{e}-B\mathbf{e} \\ 0 & 1 \end{pmatrix}$, where phase $m+1$ of the Markov chain is (θ, 1– $\theta\mathbf{e}$), where θ is a substochastic vector of order m.

ii) Define $\xi_{\mathbf{h},0} = 0$ for $\mathbf{h} \in C^0$. If $I_n = i \le m$, then at time $n+1$,

   a. for $\mathbf{h} \in C^0$, $\xi_{\mathbf{h},n+1} = \xi_{\mathbf{h},\mathrm{n}} + 1$, $\xi_{\mathbf{u},n+1} = \xi_{\mathbf{u},n}$, for $\mathbf{u} \neq \mathbf{h}$, $\mathbf{u} \in C^0$ and $I_{n+1} = j \le m$ with probability $(B_\mathbf{h})_{i,j}$;
   b. $\xi_{\mathbf{u},n+1} = \xi_{\mathbf{u},n}$, for all $\mathbf{u} \in C^0$ and$I_{n+1} = j \le m$ with probability $(B_0)_{i,j}$; and
   c. $\xi_{\mathbf{u},n+1} = \xi_{\mathbf{u},n}$, for all $\mathbf{u} \in C^0$ and $I_{n+1} = m+1$ with probability $1–(B\mathbf{e})_i$.

iii) If $I_n = m+1$, then the process is terminated. Define $X_\mathbf{h} = \xi_{\mathbf{h},n}$, for $\mathbf{h} \in C^0$.

In this paper, we also call $\{X_\mathbf{h}, \mathbf{h} \in C^0\}$ a discrete *MPH*. Intuitively, we interpret $X_\mathbf{h}$ as the number of type $\mathbf{h}$ batches arrived before the underlying Markov chain is absorbed into phase $m+1$.

We remark that the above definition of discrete time *MPH*-distributions is similar to the terminating marked Markovian arrival processes introduced and studied in He and Neuts (1998) (Also see Li (2003), Lucantoni (1991), and Neuts (1979)). A special case with one type of items is studied in Latouche et al. (2003). In fact, it was the analogy between the two types of processes that inspired us utilizing the discrete *MPH*-distributions to study multivariate claims processes in risk analysis in Section 3.

For $1 \le k \le K$, let Y_k be the total number of type k items arrived before the underlying process is absorbed into phase $m+1$. Then we have defined another discrete multivariate random vector $\{Y_k, 1 \le k \le K\}$, which can be expressed in terms of $\{X_\mathbf{h}, \mathbf{h} \in C^0\}$ as

$$Y_k = \sum_{\mathbf{h}\in C^0} |\mathbf{h}|_k X_\mathbf{h}, \quad 1 \le k \le K, \tag{2.1}$$

where $|\mathbf{h}|_k$ is the number of times that k appears in the string $\mathbf{h}$. Note that we must have $|\mathbf{h}| = \sum_{k=1}^{K} |\mathbf{h}|_k$, for $\mathbf{h} \in C^0$.

The random vectors $\{X_\mathbf{h}, \mathbf{h} \in C^0\}$ and $\{Y_k, 1 \le k \le K\}$ can be useful in insurance risk and other areas. For example, Cummins and Wiltbank (1983) commented that:

> *an organization could be subject to losses from several perils and each loss event from a given peril could result in claims of more than one type. For example, in workers' compensation insurance, a single loss event could give rise to $r_1$medical claims and $r_2$income replacement claims....*

For this case, $\{X_\mathbf{h}, \mathbf{h} \in C^0\}$ can be used to model the number of loss events that cause the claim combination $\{k_1, k_2, \ldots, k_\mathbf{h}\}$ and $\{Y_k, 1 \le k \le K\}$ represents the total number of different types of claims. Note that the arrivals of different types of claim batches can be dependent, because, for example, accidents could be more likely to occur in certain time periods than others.

Let J_n be the type of the batch arrived in period n, if there is a batch arrived; $J_n = \varphi$, otherwise. Then $X_\mathbf{h}$ and Y_k can be rewritten as follows:

$$\begin{aligned} X_\mathbf{h} &= \sum_{n=1}^{\infty} I_{\{J_n=\mathbf{h},\ I_n \le m\}}, \quad \mathbf{h} \in C^0; \\ Y_k &= \sum_{n=1}^{\infty} \sum_{\mathbf{h}\in C^0} |\mathbf{h}|_k I_{\{J_n=\mathbf{h},\ I_n \le m\}}, \quad 1 \le k \le K, \end{aligned} \tag{2.2}$$

where $I_{\{.\}}$ is the indicator function. The expressions in Eq. (2.2) make an explicit connection between Definition 2.1 and the continuous *MPH*-distribution introduced in Kulkarni (1989). This can be seen if a time period is marked as $\mathbf{h}$ when the last transition is due to $B_\mathbf{h}$, $\mathbf{h} \in C^0$.

Example 2.1 Assume that Y_1 and Y_2 are two independent discrete phase-type distributions with *PH*-representations (α_1, T_1) and (α_2, T_2) with $\alpha_1\mathbf{e} = \alpha_2\mathbf{e} = 1$, respectively. Define a discrete Markov chain with an absorption state and transition probability B given as follows:

$$B = \begin{pmatrix} T_1 \otimes T_2 & (I - T_1)\mathbf{e} \otimes T_2 & T_1 \otimes (I - T_2)\mathbf{e} & (I - T_1)\mathbf{e} \otimes (I - T_2)\mathbf{e} \\ 0 & T_2 & 0 & (I - T_2)\mathbf{e} \\ 0 & 0 & T_1 & (I - T_1)\mathbf{e} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

where "$\otimes$" is for the Kronecker product operation of matrices. Consider a case with $C^0 = \{\{1\}, \{2\}, \{12\}\}$, $m = 2$, and define random vector $(X_{\{1\}}, X_{\{2\}}, X_{\{12\}})$ to be an discrete *MPH* distribution with representation $\theta = (\alpha_1 \otimes \alpha_2, 0, 0)$,

$$B_{\{1\}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_1 \end{pmatrix}, \; B_{\{2\}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_{\{12\}} = \begin{pmatrix} T_1 \otimes T_2 & (I - T_1)\mathbf{e} \otimes T_2 & T_1 \otimes (I - T_2)\mathbf{e} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

It can be shown that $Y_1 = X_{\{12\}} + X_{\{1\}}$ and $Y_2 = X_{\{12\}} + X_{\{2\}}$. It is also easy to see that $X_{\{12\}} = \min\{Y_1, Y_2\}$. Random vectors $(X_{\{1\}}, X_{\{2\}}, X_{\{12\}})$ and (Y_1, Y_2) are both discrete *MPH*. While $\{X_{\{1\}}, X_{\{2\}}, X_{\{12\}}\}$ are dependent, $\{Y_1, Y_2\}$ are independent (by assumption).

2.1 Discrete *MPH* Random Vector $\{X_{\mathbf{h}}, \mathbf{h} \in C^0\}$

Define

$$\begin{aligned}
p_i\left(x_{\mathbf{h}},\ \mathbf{h} \in C^0\right) &= P\left\{X_{\mathbf{h}} = x_{\mathbf{h}},\ \mathbf{h} \in C^0 | I_0 = i\right\}, \quad \text{for } x_{\mathbf{h}} \ge 0,\ \mathbf{h} \in C^0,\ 1 \le i \le m; \\
\mathbf{p}\left(x_{\mathbf{h}},\ \mathbf{h} \in C^0\right) &= \left(p_i\left(x_{\mathbf{h}},\ \mathbf{h} \in C^0\right)\right)_{m\times 1}; \\
p_i^*\left(z_{\mathbf{h}},\ \mathbf{h} \in C^0\right) &= E\left[\prod_{\mathbf{h}\in C^0} z_{\mathbf{h}}^{X_{\mathbf{h}}} \middle| I_0 = i\right] \\
&= \sum_{\{x_{\mathbf{h}}\ge 0,\ \mathbf{h}\in C^0\}} \left(\prod_{\mathbf{h}\in C^0} z_{\mathbf{h}}^{x_{\mathbf{h}}}\right) p_i(x_{\mathbf{h}},\ \mathbf{h} \in C^0), \quad 1 \le i \le m; \\
\mathbf{p}^*\left(z_{\mathbf{h}},\ \mathbf{h} \in C^0\right) &= \left(p_i^*\left(z_{\mathbf{h}},\ \mathbf{h} \in C^0\right)\right)_{m\times 1},
\end{aligned} \tag{2.5}$$

for $0 \le z_{\mathbf{h}} \le 1$, $\mathbf{h} \in C^0$. Note that $\mathbf{p}\left(x_{\mathbf{h}}, \mathbf{h} \in C^0\right)$ and $\mathbf{p}^*\left(z_{\mathbf{h}}, \mathbf{h} \in C^0\right)$ are column vectors of order m.

