Matrix-Analytic Methods – An Algorithmic Approach to Stochastic Modelling and Analysis

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Abstract The field of matrix analytic methods (MAM) was pioneered by Dr. Marcel F. Neuts in the middle of the 1970s for the study of queueing models. In the past 40 years, the theory on MAM has been advanced in parallel with its applications significantly.

Matrix-analytic methods contain a set of tools fundamental to the analysis of a family of Markov processes rich in structure and of wide applicability. Matrix-analytic methods are extensively used in science, engineering, and statistics for the modelling, performance analysis, and design of computer systems, telecommunication networks, network protocols, manufacturing systems, supply chain management systems, risk/insurance models, etc.

1 Introduction

The field of matrix analytic methods (MAM) was pioneered by Dr. Marcel F. Neuts in the middle of the 1970s for the study of queueing models. In the past 40 years, the theory on MAM has been advanced in parallel with its applications significantly (See Neuts [20, 21], Latouche and Ramaswami [14], He [11]).

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2 Basic Theory and Tools

The fundamental ideas of MAM are (i) The use of *phase* in stochastic modeling and analysis; and (ii) The *algorithmic* approach to stochastic analysis. Based on the basic ideas, a set of fundamental tools was introduced and developed in MAM: (1) Phase-type (PH) distributions; (2) Markovian arrival processes (MAP); (3) Paradigms of structured Markov processes (e.g., QBD, M/G/1, and GI/M/1 paradigms); and (4) Markov modulated fluid flow (MMFF) processes.

1. *PH-distributions* were introduced by Marcel Neuts in 1975 as a generalization of the exponential distribution for the study of queueing models. A PH-distribution is defined as the probability distribution of the absorption time in a finite state Markov chain. This class of distributions can approximate any distribution with nonnegative support, which implies the wide scope of applications of PH-distributions. This class of distributions possesses the partial memoryless property, which leads to analytically and numerically tractable models for a variety of stochastic systems. Applications of PH-distributions have gone far beyond queueing theory to fields such as biology and statistics. See Neuts [18], O'Cinneide [22], Buchholz et al. [6], He [11] for details and references.
2. *MAP*s were introduced by Marcel Neuts in 1979, as a generalization of the Poisson process. An MAP is a counting process defined by counting marked transitions in a finite state Markov chain. This class of counting processes can approximate any stochastic counting process, which implies the wide applicability of MAPs. Similar to PH-distributions, MAPs have a wide scope of applications and lead to analytically and numerically tractable models for stochastic systems. See Neuts [19], Asmussen and Koole [1], Buchholz et al. [6], He [11] for details and references.

   For both PH-distributions and MAPs, parameter estimation and fitting are fundamental to their applications. Moment matching and expectation-maximization (EM) algorithms have been developed for that purpose. See Asmussen et al. [3] and Buchholz et al. [6] for details and references.
3. The *structured Markov processes* of interest include the *QBD, M/G/1, and GI/M/1* paradigms. The QBD processes are generalizations of the birth-and-death process. Two fundamental matrices, R and G, with explicit probabilistic interpretations are introduced. Using those matrices, *matrix-geometric and matrix-exponential solutions* have been obtained for the transient and stationary solutions of QBD processes. Numerically robust and efficient algorithms have been developed for computing various quantities. The M/G/1 and GI/M/1 paradigms are generalizations of the QBD paradigm, and contain Markov processes skip-free to either the left (down) or right (up). Similar to the QBD case, matrix-geometric/exponential solutions have been found for those generalizations. In addition, matrix-exponential models and Markov models with a tree structure are introduced as further generalizations of those paradigms. See Neuts [20, 21], Latouche [13], Latouche and Ramaswami [14], Sengupta [25], ~~and~~ Asmussen and Bladt (1996), and He [11] for details and references.

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4. MAM was used in the analysis of *MMFF*s in 1990's (Asmussen [2] and Ramaswami [24]). MAM injected new ideas and solution approaches in the study of MMFFs and their applications in queueing theory and risk analysis, which lead to more efficient algorithms for computing quantities and extensions to much more sophisticated models. While Markov chains, such as birth-and-death processes, QBD, M/G/1, and GI/M/1, can be used effectively in the analysis of stochastic models with a relatively simple structure, MMFFs are an indispensable tool for the investigation of some complicated systems. See Bean et al. [4], da Silva Soares and Latouche [8], and Latouche and Nguyen [15] for more details and references.

3 Applications

Since the 1970's, MAM has found applications in many areas.

1. MAM started from research in *queueing theory*. Today, MAM has been used successfully in the investigation of a variety of queueing models, including but not limited to retrial queues, vacation queues, queues with customer abandonments, feedback queue, queues with server repairs, and multi-server queues. In such studies, PH-distributions are usually used to model the service times, MAPs are used to model the customer arrival processes, and (embedded) QBD, M/G/1, and GI/M/1 types of Markov chains are used to describe the dynamics of the stochastic systems at specifically selected epochs. Matrix-geometric or matrix-exponential solutions are utilized for finding various queueing quantities of interest. For most of those cases, quantities of queueing interest can be computed efficiently. See Ramaswami [23], Neuts [20, 21], Lucantoni et al. [16], He [9], Takine [26] Kroese et al. [12], Miyazawa and Zhao [17], and Xia et al. [27] for details and references.
2. MAM found applications in *reliability theory*. PH-distributions can be used to model the times to failure of components in a reliability system. MAPs can be used to model the arrivals of events such as shock-waves. For many reliability systems, the closure properties of PH-distributions and MAPs can be used to obtain various system performance quantities such as system time to failure, system availability, and system failure rate.
3. Due to its similarity to queueing systems, many *inventory and supply chain models* can be analyzed by MAM. MAM can be used not only to analyze such stochastic systems, but also to develop algorithms for computing the optimal control policy. This expands MAM's application significantly. See Chen and Song [7] and He [11] for details and references.
4. *Risk and insurance analysis* is another area where MAM finds ample applications, especially after MAM is applied to MMFFs. Similar to queueing applications, MAM finds closed-form solutions for many classical risk models and quantities. Algorithms have been developed for computing many quantities

efficiently. This is the most active area of research in MAM in the last decade. See Asmussen [2] and Badescu and Landriault [5] for details and references.

4 Summary

MAM has been proven to be a set of powerful tools in stochastic models in the past 40 years. The interest in MAM has been growing. While there is still potential for MAM to grow in its totality, some of its components such as *PH-distributions* and *MAPs* stand out independently as useful tools in many branches of science and engineering. MAM, which takes advantages of the ever-increasing computing power, will continue to find new applications in science and engineering (e.g., machine/deep learning and artificial intelligence (AI)).

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AQ1. Please provide details for Asmussen and Bladt (1996) in reference list.
AQ2. Ref. [10] was not cited anywhere in the text. Please provide in text citation or delete the reference from the reference list.