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Incentive compatibility in prediction markets: Costly actions and external incentives

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ABSTRACT

We consider a multi-round prediction market in which two agents, Alice and Bob, are trading on an event on which each may take action to influence its outcome. The existing literature assumes that there is no net difference between the costs of different actions the agents may take outside the prediction market when external incentives exist. For example, the cost for Alice to work hard to complete the project is the same as it is for her to “loaf” and not work hard. In this work we consider first a two-round and later a four-round setting in which the agents’ costs of external actions differ between actions. We show that a prediction market is incentive-compatible when external action costs differ as long as they remain within a proper range, regardless of the initial market estimate, something that is not shown in the existing literature.

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1. Background introduction

Prediction markets are created for the purpose of aggregating information from individuals about uncertain events of interest. For example, a prediction-market-like mechanism is known to predict the financial value of soccer players in transfer markets better than other standard forecasting methods (Peeters, 2018). However, some decision makers are suspicious about the use of prediction markets in certain settings, such as corporate settings. In corporate settings, many managers are concerned that workers may have a greater incentive to take adverse actions within the workplace when a prediction market is present than when it is not. One public event in which external actions may influence a traded event is an election. Depending on the stakes within a prediction market, some voters may change their votes in order to maximize their total returns, both within and outside the prediction market. However, the preceding argument, though theoretically valid, has not yet been shown to hold any merit in practice. In fact, prediction

markets are used extensively for eliciting and aggregating opinions on political events (Graefe, Armstrong, Jones, & Cuzán, 2014; Hanea et al., 2017; Rothschild, 2015). It is generally assumed that though agents who participate in prediction markets by trading may have superior information about the relevant event, they have no direct control over the event outcome (e.g., one individual vote is not likely to sway an entire election). However, prediction markets are often used in situations in which this assumption is violated (Chakraborty & Das, 2016) (e.g., student evaluations). This study considers the impact of external incentives on the efficacy of prediction markets, especially when it requires costly actions to act on external incentives. The prediction markets that are deployed in corporate settings consist of a market maker, a center with which all participants, or *agents*, trade, that is present in order to facilitate trade and boost the market liquidity. In our study, the market maker, who is also the market participants’ employer, does not want agents to take undesirable actions at work that may impact the outcome of the traded event.

There is substantial evidence that prediction markets can help to produce forecasts of event outcomes that have lower prediction errors than conventional forecasting methods (Arrow et al., 2008). However, it is also possible

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that agents might bluff and deceive other players by not revealing their true beliefs, hoping to correct the prediction probability later and benefit in the market as a result (Chen et al., 2010; Dimitrov & Sami, 2008). In addition to bluffing in order to maximize the prediction market payoff, an agent may also change her behavior outside the prediction market to maximize her reward both within and outside the market. For example, employees might have an incentives to “slack off” when working on a project just because their prediction market position is favorable to the project being delayed and they would somehow benefit more in the prediction market by working less hard in the workplace. This may seem like a far-fetched idea, but not long ago the U.S. congress was worried that terrorists might have a higher incentive to actually conduct terrorist attacks if they could trade in a relevant prediction market (Looney, 2004). Eventually, these proposed “terrorist markets” were shut down by the U.S. congress due to these concerns. There are additional canonical real-world examples which show that having agents trade in a prediction market distorts their incentives both within and outside the prediction market (Chakraborty & Das, 2014), such as instructor prediction markets (Chakraborty, Das, Lavoie, Magdon-Ismail, & Naamad, 2013). In fact, there is no *ex post* way of determining how the presence of a prediction market changes agents’ probability estimates, without considering the equilibrium strategies of agents within prediction markets. In this study, we use equilibrium analysis to see whether prediction markets may indeed cause deviant agent behavior when agents’ external actions to the market are costly.

In conducting an equilibrium analysis in the presence of external incentives in prediction markets, we find that when external actions have asymmetric costs (that is, one action costs more or less than another), these asymmetries may actually lead agents to behave truthfully. Specifically, we determine the equilibrium strategies for two agents in first a two-round and later a four-round setting, where the agents trade in a prediction market with a final value that is contingent on a traded event, on the likelihood of which the same two agents have a direct impact. We prove that equilibrium strategies lead participants to always take desirable/undesirable actions in relation to the project (work hard or “loaf”) and be truthful when reporting in the prediction market, just as they would have done had the prediction market not existed. Our result is important in the face of decision makers’ concerns regarding prediction markets potentially inducing undesirable actions.

The remainder of this document is arranged as follows. We introduce related work in Section 2 and define our general model in Section 3. We next show that when the cost of exerting high effort is positive, agents do not work hard, but are truthful in the prediction market (Section 4.2); similarly, we show that when the cost of exerting high effort is negative, agents work hard and are still truthful in the prediction market (Section 4.3). However, Section 4.4 then shows that when there is no cost for efforts, there is a possibility that agents will not always work hard and will not report truthfully in the prediction market. Section 4.5 extends our two-round setting to four rounds and proves that our conclusion still holds. Section 5 discusses future research directions and concludes the paper.