Proposition 2.1 *For the discrete MPH-distribution defined in Definition 2.1, the conditional joint probability generating functions of* $\{X_{\mathbf{h}}, \boldsymbol{h} \in C^0\}$ *are given by*

$$\mathbf{p}^*(z_{\mathbf{h}},\ \mathbf{h} \in C^0) = \left(I - B_0 - \sum_{\mathbf{h}\in C^0} z_{\mathbf{h}} B_{\mathbf{h}}\right)^{-1} (I - B)\mathbf{e}, \quad \text{for } 0 \le z_{\mathbf{h}} \le 1,\ \mathbf{h} \in C^0. \tag{2.6}$$

where I is the identity matrix.

Proof Recall that J_n is the type of the batch arrived at time n, if there is a batch arrived; $J_n = \varphi$, otherwise. By conditioning on the first transition of the underlying Markov chain, we obtain

$$
\begin{aligned}
&p_i\left(x_{\mathbf{h}},\ \mathbf{h}\in C^0\right)\\
&= P\{X_{\mathbf{h}}=x_{\mathbf{h}},\ \mathbf{h}\in C^0|I_0=i\}\\
&= \sum_{j=1}^{m+1}\sum_{\mathbf{u}\in\{\varphi\}\cup C^0} P\{X_{\mathbf{h}}=x_{\mathbf{h}},\ \mathbf{h}\in C^0, J_1=\mathbf{u}, I_1=j|I_0=i\}\\
&= \sum_{j=1}^{m+1}\sum_{\mathbf{u}\in\{\varphi\}\cup C^0} P\{J_1=\mathbf{u}, I_1=j|I_0=i\}P\{X_{\mathbf{h}}=x_{\mathbf{h}},\ \mathbf{h}\in C^0|J_1=\mathbf{u}, I_1=j, I_0=i\}\\
&= \begin{cases}
P\{I_1=m+1|I_0=i\}\\
\quad+\sum_{j=1}^{m} P\{J_1=\varphi, I_1=j|I_0=i\}P\{X_{\mathbf{h}}=0,\mathbf{h}\in C^0|I_1=j\}, \text{ if } x_{\mathbf{h}}=0, \text{ for all } \mathbf{h}\in C^0;\\
\sum_{j=1}^{m}\sum_{\mathbf{u}\in C^0, x_{\mathbf{u}}\geq 1,} P\{J_1=\mathbf{u}, I_1=j|I_0=i\}P\{X_{\mathbf{u}}=x_{\mathbf{u}}-1, X_{\mathbf{h}}=x_{\mathbf{h}}, \mathbf{h}\neq\mathbf{u}, \mathbf{h}\in C^0|I_1=j\}\\
\quad+\sum_{j=1}^{m} P\{J_1=\varphi, I_1=j|I_0=i\}P\{X_{\mathbf{h}}=x_{\mathbf{h}}, \mathbf{h}\in C^0|I_1=j\}, \quad \text{otherwise.}
\end{cases}
\end{aligned}
\tag{2.7}
$$

Note that $P\{I_1 = m+1\ |I_0 = i\} = 1–(B\mathbf{e})_i$, $P\{\ J_1 = \varphi,\ I_1 = j\ |I_0 = i\} = (B_0)_{i,j}$ and $P\{\ J_1 =\mathbf{u},\ I_1 = j\ |I_0 = i\} = (B_{\mathbf{u}})_{i,j}$. In matrix form, Eq. (2.7) can be written as

$$
\begin{aligned}
&\mathbf{p}\left(x_{\mathbf{h}}=0,\ \mathbf{h}\in C^0\right) = (I-B)\mathbf{e} + B_0\mathbf{p}\left(x_{\mathbf{h}}=0,\ \mathbf{h}\in C^0\right);\\
&\mathbf{p}\left(x_{\mathbf{h}},\ \mathbf{h}\in C^0\right) = B_0\mathbf{p}\left(x_{\mathbf{h}},\ \mathbf{h}\in C^0\right) + \sum_{\mathbf{u}\in C^0:x_{\mathbf{u}}\geq 1} B_{\mathbf{u}}\mathbf{p}\left(x_{\mathbf{u}}-1, x_{\mathbf{h}},\ \mathbf{h}\neq\mathbf{u}, \mathbf{h}\in C^0\right).
\end{aligned}
\tag{2.8}
$$

Then Eq. (2.6) can be obtained from Eq. (2.8) by routine calculations for probability generating functions. This completes the proof of Proposition 2.1.

Equation (2.8) directly leads to the following recursive formulas for computing the conditional joint probabilities of $\{X_{\mathbf{h}}, \mathbf{h}\in C^0\}$. □

Proposition 2.2 *The conditional joint probabilities of* $\{\ X_{\mathbf{h}}, \boldsymbol{h}\in C^0\}$ *satisfies, for* $x_{\mathbf{h}}\geq 0$, $\boldsymbol{h}\in C^0$,

$$
\begin{aligned}
&\mathbf{p}\left(x_{\mathbf{h}}=0,\ \text{for all } \mathbf{h}\in C^0\right) = (I-B_0)^{-1}(I-B)\mathbf{e};\\
&\mathbf{p}\left(x_{\mathbf{h}},\ \mathbf{h}\in C^0\right) = \sum_{\mathbf{u}\in C^0:\ x_{\mathbf{u}}\geq 1} (I-B_0)^{-1}B_{\mathbf{u}}\mathbf{p}\left(x_{\mathbf{u}}-1, x_{\mathbf{h}},\ \mathbf{h}\neq\mathbf{u},\ \mathbf{h}\in C^0\right).
\end{aligned}
\tag{2.9}
$$

We remark that Eq. (2.6) can be rewritten as

$$
\mathbf{p}^*\left(z_{\mathbf{h}},\ \mathbf{h}\in C^0\right) = (I-B)\mathbf{e} + \left(B_0 + \sum_{\mathbf{h}\in C^0} z_{\mathbf{h}}B_{\mathbf{h}}\right)\left(I - B_0 - \sum_{\mathbf{h}\in C^0} z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}(I-B)\mathbf{e},
\tag{2.10}
$$

which is equivalent to

$$\mathbf{p}^*\left(z_{\mathbf{h}},\ \mathbf{h}\in C^0\right)=(I-B)\mathbf{e}+\left(B_0+\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)\mathbf{p}^*\left(z_{\mathbf{h}},\ \mathbf{h}\in C^0\right). \tag{2.11}$$

Taking derivatives of both sides of Eq. (2.11) with respect to $z_{\mathbf{u}}$ and $z_{\mathbf{v}}$, for $\mathbf{u}$, $\mathbf{v}\in C^0$, we obtain

$$\begin{aligned}
\frac{\partial\mathbf{p}^*(z_{\mathbf{h}},\ \mathbf{h}\in C^0)}{\partial z_{\mathbf{u}}}&=\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}B_{\mathbf{u}}\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}(I-B)\mathbf{e};\\
\frac{\partial^2\mathbf{p}^*(z_{\mathbf{h}},\ \mathbf{h}\in C^0)}{\partial z_{\mathbf{v}}\partial z_{\mathbf{u}}}&\\
=\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}&B_{\mathbf{v}}\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}B_{\mathbf{u}}\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}(I-B)\mathbf{e}\\
+\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}&B_{\mathbf{u}}\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}B_{\mathbf{v}}\left(I-B_0-\sum_{\mathbf{h}\in C^0}z_{\mathbf{h}}B_{\mathbf{h}}\right)^{-1}(I-B)\mathbf{e}.
\end{aligned} \tag{2.12}$$

Letting $z_{\mathbf{h}}=1$ for all $\mathbf{h}\in C^0$ in Eq. (2.12), it is easy to obtain the following result.

Proposition 2.3 *The conditional first and second moments of* $\{X_{\mathbf{h}},\mathbf{h}\in C^0\}$ *are given by*

$$\begin{aligned}
&(E[X_{\mathbf{u}}|I_0=i])_{m\times1}=(I-B)^{-1}B_{\mathbf{u}}\mathbf{e},\quad \mathbf{u}\in C^0;\\
&(E[X_{\mathbf{u}}(X_{\mathbf{u}}-1)|I_0=i])_{m\times1}=2\,(I-B)^{-1}B_{\mathbf{u}}\,(I-B)^{-1}B_{\mathbf{u}}\mathbf{e},\quad \mathbf{u}\in C^0;\\
&(E[X_{\mathbf{u}}X_{\mathbf{v}}|I_0=i])_{m\times1}=(I-B)^{-1}\left(B_{\mathbf{u}}\,(I-B)^{-1}B_{\mathbf{v}}+B_{\mathbf{v}}\,(I-B)^{-1}B_{\mathbf{u}}\right)\mathbf{e},\quad \mathbf{u}\neq\ \mathbf{v}\in C^0.
\end{aligned} \tag{2.13}$$

Once the initial distribution of the underlying Markov chain $\{I_n, n\geq 0\}$ is given, the joint probability distribution, means, variances, co-variances and coefficients of correlations of $\left\{X_{\mathbf{h}},\mathbf{h}\in C^0\right\}$ can be found by using Propositions 2.2 and 2.3. Details are omitted.