2. Literature review

A large proportion of the research on prediction markets and scoring rules, which forms a building block for the prediction markets that we consider in our paper, does not consider outside incentives. Brier (1950) defines the quadratic scoring rule, assuming implicitly that agents cannot influence the outcome of the predicted event (tomorrow’s weather). Bickel (2007) compares three commonly-used strictly proper scoring rules, namely the quadratic, spherical and logarithmic scoring rules, and shows that the logarithmic scoring rule is preferred when one needs to account for non-linear utilities and rank ordering. Merkle and Steyvers (2013) demonstrate that different strictly proper scoring rules yield considerably different rankings of forecasters based on their scoring rule scores. However, we assume that agents interact in the market scoring rule market proposed by Hanson (2002), which is derived from the difference of sequential scoring rules. Hanson’s market scoring rule (MSR) gives risk-neutral and myopic agents an incentive to reveal their probabilistic estimates truthfully by ensuring that truthful reporting maximizes an agent’s expected payoffs; thus, it is said to be *incentive-compatible*. Das (2008) implies that market making can serve as an effective trading strategy for individual agents who do not possess superior information but are willing to learn from prices. However, unlike Das (2008), we consider MSR prediction markets in which forward-looking agents may take costly external actions outside the market in order to influence the likelihood of the traded event (for example, not work hard or vote for another candidate in an election). The agents in our setting are informed and do not simply learn from the traded prices.

It is generally assumed that the agents who participate in a prediction market by trading may have superior information about the traded event, but have no direct control over the outcome, as was noted by Chakraborty and Das (2016). They show that if an agent does not participate in a prediction market, they behave truthfully outside the prediction market, but that they may change their outside actions as a function of their earlier prediction market report. Chakraborty and Das consider a voting game in which the voting actions have equal costs and voting occurs after all agents have finished interacting in the prediction market. Unlike Chakraborty and Das (2016), we consider iterative interactions between agents’ outside actions, and agents’ reports, within the prediction market. In addition (and this is perhaps the most important difference), we also assume an asymmetric cost of external actions; for example, not voting is preferred to voting (in practice, there may be effort involved in voting that is saved when not voting). Using the taxonomy of Chakraborty and Das (2016), one can say our study considers both *price manipulation* (altering the market price due to external rewards) and *outcome manipulation* (altering external actions based on market prices). As such, we now discuss both of these form of prediction market manipulation.

Similarly to Chakraborty and Das (2016), many papers assume that agents may influence the outcomes of events traded in a prediction market, or have some other incentive outside the prediction market (Chakraborty & Das,

2014; Dimitrov & Sami, 2010; Huang & Shoham, 2014; Shi, Conitzer, & Guo, 2009). Shi et al. (2009) indicate that one potential downside of prediction markets is that they may give agents an incentive to take undesirable actions, and prove that there exist principle-aligned prediction mechanisms that do not incentivize undesirable actions with an ‘over-payment result’. In particular, unlike this paper, Shi et al. (2009) do not use a market scoring rule mechanism, but instead use sequential scoring rules, and have a linear subsidy (in the number of agents). Chakraborty and Das (2014) give a two-round example to help us understand when markets may be prone to manipulation due to different outside incentives, and how much the resulting prediction probabilities can be trusted. However, again unlike this paper, Chakraborty and Das (2014) do not consider costly actions. Huang and Shoham (2014) assume profit-indifferent manipulators and propose a modification of market scoring rules in the form of trade limits, in order to reduce the extent of the manipulation of prediction markets due to external incentives. However, the limitation of trade amounts may also interfere with the elicitation of agents’ true beliefs (depending on a market’s liquidity, bounded budgets may lead to agents not revealing all of their information in a market), though this limitation is not an issue here, as we do not bound agents’ budgets.

Chen, Gao, Goldstein, and Kash (2015) employ a two-player market scoring rule setting in which a manipulator with outside incentives trades first, followed by a truthful trader. We likewise consider a two-player market setting, but the two agents in our model are both strategic traders with outside incentives. Unlike the papers cited above, we show that when non-myopic agents’ expected payoffs consist not only of payoffs from the prediction market, but also of the costs of their related actions that decide the outcome of the market, the quadratic scoring rule, used as the market reward mechanism, incentivizes agents to report truthfully in the prediction market and to take action as if the prediction market was not present. We realize that the claims made regarding our contribution are very broad, and, as with any analytical result, there are various assumptions and inherent limitations regarding the generalizability of the results. We discuss the limitations and shortcomings of our results explicitly in Section 3, after presenting our model. We then discuss the limitations of our results further in Section 5. These discussions are presented not only to encourage researchers to consider these limitations when deciding whether to apply our model and results, but also to identify future research directions.