2.2 Discrete *MPH* Random Vector $\{Y_k, 1\leq k\leq K\}$

The joint probabilities, means, variances, co-variances and correlations of $\{Y_k, 1\leq k\leq K\}$ can be obtained directly from that of $\left\{X_{\mathbf{h}},\mathbf{h}\in C^0\right\}$ by using relationship (2.1). The conditional joint probability generating functions of $\{Y_k, 1\leq k\leq K\}$ are defined as $p^*_{\mathbf{Y},i}(z_1,...,z_K)=E\left[\prod_{k=1}^K z_k^{Y_k}\,\middle|\, I_0=i\right]$, $1\leq i\leq m$ and similar to Proposition 2.1, can be represented in vector form as

$$\mathbf{p}^*_{\mathbf{Y}}(z_1,z_2,...,z_K)=\left(I-B_0-\sum_{\mathbf{h}\in C^0}\left(\prod_{j=1}^{|h|}z_{k_j}\right)B_{\mathbf{h}}\right)^{-1}(I-B)\mathbf{e}. \tag{2.14}$$

The joint probabilities $\mathbf{p}_{\mathbf{Y}}(y_k, 1\leq k\leq K)=(P\{Y_k=y_k, 1\leq k\leq K\ |I_0=i\})_{m\times1}$ of $\{Y_k, 1\leq k\leq K\}$ can be obtained recursively as

$$\begin{aligned}
&\mathbf{p}_{\mathbf{Y}}(y_k=0,\ k=1,...,K)=(I-B_0)^{-1}(I-B)\mathbf{e};\\
&\mathbf{p}_{\mathbf{Y}}(y_1,...,y_K)=\sum_{\mathbf{u}\in C^0:\ y_k\geq|\mathbf{u}|_k,k=1,...,K}(I-B_0)^{-1}B_{\mathbf{u}}\mathbf{p}_{\mathbf{Y}}(y_1-|\mathbf{u}|_1,\ ...,\ y_K-|\mathbf{u}|_K).
\end{aligned} \tag{2.15}$$

By Eq. (2.1), the first and second moments of $\{Y_k, 1 \le k \le K\}$ can be obtained from that of $\{X_{\mathbf{h}}, \mathbf{h} \in C^0\}$ as follows:

$$E[Y_k] = \sum_{\mathbf{h} \in C^0} |\mathbf{h}|_k E[X_{\mathbf{h}}], \quad 1 \le k \le K;$$
$$E[Y_k Y_j] = \sum_{\mathbf{u} \in C^0} \sum_{\mathbf{h} \in C^0} |\mathbf{u}|_j |\mathbf{h}|_k E[X_{\mathbf{u}} X_{\mathbf{h}}], \quad 1 \le k, j \le K. \tag{2.16}$$

3 Risk Analysis: Multivariate Claim Processes

In this section, we introduce a multivariate insurance claim process in which K types of claims, denoted as type $1, 2, \ldots, K$, are considered. Different types of claims arrive according to a marked Markovian arrival process (He 1996; He and Neuts 1998). Each batch of claims may consist of different types of claims. For example, a claim batch can be comprised of n_1 type 1 claims and n_2 type 2 claims. In addition, the joint distribution of the sizes of claims in a batch is assumed to be a discrete *MPH*, thus allowing dependency between them. The analysis on claims can be done at both batch level and individual claim level. The model presented in this paper is quite general since Markovian arrival processes can be used to approximate general arrival processes (Asmussen and Koole 1993) and phase-type distributions can be used to approximate any distribution functions with a nonnegative support (Asmussen 2000) and (O'Cinneide 1990).

We next define the multivariate claim processes.

Assumption A The claim process is defined as follows.

A.1) The claims arrive in batches. The set of possible batches, represented by strings of integers between 1 and K and denoted as C^0, has finite many elements. The length of each element (string of integers) in C^0 is finite.

A.2) The batches of claims arrive according to continuous time marked Markovian arrival process (*MMAP*) $\{D_0, D_{\mathbf{h}}, \mathbf{h} \in C^0\}$, where $\{D_{\mathbf{h}}, \mathbf{h} \in C^0\}$ are nonnegative matrices of order m_a and D_0 is a matrix of order m_a and has negative diagonal elements and nonnegative off-diagonal elements. Let $D = D_0 + \sum_{\mathbf{h} \in C^0} D_{\mathbf{h}}$. Then D is the infinitesimal generator of the underlying continuous time Markov chain of the arrival process. We assume that the initial phase distribution of the underlying Markov chain of the arrival process is stochastic vector η (i.e., $\eta \ge 0$ and $\eta \mathbf{e} = 1$).

A.3) Let $\{X_{\mathbf{h},j}, 1 \le j \le |\mathbf{h}|\}$ be the vector of the sizes of the claims within a type $\mathbf{h}$ batch, which is assumed to follow a discrete *MPH*–distribution with matrix representation $(\beta_{\mathbf{h}}, T_{\mathbf{h},j}, 1 \le j \le |\mathbf{h}|)$ of order $m_{\mathbf{h}}$, for $\mathbf{h} \in C^0$.

A.4) The random time horizon τ has a continuous time phase-type distribution with representation (γ, C) of order m_τ.

Under these assumptions, let $S_j(\tau)$ be the total claims (or losses) of type j occurring in $[0, \tau]$, for $j = 1, 2, \ldots, K$. We define the vector of all types of losses during time interval $[0, \tau]$ by $(S_1(\tau)), \ldots, S_K(\tau))$.

Note 1: Assumptions A.2) and A.4) can be combined into a terminating *MMAP* for the claim arrival process (see Example 3.2). This approach can capture the possible dependency between the arrivals and the terminating time. All results in Sections 3.1, 3.2 and 3.3 still hold.

Note 2: The amount of claims $(S_1(t)), \ldots, S_K(t))$ in a fixed time interval $[0, t]$ can be approximated by $(S_1(\tau)), \ldots, S_K(\tau))$, where τ has an Erlang distribution with parameters $\{m, m/t\}$ for a large m (Ramaswami et al. 2008). This is because the mean of τ is t and the variance of τ is t^2/m. The random variable τ approaches constant t as m goes to infinity.

Assumption A.3) implies that the total amount of claims in a batch has a discrete *PH*-distribution with matrix representation $(\beta_{\mathbf{h}}, T_{\mathbf{h}})$, where $T_{\mathbf{h}} = T_{\mathbf{h},1+\mathrm{T}\mathbf{h},2+} \cdots + T_{\mathbf{h},|\mathbf{h}|}$. In Section 3.3, in order to analyze the amounts of claims of individual types, we further assume that $\{T_{\mathbf{h},j}, 1 \leq j \leq |\mathbf{h}|\}$ have the following structure:

$$T_{h,j} = \begin{pmatrix} 0 & \ldots & 0 & 0 & \cdots & 0 \\ T_{\mathbf{h},(j,1)} & \ldots & T_{\mathbf{h},(j,j)} & T_{\mathbf{h},(j,j+1)} & \cdots & T_{\mathbf{h},(j,|\mathbf{h}|)} \\ 0 & \ldots & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 1 \leq j \leq |\mathbf{h}|. \tag{3.1}$$

Intuitively, Eq. (3.1) indicates that phases of the underlying Markov chain are divided into non-overlapping groups, each group of phases corresponds a specific type of claims. An important special case of the distributions of claims is given in the following example.