3. Model description

This section formally proposes and describes our setting. We consider a company *C*, in which two employees, who we will call Alice and Bob, are assigned to a time-limited project *E*, which they are to complete together. We consider each week from the beginning of this project as a round, and the scheduled time for completing the project is *T* weeks. As our model is designed for a multi-round setting, the number of rounds should be equal to or larger than two. In each round, Alice and Bob must decide individually whether to give a high, later denoted 1, or

low, later denoted 0, effort to project *E* during each week. After *T* rounds, Alice and Bob’s combined total efforts will determine the project’s likelihood of success (e.g., meeting its scheduled delivery date). The project *E* has a binary outcome, as *E* either occurs or does not occur. If the project succeeds by the end of *T* rounds, we say that *E* occurs; if the project fails by the end of *T* rounds, we say that *E* does not occur. At the end of each round, every high effort will bring some payoff score (negative scores are equivalent to net costs and positive scores to net rewards) to the player who exerted this effort, while low efforts will not bring any payoff score to the players.¹ In our model, effort is rewarded immediately after each round. This is a simplifying assumption, and need not necessarily hold in practice. In fact, in practice, a manager can tell if an employee is not working hard over time, and for ease of modeling, we assume this observation is immediate, but may not be so in practice. If the reward is delayed, it is discounted appropriately.

At the same time, in a prediction market, Alice and Bob also trade in security *F*, the ultimate value of which is contingent on the outcome of *E*. We assume that the prediction market is a market scoring rule market, as we find is used in practice.² If there is no related prediction market in *C*, then employees will be inspired to always devote high/low efforts to *E* in order to gain the maximal expected payoff. However, when a prediction market contingent on *E* exists, the employer may worry that employees will change their effort levels in order to benefit more from the rewards procured in the prediction market.

For each round *i* (*i* = 1, . . . , *T*), Alice and Bob devote efforts $e_A^{(i)}$ and $e_B^{(i)}$ to projecting *E* and report the prediction probabilities in the prediction market as $r_A^{(i)}$ and $r_B^{(i)}$, respectively. When *E* occurs, a report of $r_A^{(i)}$ earns Alice a net score $\rho_s^{(i)}(r_A^{(i)}) = s(E, r_A^{(i)}) - s(E, r_B^{(i-1)})$, where *s* is a proper scoring rule; when *E* does not occur, the report earns Alice a net score $\rho_f^{(i)}(r_A^{(i)}) = s(E, \bar{r}_A^{(i)}) - s(E, \bar{r}_B^{(i-1)})$ instead ($\rho_s^{(i)}(r_B^{(i)})$ and $\rho_f^{(i)}(\bar{r}_B^{(i)})$ are defined similarly). Note that $\bar{r}_A^{(i)} = 1 - r_A^{(i)}$. In the remainder of this paper, we will define $\bar{w} = 1 - w$ analogously for any variable *w*. In addition, *s*(·) is said to be a proper scoring rule if, for a risk-neutral agent with belief *p* and report *r* on an event,

$$\frac{d}{dr} (ps(r) + \bar{p}s(\bar{r})) \Big|_{r=p} = 0, \tag{1}$$

and

$$\frac{d^2}{dr^2} (ps(r) + \bar{p}s(\bar{r})) \leq 0. \tag{2}$$

When an agent’s score-maximizing report is equal to her true belief, a proper scoring rule, and in turn a market

¹ Payoff scores need not be linear in effort, and the constant α is used to convert the functional form of effort into payoff scores, in order for them to be compared with the scores earned in the prediction market. When the payoff score is negative, high efforts bring net costs to agents; when it is positive, high efforts bring net rewards to agents. The total scores earned from exerting efforts and reporting in the prediction market could be converted into financial costs or rewards that will be given to the agents.

² See the website of PredictIt: <https://www.predictit.org/About/HowItWorks>.

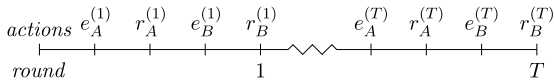


Fig. 1. Model dynamics timeline.

scoring rule, is said to be *incentive-compatible*. Here, we assume that Alice (Bob) assumes that the other player is myopic when she (he) sees his (her) previous prediction in the market. We also use ν_A ($0 < \nu_A < 1$) to describe Alice's impact on the likelihood of success of project E . We denote by $h_A^{(i)}$ ($h_B^{(i)}$) the accumulated number of high efforts exerted by Alice (Bob) by the end of round i . $p_A^{(i)}$ ($p_B^{(i)}$) is the belief regarding the likelihood of E occurring that is held by Alice (Bob) in round i before she (he) takes any actions in round i . $\pi_A^{(i)}$ ($\pi_B^{(i)}$) is the payoff earned between the current round, i , and the final round T by Alice (Bob) by exerting effort towards E and making reports in the prediction market in all rounds i to T . $I_A^{(i)}$ ($I_B^{(i)}$) is the system state that Alice (Bob) has formed in round i after she (he) observes the most recent prediction probability but before she (he) takes any actions in round i . $\mathbb{E} [\pi_A^{(i)}(I_A^{(i)}, a_A^{(i)}) | \mathbb{E} [\pi_B^{(i)}(I_B^{(i)}, a_B^{(i)})]]$ is the expected payoff given the current system state $I_A^{(i)}$ ($I_B^{(i)}$) and the action set $a_A^{(i)}$ ($a_B^{(i)}$) that Alice (Bob) would take in round i ; while $\mathbb{E} [\pi_A^{(i)*}(I_A^{(i)}) | \mathbb{E} [\pi_B^{(i)*}(I_B^{(i)})]]$ is the corresponding maximal expected payoff score for a given state $I_A^{(i)}$ ($I_B^{(i)}$) for Alice (Bob). $a_A^{(i)*} = (e_A^{(i)*}, r_A^{(i)*})$ ($a_B^{(i)*} = (e_B^{(i)*}, r_B^{(i)*})$) is the optimal action set that Alice (Bob) takes in round i in order to receive $\mathbb{E} [\pi_A^{(i)*}(I_A^{(i)}) | \mathbb{E} [\pi_B^{(i)*}(I_B^{(i)})]]$.

Fig. 1 shows the dynamics of our model. We assume that in each round Alice first determines her effort level, $e_A^{(i)}$, then makes a report in the prediction market, $r_A^{(i)}$; next, Bob determines the effort level he will exert in this round, $e_B^{(i)}$; finally, the round concludes by Bob reporting in the prediction market, $r_B^{(i)}$. In practice, during each round, each agent determines her (his) effort level and reported belief for this round simultaneously, though the actions exerted towards the project and the belief reported in the prediction market are conducted sequentially. Alice's and Bob's reports in the prediction market are always common knowledge to both agents in all cases. However, their effort levels are not common knowledge.

Given that in each round, two risk-neutral forward-looking agents are maximizing their expected profits from the current round to the end of the project horizon, T , we can write down the Bellman equation for determining the payoff for each round, for each agent. However, before writing down the equations for each round, we first define the payoff that each agent will receive in each round. In round i , Alice (Bob) will receive $\rho_e^{(i)}(e_A^{(i)})$ ($\rho_e^{(i)}(e_B^{(i)})$) from devoting effort $e_A^{(i)}$ ($e_B^{(i)}$) to the project, and will receive $p_A^{(i)} \cdot \rho_s^{(i)}(r_A^{(i)}) + \bar{p}_A^{(i)} \cdot \rho_f^{(i)}(\bar{r}_A^{(i)})$ ($p_B^{(i)} \cdot \rho_s^{(i)}(r_B^{(i)}) + \bar{p}_B^{(i)} \cdot \rho_f^{(i)}(\bar{r}_B^{(i)})$) from reporting probability estimate $r_A^{(i)}$ ($r_B^{(i)}$) in the prediction market. As was mentioned earlier, the value of the payoff function of effort exerted, $\rho_e^{(i)}(e_A^{(i)})$ ($\rho_e^{(i)}(e_B^{(i)})$), is negative

when high efforts bring net costs, and positive when high efforts bring net rewards. Our model assumes that the likelihood of E occurring is determined by the total number of high efforts over all rounds, weighted by each agent's impact on determining the likelihood of E :

$$\frac{1}{T} \left[\nu_A \sum_{n=1}^T e_A^{(n)} + (1 - \nu_A) \sum_{n=1}^T e_B^{(n)} \right]. \tag{3}$$

Before Alice takes any actions in round i , the number of accumulated high efforts exerted by her by the end of round $i - 1$ ($h_A^{(i-1)}$) is known to herself, but the number of accumulated high efforts exerted by Bob by the end of round $i - 1$ ($h_B^{(i-1)}$) is not observed by her directly. However, $\sum_{n=i+1}^T \tilde{e}_A^{(n)}$ and $\sum_{n=i}^T \tilde{e}_B^{(n)}$ are all future efforts, and $e_A^{(i)}$ is the effort-level decision she needs to make in the current round, i . For Bob, the situation before he takes any actions in round i is slightly different. Because Alice has already taken her actions in this round, the total number of high efforts exerted by her is $h_A^{(i)}$ instead. To be more specific, we can write down the definitions of $p_A^{(i)}$ and $p_B^{(i)}$ as:

$$p_A^{(i)} = \frac{1}{T} \left[\nu_A \left(h_A^{(i-1)} + e_A^{(i)} + \sum_{n=i+1}^T \tilde{e}_A^{(n)} \right) + (1 - \nu_A) \left(h_B^{(i-1)} + \sum_{n=i}^T \tilde{e}_B^{(n)} \right) \right], \tag{4a}$$

$$p_B^{(i)} = \frac{1}{T} \left[\nu_A \left(h_A^{(i)} + \sum_{n=i+1}^T \tilde{e}_A^{(n)} \right) + (1 - \nu_A) \left(h_B^{(i-1)} + e_B^{(i)} + \sum_{n=i+1}^T \tilde{e}_B^{(n)} \right) \right]. \tag{4b}$$