Example 3.1 Assume that i) the size of type k claims in any batch has a discrete phase-type distribution with *PH*-representation (β_k, T_k) of order m_k, $1 \leq k \leq K$; and ii) the size of individual claims in any batch are independent random variables. By definition, a batch $\mathbf{h} = \{k_1 k_2 \ldots k_{|\mathbf{h}|}\} \in C^0$ consists of $|\mathbf{h}|$claims of the types $k_1, \ldots$ and $k_{|\mathbf{h}|}$. It is easy to see that the amounts of claims of individual types have an *MPH*-distribution with matrix representation $(\beta_{\mathbf{h}}, T_{\mathbf{h},j}, 1 \leq j \leq |\mathbf{h}|)$, where

$$\beta_{\mathbf{h}} = \left(\beta_{k_1},\ (1-\beta_{k_1}\mathbf{e})\beta_{k_2},\ (1-\beta_{k_1}\mathbf{e})(1-\beta_{k_2}\mathbf{e})\beta_{k_3}, \cdots,\ \left(\prod_{j=1}^{|\mathbf{h}|-1}(1-\beta_{k_j}\mathbf{e})\right)\beta_{k_{|\mathbf{h}|}}\right); \tag{3.2}$$

$$T_{\mathbf{h},j} = \begin{pmatrix} 0 \ldots 0 & 0 & 0 & \cdots & 0 \\ 0 \ldots 0 & T_{k_j} & (I-T_{k_j})\mathbf{e}\beta_{k_{j+1}} & \cdots & (I-T_{k_j})\mathbf{e}\left(\prod_{i=j+1}^{|\mathbf{h}|-1}(1-\beta_{k_i}\mathbf{e})\right)\beta_{k_{|\mathbf{h}|}} \\ 0 \ldots 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 1 \leq j \leq |\mathbf{h}|. \tag{3.3}$$

The total amount of claims in a batch is the sum of the amounts of individual types of claims in that batch. Then the total amount of claims in the batch is the sum of $|\mathbf{h}|$independent claims of the type $k_1, \ldots$ and $k_{\mathbf{h}}$, which is also a *PH*-distribution with matrix representation $(\beta_{\mathbf{h}}, T_{\mathbf{h}})$, where $T_{\mathbf{h}}$ has a upper triangular structure and can be given by

$$T_{\mathbf{h}} = \begin{pmatrix} T_{k_1} & (I-T_{k_1})\mathbf{e}\beta_{k_2} & (I-T_{k_1})\mathbf{e}(1-\beta_{k_2}\mathbf{e})\beta_{k_3} & \cdots & (I-T_{k_1})\mathbf{e}\left(\prod_{j=2}^{|\mathbf{h}|-1}(1-\beta_{k_j}\mathbf{e})\right)\beta_{k_{|\mathbf{h}|}} \\ 0 & T_{k_2} & (I-T_{k_2})\mathbf{e}\beta_{k_3} & \cdots & (I-T_{k_2})\mathbf{e}\left(\prod_{j=3}^{|\mathbf{h}|-1}(1-\beta_{k_j}\mathbf{e})\right)\beta_{k_{|\mathbf{h}|}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{k_{|\mathbf{h}|-1}} & (I-T_{k_{|\mathbf{h}|-1}})\mathbf{e}\beta_{k_{|\mathbf{h}|}} \\ 0 & \cdots & \cdots & 0 & T_{k_{|\mathbf{h}|}} \end{pmatrix}. \tag{3.4}$$

It is easy to see that $m_{\mathbf{h}} = \sum_{j=1}^{|\mathbf{h}|} m_{k_j}$, for $\mathbf{h} = \{k_1 k_2 \ldots k_{\mathbf{h}}\} \in C^0$. This completes Example 3.1.

Our analysis on $(S_1(\tau)), \ldots, S_K(\tau))$ consists of the following parts. From Section 3.1 to Section 3.3, we study i) the numbers of different types of batches (or individual claims) arrived in $[0, \tau]$; ii) the total amounts of claims associated with different types of batches; and iii) the total amounts of different types of claims $(S_1(\tau)), \ldots, S_K(\tau))$. For each case, we show that the random vector(s) involved has a discrete *MPH*-distribution and we construct a matrix-representation for it explicitly. In Section 3.4, a few numerical examples are presented. In Section 3.5, the value at risk is analyzed.

3.1 The Numbers of Claims: A Discrete *MPH*-Distribution

For $\mathbf{h} \in C^0$, let $N_{\mathbf{h}}(t)$ be the number of type $\mathbf{h}$ batches arrived in $[0, t]$. Then the number of type $\mathbf{h}$ batches arrived in $[0, \tau]$ is given by $N_{\mathbf{h}}(\tau)$. Let $N_k(t)$ be the number of type k claims arrived in $[0, t]$, $1 \le k \le K$. Then the number of type k claims arrived in $[0, \tau]$ is given by $N_k(\tau)$. The relationship between $\{N_{\mathbf{h}}(t), \mathbf{h} \in C^0\}$ and $\{N_k(t), 1 \le k \le K\}$ is shown in Eq. (2.1). We are interested in random vectors $\{N_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ and $\{N_k(\tau), 1 \le k \le K\}$. Since τ has a phase-type distribution by He and Neuts (1998), the random vector $\{N_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ is associated with a terminating *MMAP* and has a discrete *MPH*-distribution.

Theorem 3.1 *(Theorem 3.4,* He and Neuts 1998*)Under Assumption A, the numbers of batches arrived in the random time horizon* $\tau, \{N_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$*, have a discrete MPH-distribution with matrix representation*

$$\left\{\eta \otimes \gamma, \quad B_0 = 0, \quad B_{\mathbf{h}} = -\left(D_0 \otimes I + I \otimes C\right)^{-1} \left(D_{\mathbf{h}} \otimes I\right), \quad \mathbf{h} \in C^0\right\} \tag{3.5}$$

The orders of the matrices in the matrix representation are $m_a m_\tau$.

The joint distribution, means, variances, co-variances and correlations of $\{N_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ and $\{N_k(\tau), 1 \le k \le K\}$ can be found using the formulas given in Section 2.

3.2 The Amount of Claims of Individual Batches: A Discrete *MPH*-Distribution

For $\mathbf{h} \in C^0$, denote by $S_{\mathbf{h}}(t)$ the total amount of claims that come from type $\mathbf{h}$ batches arrived in $[0, t]$. Then $S_{\mathbf{h}}(\tau)$ is the total amount of claims that come from type $\mathbf{h}$ batches arrived in $[0, \tau]$. Note the difference between $S_1(t)$ and$S_{\{1\}}(t)$: $S_1(t)$ is the total amount of type 1 claims (which may arrive with different types of batches) in $[0, t]$, while $S_{\{1\}}(t)$ is the total amount of batch $\{1\}$ claims in $[0, t]$.

We show that the random vector $\{S_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ has a discrete *MPH*-distribution. For that purpose, denote by $\{I_{a,n}, n \ge 0\}$ the underlying Markov chain of the discrete *MPH* random vector $\{N_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ defined in Section 3.1. Denote by $\{I_{\mathbf{h},n}, n \ge 0\}$ the underlying Markov chain of the discrete *PH*–distribution $(\beta_{\mathbf{h}}, T_{\mathbf{h}})$. We introduce a new underlying Markov chain such that, after each transition in $\{I_{a,n}, n \ge 0\}$, check the type of the batch just arrived. Since $B_0 = 0$, there is always a batch arrives after each transition. If the type is $\mathbf{h}$, then the Markov chain $\{I_{\mathbf{h},n}, n \ge 0\}$ is initialized. If $\{I_{\mathbf{h},n}, n \ge 0\}$ is initialized in one of its transient phase (i.e., a non-absorption phase), the clock of the Markov chain $\{I_{a,n}, n \ge 0\}$ is stopped. When $\{I_{\mathbf{h},n}, n \ge 0\}$ enters its absorption phase, the Markov chain $\{I_{a,n}, n \ge 0\}$ takes another transition, the type of the next batch is observed and the next cycle begins. If $\{I_{\mathbf{h},n}, n \ge 0\}$ is initialized in its absorption phase (i.e., the total amount of the claims in the batch is zero), the Markov chain $\{I_{a,n}, n \geq 0\}$ continues to its next transition. If $\{I_{a,n}, n \geq 0\}$ enters its absorption phase, the whole process is terminated. It is easy to see that $S_{\mathbf{h}}(\tau)$ is the total time that a type **h** batch is observed while the underlying Markov chain coincides with $\{I_{\mathbf{h},n}, n \geq 0\}$.

The state space of the above underlying Markov chain has $m_{\text{tot}} = m_a m_\tau + m_a m_\tau \sum_{\mathbf{h}\in C^0} m_{\mathbf{h}}$ phases: the first $m_a m_\tau$ phases correspond to transitions with zero claims and the $m_a m_\tau m_{\mathbf{h}}$ phases correspond to transitions with claims from the batch **h**, for $\mathbf{h} \in C^0$.

Let $\hat{B} = \sum_{\mathbf{h}\in C^0} B_{\mathbf{h}}(1 - \beta_{\mathbf{h}}\mathbf{e})$, which corresponds to phase transitions in $\{I_{a,n}, n \geq 0\}$ if the amount of claims in the batch is zero. For such a case, the clock of the Markov chain $\{I_{a,n}, n \geq 0\}$ continues without being stopped. The Markov chain $\{I_{a,n}, n \geq 0\}$ continues until either a nonzero claim batch is observed or it enters its absorption phase. We remark that the first $m_a m_\tau$ phases of the underlying Markov chain are not necessary if $\hat{B} = 0$.