Eq. (4a) shows the formal definition of $p_A^{(i)}$ to be the total efforts of Alice, past and future, $h_A^{(i-1)} + e_A^{(i)} + \sum_{n=i+1}^T \tilde{e}_A^{(n)}$, plus the total efforts of Bob, past and future, $h_B^{(i-1)} + \sum_{n=i}^T \tilde{e}_B^{(n)}$, all weighted by ν_A . From Alice's perspective, she knows her past effort levels, and thus $h_A^{(i-1)}$ is known and is some natural number between 0 and $i - 1$. Similarly, as we will see in our analysis, Alice may infer Bob's effort levels from his reports in the prediction market, and again $h_B^{(i-1)}$ is known to Alice. The $\tilde{e}_A^{(n)}$ and $\tilde{e}_B^{(n)}$ effort values, for $n \geq i + 1$, are not necessarily binary, but instead are real numbers over $[0, 1]$, to account for the fact that Alice's and Bob's equilibrium effort strategies are mixed. As effort levels are not common knowledge, we need to define how each agent interprets the reported probability of the other agent. One way of interpreting reported probabilities is to use Bayesian updating, given current prior beliefs. However, as the probability of E depends not only on market estimates, but also on current and future effort levels, defining the Bayesian updating policy is quite convoluted. To simplify our analysis, we define $r_B(A)^{(i)}$ as Bob's estimate of the likelihood of E up to and including round i after observing Alice's last report. However, when Alice makes the report in round i , Bob has not yet taken actions in this round, so the expectation of his effort value of round i as perceived

by Alice is denoted as $\mathbb{E}_A [\tilde{e}_B^{(i)}]$. Similarly, we define $r_{A(B)}^{(i)}$ as Alice’s estimate of the likelihood of E ’s occurring up to and including round i after observing Bob’s last report. We define the two notations formally as:

$$r_B(A)^{(i)} = \frac{1}{i} \left[\nu_A h_A^{(i)} + (1 - \nu_A) \left(h_B^{(i-1)} + \mathbb{E}_A(\tilde{e}_B^{(i)}) \right) \right], \quad (5a)$$

$$r_A(B)^{(i)} = \frac{1}{i} \left[\nu_A h_A^{(i)} + (1 - \nu_A) h_B^{(i)} \right]. \quad (5b)$$

Here, we assume that at the beginning of round i , Alice has no information that she can use to predict Bob’s effort level for this round, which implies $\mathbb{E}_A(\tilde{e}_B^{(i)}) = 0.5$ for any round i . Then, the notations of $r_B(A)^{(i)}$ and $r_A(B)^{(i)}$ can be defined further as:

$$r_B(A)^{(i)} = \frac{1}{i} \left[\nu_A h_A^{(i)} + (1 - \nu_A)(h_B^{(i-1)} + 0.5) \right], \quad (6a)$$

$$r_A(B)^{(i)} = \frac{1}{i} \left[\nu_A h_A^{(i)} + (1 - \nu_A) h_B^{(i)} \right]. \quad (6b)$$

Note that $r_B(A)^{(i)} = r_A^{(i)}$ and $r_A(B)^{(i)} = r_B^{(i-1)}$, as we are simply presenting how each of the agents interprets the other agent’s observed prediction market probabilities. In Eq. (4a), Alice’s number of previous high efforts, $h_A^{(i-1)}$, is known to herself, while the number of Bob’s previous high efforts, $h_B^{(i-1)}$, could not be observed directly but could be inferred using his last prediction probability $r_B^{(i-1)}$ as

$$h_B^{(i-1)} = \frac{(i - 1) \cdot r_B^{(i-1)} - \nu_A h_A^{(i-1)}}{1 - \nu_A}.$$

For Eq. (4b), the unobservable number of previous high efforts exerted by Alice could likewise be inferred using $r_A^{(i)}$ as

$$h_A^{(i)} = \frac{i r_A^{(i)} - (1 - \nu_A)(h_B^{(i-1)} + 0.5)}{\nu_A}.$$

By inserting the expressions of $h_B^{(i-1)}$ and $h_A^{(i)}$ into Eqs. (4a) and (4b) separately, we can get a further expression of the agents’ beliefs regarding the likelihood of E ’s occurrence during any round i as:

$$p_A^{(i)} = \frac{1}{T} \left[(i - 1) r_B^{(i-1)} + \nu_A \left(e_A^{(i)} + \sum_{n=i+1}^T \tilde{e}_A^{(n)} \right) + (1 - \nu_A) \sum_{n=i}^T \tilde{e}_B^{(n)} \right], \quad (7a)$$

$$p_B^{(i)} = \frac{1}{T} \left[i r_A^{(i)} + \nu_A \sum_{n=i+1}^T \tilde{e}_A^{(n)} + (1 - \nu_A) \times \left(e_B^{(i)} + \sum_{n=i+1}^T \tilde{e}_B^{(n)} - 0.5 \right) \right]. \quad (7b)$$

With the payoff scores being collected in each round, each agent maximizes the current round’s payoff scores

plus the discounted future rounds’ payoff scores.

$$\begin{aligned} \mathbb{E} \left[\pi_A^{(i)*}(I_A^{(i)}) \right] &= \max_{(e_A^{(i)}, r_A^{(i)})} \left\{ \underbrace{\delta \mathbb{E}[\pi_A^{(i+1)*}]}_{\text{discounted future payoff}} \right. \\ &\quad \left. + \underbrace{\rho_e^{(i)}(e_A^{(i)})}_{\text{effort payoff}} + \underbrace{p_A^{(i)} \rho_s^{(i)}(r_A^{(i)}) + \bar{p}_A^{(i)} \rho_f^{(i)}(\bar{r}_A^{(i)})}_{\text{prediction market payoff}} \right\}, \quad (8) \\ \mathbb{E} \left[\pi_B^{(i)*}(I_B^{(i)}) \right] &= \max_{(e_B^{(i)}, r_B^{(i)})} \left\{ \delta \mathbb{E} \left[\pi_B^{(i+1)*} \right] + \rho_e^{(i)}(e_B^{(i)}) \right. \\ &\quad \left. + \underbrace{p_B^{(i)} \rho_s^{(i)}(r_B^{(i)}) + \bar{p}_B^{(i)} \rho_f^{(i)}(\bar{r}_B^{(i)})}_{\text{prediction market payoff}} \right\}. \end{aligned}$$

3.1. Applicability of the developed model

Before moving to determining agents’ equilibrium decisions in various settings, we will discuss the applicability of the developed model. Recall that our paper is motivated by the fact that prediction market agents who are participating in a market at work are paid for “doing their job”, e.g., exerting a high effort at work. As agents receive compensation for exerting a high effort, accounting for this compensation when determining an agent’s equilibrium actions in a prediction market is necessary, and the purpose of this paper. However, as with most analytical work, there are shortcomings not only with our model, which we will now discuss, but in turn with the conclusions, as we discuss in Section 5.

One clear shortcoming with the developed model is that it considers only two agents trading on a binary event (E either occurs or does not occur). As has been noted in other research on prediction markets, results that hold for binary events do not always map to the n -nary outcome space (Karimi & Dimitrov, 2018). The reason why binary events behave differently is that, for them, a given change in one report is equivalent to a change in the opposite direction in the other report. With three or more outcomes, a change in one report does not necessarily imply an equal and opposite change in any other one report, but instead changes in the aggregate of multiple other reports.

Another clear shortcoming of the developed model is the agents. The agents are risk-neutral, something that is known not to be the case in practice (Carlsson, Daruvala, & Johansson-Stenman, 2005; Holt & Laury, 2002; Rosati & Hare, 2016). It is not clear how/whether risk attitudes impact agents’ actions when they are exposed to external incentives. There is still scant research in risk-aversion in prediction markets without external incentives (Dimitrov, Sami, & Epelman, 2015; Karimi & Dimitrov, 2018; Ottaviani & Sørensen, 2007), and none that we are aware of on risk-aversion in the presence of external incentives. In addition to risk-neutral agents, we also assume that agents are homogeneous, in that they both receive either a positive or a negative cost for exerting high effort. In practice, the relative magnitudes of agents’ costs may impact their actions, especially when they have different signs.

The final significant shortcoming of our presented model is the stylized learning model for agents. The learning model allows us to have a tractable model, but results in agents not accounting for all of the other agent’s future

actions. This may not happen in practice, and may skew our results.

Even with all of the above shortcomings, we think that the work presented in this paper is a necessary first step in further addressing external incentives in prediction markets. It is clear that agents who participate in corporate prediction markets are paid for taking certain external actions while being penalized for taking others. If this situation is not considered and modeled, then managers may still believe that using prediction markets within a workplace is a bad idea. While we make multiple simplifying assumptions that lead to shortcomings in our model, the results, as we will soon see, show that the cost of acting on external incentives may restore incentive-compatibility in prediction markets. The results are promising, and future work, in addition to addressing the shortcoming outlined above, would do well to consider field or laboratory experiments for verifying whether costly actions do actually restore incentive-compatibility in prediction markets.

In the remainder of this paper, we will determine the values of $\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)})]$ and $\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)})]$ and the equilibrium strategies of both agents.

4. Results and analysis

This section begins by introducing one of the most commonly used scoring rules, the quadratic scoring rule, that will be used in all of the following cases. We then provide a mathematical analysis of cases 1 (high efforts bring a net cost to agents), 2 (high efforts bring a net reward instead), and 3 (external incentives do not exist). We assume $T = 2$ for the first three cases, but discuss results for $T = 4$ in case 4.

We show here that, with external incentives (net payoff scores are given to exerted efforts), the quadratic scoring applied in the prediction market is incentive-compatible (agents will report truthfully in the prediction market). In fact, we prove that, in a two-round setting, agents will not take desirable actions (always exert high efforts) when high efforts bring net costs in case 1, but will take desirable actions (always exert high efforts) when high efforts bring net rewards in case 2. When external incentives exist, agents will behave as if the prediction market was not present, which indicates that the prediction market will not change agents' incentives outside the market. However, when external incentives do not exist, case 3 proves that agents have incentives to bluff (report untruthfully) in the prediction market when they are forward-looking. This result is aligned with the conclusions from previous work (Chen et al., 2010; Dimitrov & Sami, 2008). Finally, case 4 recreates case 2 in the four-round setting.