The above analysis leads to the first main result of the paper. For notational convenience, we assume that the elements in C^0 are arranged in certain order (say **u** is the first batch and **v** is the last batch), which is followed whenever the elements appear in vectors or matrices.

Theorem 3.2 *The amounts of claims* $\{S_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ *have a discrete MPH-distribution with matrix representation* $(\theta, L_0, L_{\mathbf{h}}, \mathbf{h} \in C^0)$*, where*

$$\theta = \Big((\eta \otimes \gamma)\hat{B}, \quad ((\eta \otimes \gamma)B_{\mathbf{h}}) \otimes \beta_{\mathbf{h}}, \ \ \mathbf{h} \in C^0 \Big), \tag{3.6}$$

$$L_0 = \begin{pmatrix} \hat{B} & B_{\mathbf{u}} \otimes \beta_{\mathbf{u}} & \cdots & B_{\mathbf{h}} \otimes \beta_{\mathbf{h}} & \cdots & B_{\mathbf{v}} \otimes \beta_{\mathbf{v}} \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \tag{3.7}$$

and, for $\mathbf{h} \in C^0$,

$$L_{\mathbf{h}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \hat{B} \otimes ((I - T_{\mathbf{h}})\mathbf{e}) & B_{\mathbf{u}} \otimes ((I - T_{\mathbf{h}})\mathbf{e}\beta_{\mathbf{u}}) & \cdots & I \otimes T_{\mathbf{h}} + B_{\mathbf{h}} \otimes ((I - T_{\mathbf{h}})\mathbf{e}\beta_{\mathbf{h}}) & \cdots & B_{\mathbf{v}} \otimes ((I - T_{\mathbf{h}})\mathbf{e}\beta_{\mathbf{v}}) \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. \tag{3.8}$$

Proof After a transition of the Markov chain $\{I_{a,n}, n \geq 0\}$, a type **h** batch arrives with (matrix) probability $B_{\mathbf{h}}$. The next batch has a zero amount of claims with probability $1-\beta_{\mathbf{h}}\mathbf{e}$ and a positive amount of claim with probability distribution $\beta_{\mathbf{h}}\mathbf{e}$. The total probability for a zero amount of claim is then given by $\hat{B} = \sum_{\mathbf{h}\in C^0} B_{\mathbf{h}}(1 - \beta_{\mathbf{h}}\mathbf{e})$. With probability $B_{\mathbf{h}} \otimes \beta_{\mathbf{h}}$, the Markov chain $\{I_{\mathbf{h},n}, n \geq 0\}$ is initialized in order to count the amount of claims brought in by that batch, which leads to L_0. Once the Markov chain $\{I_{\mathbf{h},n}, n \geq 0\}$ is initialized, the underlying Markov chain is governed by $I \otimes T_{\mathbf{h}}$, since during this period of time, the phase of $\{I_{a,n}, n \geq 0\}$ remains the same and the phase of $\{I_{\mathbf{h},n}, n \geq 0\}$ is governed by $T_{\mathbf{h}}$. When the counting process of the claims in batch **h** is over, with probability $(I - T_{\mathbf{h}})\mathbf{e}$, $\{I_{a,n}, n \geq 0\}$ makes its next move and the next batch is **u** with (matrix) probability $B_{\mathbf{u}}$. The counting process for the amount of claims in the batch **u** is initialized with (matrix) probability $B_{\mathbf{u}}\otimes((I - T_{\mathbf{h}})\mathbf{e}\beta_{\mathbf{u}})$. The Markov chain continues its transition without counting claims with (matrix) probability $\hat{B}$, for which case the amount of claims in the batch **h** (and all possible batches) is zero. Then the matrix $L_{\mathbf{h}}$ is obtained. The initial distribution vector θ can be obtained similarly. This completes the proof of Theorem 3.2. □

Then $(\theta, L_0, L_{\mathbf{h}}, \mathbf{h} \in C^0)$ gives a matrix representation of $\{S_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$. The joint probabilities, means, variances, co-variances and correlations of $\{ S_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ can then be computed using formulas provided in Sections 2.1. The implementation of the computational methods is straightforward. Let $p_j(n_{\mathbf{h}}, \mathbf{h} \in C^0)$ be the probability that $S_{\mathbf{h}}(\tau) = n_{\mathbf{h}}$, for all $\mathbf{h} \in C^0$, given that the underlying Markov chain is in phase j initially. Let $\mathbf{p}(n_{\mathbf{h}}, \mathbf{h} \in C^0) = (p_j(n_{\mathbf{h}}, \mathbf{h} \in C^0))_{m_{tot} \times 1}$. Then Eq. (2.9) leads to

$$\mathbf{p}\left(n_{\mathbf{h}},\ \mathbf{h} \in C^0\right) = \begin{cases} (I - L_0)^{-1}\left(I - L_0 - \sum\limits_{\mathbf{h} \in C^0} L_{\mathbf{h}}\right)\mathbf{e}, & \text{if } n_{\mathbf{h}} = 0 \ for\ \mathbf{h} \in C^0; \\ \sum\limits_{\mathbf{u} \in C^0,\ n_{\mathbf{u}} \geq 1} (I - L_0)^{-1} L_{\mathbf{u}} \mathbf{p}(n_{\mathbf{u}} - 1,\ n_{\mathbf{h}},\ \mathbf{h} \neq \mathbf{u},\ \mathbf{h} \in C^0), & \text{otherwise.} \end{cases} \tag{3.9}$$

Since the matrices L_0 and $\{L_{\mathbf{h}}, \mathbf{h} \in C^0\}$ have a special structure shown in Eqs. (3.7) and (3.8), the recursion (3.9) can be further simplified as follows. Decompose $\mathbf{p}(n_{\mathbf{h}},\mathbf{h} \in C^0)$ into $\left(\mathbf{p}_0\left(n_{\mathbf{h}}, \mathbf{h} \in C^0\right), \mathbf{p}_{\mathbf{u}}\left(n_{\mathbf{h}}, \mathbf{h} \in C^0\right), \mathbf{u} \in C^0\right)$, where vector $\mathbf{p}_0\left(n_{\mathbf{h}}, \mathbf{h} \in C^0\right)$ is of order $m_a m_\tau$ and $\mathbf{p}_{\mathbf{u}}\left(n_{\mathbf{h}}, \mathbf{h} \in C^0\right)$ is of order $m_a m_\tau m_{\mathbf{u}}$.

Corollary 3.3 *For $\boldsymbol{u} \in C^0$, we have, for $n_{\mathbf{h}} \neq 0$ for at least one $\boldsymbol{h} \in C^0$,*

$$\begin{aligned} &\mathbf{p}_0\left(n_{\mathbf{h}},\ \mathbf{h} \in C^0\right) \\ &\quad = \sum_{\mathbf{u} \in C^0:\ n_{\mathbf{u}} \geq 1} \left(\left(I - \hat{B}\right)^{-1} B_{\mathbf{u}} \otimes \beta_{\mathbf{u}}\right)\left(\hat{B} \otimes (I - T_{\mathbf{u}})\mathbf{e}\right)\mathbf{p}_0\left(n_{\mathbf{u}} - 1, n_{\mathbf{h}}, \mathbf{h} \neq \mathbf{u}, \mathbf{h} \in C^0\right) \\ &\qquad + \sum_{\mathbf{u} \in C^0:\ n_{\mathbf{u}} \geq 1} \left(\left(I - \hat{B}\right)^{-1} B_{\mathbf{u}} \otimes \beta_{\mathbf{u}}\right)(I \otimes T_{\mathbf{u}})\mathbf{p}_{\mathbf{u}}\left(n_{\mathbf{u}} - 1, n_{\mathbf{h}}, \mathbf{h} \neq \mathbf{u}, \mathbf{h} \in C^0\right) \\ &\qquad + \sum_{\mathbf{u} \in C^0:\ n_{\mathbf{u}} \geq 1} \sum_{\mathbf{v} \in C^0} \left(\left(I - \hat{B}\right)^{-1} B_{\mathbf{u}} \otimes \beta_{\mathbf{u}}\right)(B_{\mathbf{v}} \otimes (I - T_{\mathbf{u}})\mathbf{e}\beta_{\mathbf{v}})\mathbf{p}_{\mathbf{v}} \\ &\qquad \times \left(n_{\mathbf{u}} - 1, n_{\mathbf{h}}, \mathbf{h} \neq \mathbf{u}, \mathbf{h} \in C^0\right), \end{aligned} \tag{3.10}$$