4.1. Application of the quadratic scoring rule

This section uses a popular scoring rules, the quadratic scoring rule, as the reward mechanism in the prediction market. As we consider extreme reports in our results, agents report probability estimates of 0 or 1; thus, we cannot use other, perhaps more popular, scoring rules, such

as the logarithmic scoring rule. Then, $s(E, r)$, introduced in Section 3, is defined as:

$$\begin{aligned} s(E, r) &= 2r - r^2 - \bar{r}^2 = -2r^2 + 4r - 1 \\ s(E, \bar{r}) &= 2\bar{r} - r^2 - \bar{r}^2 = -2r^2 + 1. \end{aligned} \tag{9}$$

$s(E, r)$ is a proper scoring rule because Eqs. (1) and (2) are satisfied. We have already defined the scores earned by reporting in the prediction market in Section 3 as:

$$\begin{aligned} \rho_s(r_A^{(i)}) &= s(E, r_A^{(i)}) - s(E, r_B^{(i-1)}) \\ \rho_f(\bar{r}_A^{(i)}) &= s(E, \bar{r}_A^{(i)}) - s(E, \bar{r}_B^{(i-1)}) \\ \rho_s(r_B^{(i)}) &= s(E, r_B^{(i)}) - s(E, r_A^{(i)}) \\ \rho_f(\bar{r}_B^{(i)}) &= s(E, \bar{r}_B^{(i)}) - s(E, \bar{r}_A^{(i)}). \end{aligned} \tag{10}$$

Using $s(E, r)$ defined in Eq. (9), we can further write the scores as:

$$\begin{aligned} \rho_s(r_A^{(i)}) &= 4r_A^{(i)} - 2(r_A^{(i)})^2 - 4r_B^{(i-1)} + 2(r_B^{(i-1)})^2 \\ \rho_f(\bar{r}_A^{(i)}) &= -2(r_A^{(i)})^2 + 2(r_B^{(i-1)})^2 \\ \rho_s(r_B^{(i)}) &= 4r_B^{(i)} - 2(r_B^{(i)})^2 - 4r_A^{(i)} + 2(r_A^{(i)})^2 \\ \rho_f(\bar{r}_B^{(i)}) &= -2(r_B^{(i)})^2 + 2(r_A^{(i)})^2. \end{aligned} \tag{11}$$

4.2. Case 1: High efforts bring net costs to agents

Case 1 assumes that the two agents' efforts together decide the outcome of E and the ultimate value of security F , but that high efforts will bring net costs to the agents who exert them. In this setting, we assume the payoff function of exerted efforts to be $\rho_e(e) = -\alpha e^2$ ($\alpha > 0$):

$$\rho_e(e_A^{(i)}) = -\alpha (e_A^{(i)})^2, \tag{12a}$$

$$\rho_e(e_B^{(i)}) = -\alpha (e_B^{(i)})^2. \tag{12b}$$

Here, we use the definition of Alice's (Bob's) belief on the likelihood of E 's final occurrence in each round when maximizing her (his) expected total payoff from Section 3, and apply the quadratic scoring rule introduced in Section 4.1. We know from Eqs. (12), (7) and (11) that $I_A^{(i)} = r_B^{(i-1)}$ and $I_B^{(i)} = r_A^{(i)}$ when deciding the values of $\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)})]$ and $\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)})]$, respectively. After inserting the payoff functions of exerted efforts (Eq. (12)), the belief on the likelihood of E 's occurring (Eq. (7)), and the payoff functions in the prediction market (Eq. (11)) into the maximization equations (Eq. (8)) from the first round to the final round, we get the following maximization equations for Alice and Bob separately when $T = 2$.

For agent Alice, based on Eqs. (7), (8), and (11), we obtain

$$\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)})] = \max_{(e_A^{(1)}, r_A^{(1)})} \left\{ \delta \mathbb{E}[\pi_A^{(2)*}(\bar{r}_B^{(1)})] - \alpha (e_A^{(1)})^2 \right\} \tag{13a}$$

$$\begin{aligned}
 &+ 2 \left(\nu_A (e_A^{(1)} + \tilde{e}_A^{(2)}) + (1 - \nu_A) \right. \\
 &\quad \times \sum_{n=1}^2 \tilde{e}_B^{(n)} \left(r_A^{(1)} - r_B^{(0)} \right) \\
 &\quad \left. + 2 \left((r_B^{(0)})^2 - (r_A^{(1)})^2 \right) \right\}, \\
 \mathbb{E} \left[\pi_A^{(2)*}(r_B^{(1)}) \right] &= \max_{(e_A^{(2)}, r_A^{(2)})} \left\{ -\alpha (e_A^{(2)})^2 \right. \\
 &\quad + 2 \left(r_B^{(1)} + \nu_A e_A^{(2)} + (1 - \nu_A) \tilde{e}_B^{(2)} \right) \\
 &\quad \times \left(r_A^{(2)} - r_B^{(1)} \right) \\
 &\quad \left. + 2 \left((r_B^{(1)})^2 - (r_A^{(2)})^2 \right) \right\}. \tag{13b}
 \end{aligned}$$