and, for $\mathbf{u} \in C^0$ and $n_{\mathbf{u}} \geq 1$,

$$\begin{aligned} &\mathbf{p}_{\mathbf{u}}\left(n_{\mathbf{h}},\ \mathbf{h} \in C^0\right) \\ &\quad = \hat{B}\mathbf{p}_0\left(n_{\mathbf{u}} - 1, n_{\mathbf{h}}, \mathbf{h} \neq \mathbf{u}, \mathbf{h} \in C^0\right) + (I \otimes T_{\mathbf{u}})\mathbf{p}_{\mathbf{u}}\left(n_{\mathbf{u}} - 1, n_{\mathbf{h}}, \mathbf{h} \neq \mathbf{u}, \mathbf{h} \in C^0\right) \\ &\qquad + \sum_{\mathbf{v} \in C^0:\ n_{\mathbf{u}} \geq 1} (B_{\mathbf{v}} \otimes (I - T_{\mathbf{u}})\mathbf{e}\beta_{\mathbf{v}})\mathbf{p}_{\mathbf{v}}\left(n_{\mathbf{u}} - 1, n_{\mathbf{h}}, \mathbf{h} \neq \mathbf{u}, \mathbf{h} \in C^0\right). \end{aligned} \tag{3.11}$$

Compared to the recursion in Eq. (3.9), the recursive formulas in Eqs. (3.10) and (3.11) are involved with matrices of order less than or equal to $m_a m_\tau \max_{\mathbf{h}}\{m_{\mathbf{h}}\}$, instead of $m_a m_\tau + m_a m_\tau \sum_{\mathbf{h} \in C^0} m_{\mathbf{h}}$.

Equations (3.9), (3.10) and (3.11) can easily be interpreted probabilistically. For the initial probability distribution θ, $\theta\mathbf{p}(n_{\mathbf{h}}, \mathbf{h} \in C^0)$ gives the probability $P\{S_{\mathbf{h}}(\tau) = n_{\mathbf{h}}$, for $\mathbf{h} \in C^0\}$. In addition, the recursive algorithm for computing the joint probabilities can be further improved by utilizing the special structure in matrices $\{T_{\mathbf{h}}, \mathbf{h} \in C^0\}$ (e.g., Eq. (3.4)). Details are omitted.

An alternative matrix representation for $\{S_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ can be found by getting rid of the first $m_a m_\tau$ phases in the matrix representation given in Theorem 3.2. Doing so, the transitions of the Markov chain $\{I_{a,n}, n \geq 0\}$ that correspond to zero amount of claims are censored.

Theorem 3.4 *The amounts of claims* $\{S_{\mathbf{h}}(\tau), \boldsymbol{h} \in C^0\}$ *have a discrete MPH-distribution with matrix representation* $(\bar{\boldsymbol{\theta}}, \bar{L}_0, \bar{L}_{\mathbf{h}}, \mathbf{h} \in C^0)$*, where*

$$\bar{\boldsymbol{\theta}} = \left(\left((\eta \otimes \gamma)(I - \hat{B})^{-1} B_{\mathbf{h}}\right) \otimes \beta_{\mathbf{h}}, \quad \mathbf{h} \in C^0 \right), \tag{3.12}$$

$\bar{L}_0 = 0$, *and , for* $\boldsymbol{h} \in C^0$,

$$\bar{L}_{\mathbf{h}} = \left(\begin{array}{ccc} 0 & \cdots & \\ (I - \hat{B})^{-1} B_{\mathbf{u}} \otimes ((I - T_{\mathbf{h}})\mathbf{e}\beta_{\mathbf{u}}) & \cdots & I \otimes T_{\mathbf{h}} \\ 0 & \cdots & \end{array} \right.$$
$$\left. \begin{array}{cccc} 0 & & 0 & \cdots 0 \\ +\, (I - \hat{B})^{-1} B_{\mathbf{h}} \otimes ((I - T_{\mathbf{h}})\mathbf{e}\beta_{\mathbf{h}}) & \cdots & (I - \hat{B})^{-1} B_{\mathbf{v}} \otimes ((I - T_{\mathbf{h}})\mathbf{e}\beta_{\mathbf{v}}) & \\ 0 & \cdots & 0 & \cdots 0 \end{array} \right). \tag{3.13}$$

Proof Since $\hat{B} = \sum_{\mathbf{h} \in C^0} B_{\mathbf{h}}(1 - \beta_{\mathbf{h}}\mathbf{e})$ includes all the probabilities that the arrived batch has zero amount of claims, then $\{(I - \hat{B})^{-1} B_{\mathbf{h}}, \mathbf{h} \in C^0\}$ give the probabilities that the claims are nonzero. Then Eqs. (3.12) and (3.13) are obtained accordingly. This completes the proof of Theorem 3.4. □

It is easy to see that the joint probabilities can be computed using Eqs. 3.10 and 3.11 with minor modifications.

Corollary 3.5 *The total amount of claims (i.e., the sum of* $\{S_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$*) has a discrete PH-distribution with matrix representation* $\left(\theta,\ L_0,\ \sum_{\mathbf{h} \in C^0} L_{\mathbf{h}}\right)$ *or* $\left(\bar{\theta},\ 0,\ \sum_{\mathbf{h} \in C^0} \bar{L}_{\mathbf{h}}\right)$.

3.3 Multivariate Claims of Individual Types: A Discrete *MPH*-Distribution

Now, we are ready to consider the random vector $(S_1(\tau)), \ldots, S_K(\tau))$. An important special case is $C^0 = \{\{1\}, \{2\}, \ldots, \{K\}\}$, where there is no difference between the batches of claims and the types of claims. For the general case, by definition, $S_{\mathbf{h}}(\tau)$ is the sum of the amount of claims of individual types included in the string $\mathbf{h}$. With the structure of $(\beta_k, T_{\mathbf{h},j}, 1 \leq j \leq |\mathbf{h}|)$ given in Eq. (3.1), we decompose matrices $\{L_{\mathbf{h}}, \mathbf{h} \in C^0\}$ as follows: $L_{\mathbf{h}} = \sum_{j=1}^{|\mathbf{h}|} L_{\mathbf{h},j}$, where $L_{\mathbf{h},j}$ corresponding to transitions associated with the j-th claim in the batch $\mathbf{h}$. Then $L_{\mathbf{h},j}$ is obtained by replacing $I - T_{\mathbf{h}}$ by $\text{diag}(0, I, 0) - T_{\mathbf{h},j}$ in Eq. (3.8). Note that diag(0, I, 0) is a matrix with a (block) diagonal structure and diagonal blocks are 0, I and 0. Also note that we have $I - T_{\mathbf{h}} = \Sigma_j(\text{diag}(0,I,0) - T_{\mathbf{h},j})$. Define

$$L_k = \sum_{\mathbf{h}=\{k_1 \ldots k_{|\mathbf{h}|}\} \in C^0} \ \sum_{j=1:\, k_j = k}^{|\mathbf{h}|} L_{\mathbf{h},j}, \quad 1 \leq k \leq K \tag{3.14}$$

It is easy to see that L_k corresponds to the amount of type k claims from all batches, $1 \leq k \leq K$. By Theorem 3.2, we obtain

Theorem 3.6 *The random vector* $(S_1(\tau)), \ldots, S_K(\tau))$ *has a discrete MPH-distribution with matrix representation* $(\theta, L_0, L_1, L_2, \ldots, L_K)$*, where* θ *is defined in* Eq. (3.6), L_0 *is defined in* Eq. (3.7) *and* $\{L_1, L_2, \ldots, L_K\}$ *are defined in* Eq. (3.14).

Similarly, we can obtain another *MPH*-representation of $(S_1(\tau)), \ldots, S_K(\tau))$ by utilizing the *MPH*-representation of $\{S_{\mathbf{h}}(\tau), \mathbf{h} \in C^0\}$ given in Theorem 3.4. Details are omitted.

Computation of the joint probabilities, means, variances, covariance and correlations of $(S_1(\tau)), \ldots, S_K(\tau))$ can be done accordingly. Details are omitted.

3.4 Examples

Example 3.2 We consider a model with $K = 2$ and $C^0 = \{\{1\}, \{2\}, \{12\}\}$. Claims arrive in batches according to continuous time *MMAP* with $m_a = 3$, $\eta = (0.5, 0, 0.5)$,

$$D_0 = \begin{pmatrix} -5 & 2 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & -10 \end{pmatrix},\ D_{\{1\}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ D_{\{2\}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix},\ D_{\{12\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

Amounts of claims of type 1 and type 2 have independent discrete phase-type distributions with *PH*-representations (β_k, T_k), respectively, where

$$\beta_1 = (0.4,\ 0.6),\ T_1 = \begin{pmatrix} 0.2 & 0.3 \\ 0.5 & 0 \end{pmatrix}; \quad \beta_2 = (0.8,\ 0.2),\ T_2 = \begin{pmatrix} 0.1 & 0.5 \\ 0.1 & 0.2 \end{pmatrix}. \tag{3.16}$$

The time horizon τ has a continuous time phase-type distribution with *PH*-representation (γ, C):

$$\gamma = (0.5,\ 0.5),\ C = \begin{pmatrix} -2 & 1 \\ 1 & -5 \end{pmatrix}. \tag{3.17}$$

Using the methods developed in Sections 3.1, 3.2 and 3.3, moments are obtained as

The means

$E[N_{\{1\}}(\tau)]$	$E[N_{\{2\}}(\tau)]$	$E[N_{\{12\}}(\tau)]$	$E[N_1(\tau)]$	$E[N_2(\tau)]$
3.8098	1.8098	0.3152	4.1250	2.125
$E[S_{\{1\}}(\tau)]$	$E[S_{\{2\}}(\tau)]$	$E[S_{\{12\}}(\tau)]$	$E[S_1(\tau)]$	$E[S_2(\tau)]$
7.2987	3.0508	1.1099	7.9292	3.5302

Covariance is given in the following matrices:

$$\begin{aligned} Covar(N_{\{1\}}(\tau), N_{\{2\}}(\tau), N_{\{12\}}(\tau)) &= \begin{pmatrix} 76.30 & -2.73 & 0.76 \\ -2.73 & 10.69 & 0.40 \\ 0.76 & 0.40 & 0.36 \end{pmatrix}; \\ Covar(N_1(\tau), N_2(\tau)) &= \begin{pmatrix} 78.19 & -1.19 \\ -1.19 & 11.87 \end{pmatrix}. \end{aligned} \tag{3.18}$$

$$\begin{aligned} Covar(S_{\{1\}}(\tau), S_{\{2\}}(\tau), S_{\{12\}}(\tau)) &= \begin{pmatrix} 302.68 & -7.46 & 6.41 \\ -7.46 & 36.32 & 2.99 \\ 6.41 & 2.99 & 5..82 \end{pmatrix}; \\ Covar(S_1(\tau), S_2(\tau)) &= \begin{pmatrix} 310.67 & -1.18 \\ -1.18 & 40.41 \end{pmatrix}. \end{aligned}$$

Note that both the numbers and amounts of type 1 and type 2 claims are negatively correlated. The joint distribution of S_1 and S_2 are illustrated in the following table.

The joint distribution of $\{S_1(\tau), S_2(\tau)\}$

$S_1(\tau)\backslash S_2(\tau)$	0	1	2	3	4	5	6	7
0	0.0438	0.0372	0.0295	0.0253	0.0213	0.0181	0.0154	0.0131
1	0.0180	0.0088	0.0054	0.0040	0.0032	0.0027	0.0023	0.0020
2	0.0129	0.0052	0.0037	0.0028	0.0022	0.0018	0.0016	0.0014
3	0.0101	0.0032	0.0025	0.0019	0.0015	0.0013	0.0011	0.0009
4	0.0085	0.0021	0.0018	0.0014	0.0011	0.0009	0.0008	0.0007
5	0.0075	0.0015	0.0013	0.0010	0.0008	0.0007	0.0006	0.0005
6	0.0069	0.0011	0.0011	0.0008	0.0006	0.0005	0.0004	0.0004
7	0.0064	0.0009	0.0009	0.0006	0.0005	0.0004	0.0004	0.0003

In Example 3.2, the claim arrival process (Assumption A.2) and the terminating time (Assumption A.4) are independent. In the next example, the claim arrival process and the terminating time are not independent and are defined by a single discrete *MPH*.

Example 3.3 Let the environment be described by a two-state terminating discrete-time Markov Chain with two phases: N (stand for normal) and R (stand for risky). The transition probabilities before termination are given by

$$P = \begin{matrix} N \\ R \end{matrix} \begin{pmatrix} 0.84 & 0.1 \\ 0.8 & 0.16 \end{pmatrix}. \tag{3.19}$$

Suppose that the auto accident rates in two areas A and B are both affected by the underlying environment. An auto accident can cause property loss claims (type 1), bodily injury claims (type 2), or both (type 12). Consider a model in which claims (losses) arrive according to the following discrete *MPH* with $C^0 = \{\{A_1\}, \{A_2\}, \{A_{12}\}, \{B_1\}, \{B_2\}, \{B_{12}\}\}$.

$$\begin{aligned} &\theta = (0.2,\ 0.8), \quad B_0 = \begin{pmatrix} 0.6 & 0.1 \\ 0.8 & 0.1 \end{pmatrix}, \\ &B_{\{A_1\}} = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.02 \end{pmatrix},\ B_{\{A_2\}} = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.01 \end{pmatrix},\ B_{\{A_{12}\}} = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.01 \end{pmatrix}, \\ &B_{\{B_1\}} = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.01 \end{pmatrix},\ B_{\{B_2\}} = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.005 \end{pmatrix},\ B_{\{B_{12}\}} = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.005 \end{pmatrix}. \end{aligned} \tag{3.20}$$

Assume the sizes of claims (losses) of individual accidents have the following discrete *MPH*-distributions:

$$\begin{aligned} &\beta_{\{A_1\}} = (1), \quad T_{\{A_1\}} = (0.1); \\ &\beta_{\{A_2\}} = (0.9), \quad T_{\{A_2\}} = (0.05); \\ &\beta_{\{A_{12}\}} = (0.2,\ 0.8), \quad T_{\{A_{12}\},1} = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}, \quad T_{\{A_{12}\},2} = \begin{pmatrix} 0 & 0 \\ 0.1 & 0.05 \end{pmatrix}; \\ &\beta_{\{B_1\}} = (1), \quad T_{\{B_1\}} = (0.15); \\ &\beta_{\{B_2\}} = (0.9), \quad T_{\{B_2\}} = (0.05); \\ &\beta_{\{B_{12}\}} = (0.4,\ 0.4), \quad T_{\{B_{12}\},1} = \begin{pmatrix} 0.1 & 0.05 \\ 0 & 0 \end{pmatrix}, \quad T_{\{B_{12}\},2} = \begin{pmatrix} 0 & 0 \\ 0.01 & 0.05 \end{pmatrix}. \end{aligned} \tag{3.21}$$

Note that if an accident causes both type 1 and type 2 claims, then the claim sizes of the two types are dependent, as indicated by $\{T_{\{A_{12}\},1},\ T_{\{A_{12}\},2},\ T_{\{B_{12}\},1},\ T_{\{B_{12}\},2}\}$. Denote by $S_{A_1}(\tau)$, $S_{A_2}(\tau)$, $S_{A_{12}}(\tau)$, $S_{B_1}(\tau)$, $S_{B_2}(\tau)$ and $S_{B_{12}}(\tau)$ the amounts of claims corresponding to accident types A_1, A_2, A_{12}, B_1, B_2 and B_{12}, respectively. We analyze i) the amounts of claims of individual accidents (batches) $\{S_{A_1}(\tau), S_{A_2}(\tau), S_{A_{12}}(\tau), S_{B_1}(\tau), S_{B_2}(\tau), S_{B_{12}}(\tau)\}$; ii) amounts of claims from areas $A$and B and iii) amounts of claims of types 1 and 2.

For part i), by Theorem 3.2 or 3.4, the random vector $(S_{A_1}(\tau), S_{A_2}(\tau), S_{A_{12}}(\tau), S_{B_1}(\tau), S_{B_2}(\tau), S_{B_{12}}(\tau))$ has a discrete *MPH*-distribution with matrix representation, say, (θ, L_0, L_{A_1}, L_{A_2}, $L_{A_{12}}$, L_{B_1}, L_{B_2}, $L_{B_{12}}$). Using the formulas given in Section 2, the means and coefficients of correlations of the losses associated with accidents are found as follows.

$$
\begin{aligned}
&\text{Mean}(S_{A_1}(\tau),\ S_{A_2}(\tau),\ S_{A_{12}}(\tau),\ S_{B_1}(\tau),\ S_{B_2}(\tau),\ S_{B_{12}}(\tau))\\
&\quad = (0.0101,\ 0.0035,\ 0.0067,\ 0.0051,\ 0.0018,\ 0.0021);\\
&\text{Correlation}(S_{A_1}(\tau),\ S_{A_2}(\tau),\ S_{A_{12}}(\tau),\ S_{B_1}(\tau),\ S_{B_2}(\tau),\ S_{B_{12}}(\tau))\\
&\quad = \begin{pmatrix} 1.0000 & 0.0898 & 0.0844 & 0.0895 & 0.0645 & 0.0574 \\ 0.0898 & 1.0000 & 0.0622 & 0.0655 & 0.0469 & 0.0421 \\ 0.0844 & 0.0622 & 1.0000 & 0.0616 & 0.0447 & 0.0395 \\ 0.0895 & 0.0655 & 0.0616 & 1.0000 & 0.0471 & 0.0419 \\ 0.0645 & 0.0469 & 0.0447 & 0.0471 & 1.0000 & 0.0302 \\ 0.0574 & 0.0421 & 0.0395 & 0.0419 & 0.0302 & 1.0000 \end{pmatrix}.
\end{aligned} \tag{3.22}
$$

For part ii), the analysis on $S_A(\tau)$ and $S_B(\tau)$ can be done by using: $S_A(\tau) = S_{A_1}(\tau) + S_{A_2}(\tau) + S_{A_{12}}(\tau)$ and $S_B(\tau) = S_{B_1}(\tau) + S_{B_2}(\tau) + S_{B_{12}}(\tau)$. Then $(S_A(\tau), S_B(\tau))$ has a bivariate *PH*-distribution with matrix representation (θ, L_0, $L_{A_1}+L_{A_2}+L_{A_{12}}$, $L_{B_1}+L_{B_2}+L_{B_{12}}$). The means and coefficients of correlations are:

$$
\begin{aligned}
&\text{Mean}\,(S_A(\tau),\ S_B(\tau)) = (0.0203,\ 0.0089);\\
&\text{Correlation}\,(S_A(\tau),\ S_B(\tau)) = \begin{pmatrix} 1 & 0.1556 \\ 0.1556 & 1 \end{pmatrix}.
\end{aligned} \tag{3.23}
$$

For part iii), $S_1(\tau) = S_{A_1}(\tau) + S_{A_{12}}(\tau) + S_{B_1}(\tau) + S_{B_{12}}(\tau)$ and $S_2(\tau) = S_{A_2}(\tau) + S_{A_{12}}(\tau) + S_{B_2}(\tau) + S_{B_{12}}(\tau)$. The means and coefficients of correlations can be found from that of $(S_{A_1}(\tau), S_{A_2}(\tau), S_{A_{12}}(\tau), S_{B_1}(\tau), S_{B_2}(\tau), S_{B_{12}}(\tau))$.

$$
\begin{aligned}
&\text{Mean}(S_1(\tau),\ S_2(\tau)) = (0.0194,\ 0.0097);\\
&\text{Correlation}(S_1(\tau),\ S_2(\tau)) = \begin{pmatrix} 1 & 0.1977 \\ 0.1977 & 1 \end{pmatrix}.
\end{aligned} \tag{3.24}
$$

Note that both the claim amounts from different areas and of different types are positively correlated. According to Proposition 2.2, the joint distributions of the above random variables can be calculated recursively. Details are omitted.

3.5 Value at Risk

Assume that the premium rate of type k insurance is μ_k; and the initial reserve for type k insurance is υ_k, k =1, 2, ..., K. The operating losses at time t for type k insurance can be

Table 1 Values at risk for $\alpha = 0.95$ and $E[\tau] = 2$

m	5	10	20	30	40	50	60	70	80	90	100	110
$Var(\tau)$	0.8	0.4	0.2	0.13	0.1	0.08	0.067	0.057	0.05	0.044	0.04	0.037
VaR_1	59	43	34	30	29	27	27	26	26	25	25	25
VaR_2	−20	−21	−21	−21	−21	−22	−22	−22	−22	−22	−22	−22
$\text{VaR}_{\text{total}}$	4	−13	−23	−27	−29	−30	−31	−32	−32	−33	−33	−33

defined as $S_k(t)–\mu_k t–v_k$. Given a confidence level $\alpha \in (0, 1)$ and a time horizon t, we define the *value at risk* (VaR) for type k insurance as

$$\text{VaR}_k(\alpha, t) = \inf\{x : \quad P\{S_k(t) - \mu_k t - v_k \le x\} \ge \alpha\}, \quad 1 \le k \le K. \tag{3.25}$$

The value at risk for the total claim $S_1(t)+\ldots+S_K(t)$ can be defined similarly.

We remark that if the VaR value is positive, then the probability that the premiums and the initial reserve cannot cover the losses is greater than the acceptable level $1–\alpha$.

As pointed out in Note 2, the distribution of $(S_1(t)), \ldots, S_K(t))$ for fixed t can be approximated by that of $(S_1(\tau)), \ldots, S_K(\tau))$, where τ *has* an Erlang distribution with parameters $\{m, m/t\}$ and m is a positive integer. The latter quantity can be computed using Proposition 2.2. The total claim $S_1(t)+\ldots+S_K(t)$ can be approximated by $S_1(\tau)+\ldots+ \quad S_K(\tau)$, with the latter having a discrete *PH*-distribution so the value at risk can be computed. Note that since Proposition 2.2 provided an algorithm for the joint distribution of $(S_1(\tau)), \ldots, S_K(\tau))$, one may actually evaluate the VaR of any combinations of the K types of losses.

Example 3.4 (Example 3.2 continued) We choose $\alpha = 0.95, t = 2, \mu_1 = 10, v_1 = 50, \mu_2 = 15$, and $\nu_2 = 30$. For the Erlang approximation, we choose $m = 5, 10, 20, \ldots,$ 100 and 110. For Example 3.2, the values at risk for different type of claims are computed and presented in the following table.

First, we comment that the values at risk seem to converge for each case, which validates the approximation approach. Second, the numerical results show that for type 2 insurance, the value at risk is negative. Thus, there is at least 95 % chance that the insurance company can use its initial capital and premium income to cover the potential losses. However, this is not true for type 1 insurance. Nevertheless, if the two are combined, the initial capital and premium income may cover all the potential claims with probability 0.95. This shows the benefit of diversification in insurance management. Third, when m becomes larger, the variance of the terminating time is smaller. Table 1 shows that, if the terminating time is less variable, the value at risk is smaller, which is consistent with intuition.

4 Discussion

The analysis in this paper can be extended easily to the following two modified models. The first modified model is obtained from the one defined in Section 3 by changing the arrival process and the time horizon from continuous time to discrete time. The analysis of the new model is almost parallel to that of Section 3. The second modified model can be obtained by changing the claims from discrete distributions to continuous distributions. For this case, the total amounts of claims are continuous. The analysis is similar to that of Section 3 with minor modifications.

References

Asmussen S (2000) Ruin probabilities. World Scientific Publishing

Asmussen S, Koole G (1993) Marked point processes as limits of Markovian arrival streams. J Appl Probab 30:365–372

Assaf D, Langberg NA, Savits TH, Shaked M (1984) Multivariate phase-type distributions. Oper Res 32(3):688–702

Cai J, Li HJ (2005) Multivariate risk model of phase type. Insurance Math Econom 36:137–152

Cummins DJ, Wiltbank LJ (1983) Estimating the total claims distribution using multivariate frequency and severity distributions. J Risk ns 50(3):377–403

He QM (1996) Queues with marked customers. Adv Appl Probab 28:567–587

He QM, Neuts MF (1998) Markov chains with marked transitions. Stoch Process Appl 74(1):37–52

Herbertsson A (2011) Modelling default contagion using multivariate phase-type distributions. Rev Deriv Res 14(1):1–36

Li HJ (2003) Association of multivariate phase-type distributions with applications to shock models. Stat Probabil Lett 64:1043–1059

Kulkarni VG (1989) A new class of multivariate phase type distributions. Oper Res 37(1):151–158

Latouche G, Remiche M-A, Taylor P (2003) Transit Markov arrival processes. Ann Appl Probab 13(2):628–640

Lucantoni DM (1991) New results on the single server queue with a batch Markovian arrival process. Stoch Model 7:1–46

Neuts MF (1975) Probabie type. In: Liber Amicorum Prof. Emeritus H. Florin. University of Louvain, pp 173–206

Neuts MF (1979) A versatile Markovian point process. J Appl Probab 16:764–779

Neuts MF (1981) Matrix-geometric solutions in stochastic models – an algorithmic approach. The Johns Hopkins University Press, Baltimore

O'Cinneide CA (1990) On the limitations of multivariate phase-type families. Oper Res 38(3):519–526

Rabehasaina L (2009) Risk processes with interest force in Markovian environment. Stoch Model 25(4):580–613

Ramaswami V, Woolford GD, Stanford AD (2008) The erlangization method for Markovian fluid flows. Ann Oper Res 160:215–225

Zaphiropoulos G, Zazanis MA (2010) Discrete-time risk processes with after-effects and association. Stoch Model 26(1):27–45