Similarly, for agent Bob, we obtain

$$\begin{aligned}
 \mathbb{E} \left[\pi_B^{(1)*}(r_A^{(1)}) \right] &= \max_{(e_B^{(1)}, r_B^{(1)})} \left\{ \delta \mathbb{E} \left[\pi_B^{(2)*}(\tilde{r}_A^{(2)}) \right] - \alpha (e_B^{(1)})^2 \right. \\
 &\quad + 2 \left(r_A^{(1)} + \nu_A \tilde{e}_A^{(2)} \right. \\
 &\quad \left. + (1 - \nu_A)(e_B^{(1)} + \tilde{e}_B^{(2)} - 0.5) \right) \\
 &\quad \times \left(r_B^{(1)} - r_A^{(1)} \right) \\
 &\quad \left. + 2 \left((r_A^{(1)})^2 - (r_B^{(1)})^2 \right) \right\}, \\
 \mathbb{E} \left[\pi_B^{(2)*}(r_A^{(2)}) \right] &= \max_{(e_B^{(2)}, r_B^{(2)})} \left\{ -\alpha (e_B^{(2)})^2 \right. \\
 &\quad + 2 \left(2r_A^{(2)} + (1 - \nu_A)(e_B^{(2)} - 0.5) \right) \\
 &\quad \times \left(r_B^{(2)} - r_A^{(2)} \right) \\
 &\quad \left. + 2 \left((r_A^{(2)})^2 - (r_B^{(2)})^2 \right) \right\}. \tag{14b}
 \end{aligned}$$

We know from the timeline that Alice is the first agent to take action in round 1. Although for Alice in round 1, $\tilde{r}_B^{(1)}$, $\tilde{e}_A^{(2)}$, $\tilde{e}_B^{(2)}$ and $\tilde{e}_B^{(1)}$ are values for future actions that are not yet known to Alice, the values could be inferred by her, since our model assumes agents to be rational, forward-looking and strategic. To be more specific, Alice can play the whole game in her mind, knowing that, in any round, both agents want to maximize the expected scores earned between the current round and the final round; she can then infer future optimal actions after exerting effort $e_A^{(1)}$ and making report $r_A^{(1)}$ in round 1. Following this logic, we can use backwards induction to solve this problem, i.e., to determine the values of Eqs. (13) and (14).

Define $f_B^{(i)}(f_A^{(i)})$ as the function of Bob's (Alice's) expected payoff score earned from the current round i , and $f_B^{(i)*}(f_A^{(i)*})$ as the corresponding optimal function value. First, we consider agent Bob in round 2. For a given $r_A^{(2)}$ from Alice in the second round (which is known to Bob), if Bob's action is

$$\begin{aligned}
 a_B^{(2)} &= (e_B^{(2)}, r_B^{(2)}), \text{ then we have } f_B^{(2)} = \mathbb{E} \left[\pi_B^{(2)}(r_A^{(2)}, a_B^{(2)}) \right] \text{ as:} \\
 f_B^{(2)} &= -\alpha (e_B^{(2)})^2 - 2 (r_B^{(2)})^2 + 2 (r_A^{(2)})^2 \\
 &\quad + 2 \left((1 - \nu_A)(e_B^{(2)} - 0.5) + 2r_A^{(2)} \right) \left(r_B^{(2)} - r_A^{(2)} \right). \tag{15}
 \end{aligned}$$

We find Bob's optimal action $a_B^{(2)*} = (e_B^{(2)*}, r_B^{(2)*})$ that maximizes $f_B^{(2)}$, in the feasible region $0 \leq e_B^{(2)}, r_B^{(2)} \leq 1$, by first finding all of the feasible Karush-Kuhn-Tucker (KKT) points of $f_B^{(2)}$, then finding the optimal solution.

Lemma 1. Assume that $\alpha > \frac{(1-\nu_A)^2}{2}$. Then, $f_B^{(2)}$ has one feasible KKT point:

$$(e_B^{(2)}, r_B^{(2)}) = \begin{cases} (0, 0) & \text{if } r_A^{(2)} \leq (1 - \nu_A)/4 \\ (0, r_A^{(2)} - (1 - \nu_A)/4) & \text{if } r_A^{(2)} > (1 - \nu_A)/4, \end{cases}$$

or, equivalently, we can write $(e_B^{(2)}, r_B^{(2)}) = (0, \max\{0, r_A^{(2)} - \frac{1}{4}(1 - \nu_A)\})$.

Proof. For notational convenience, in this proof, we let $x = e_B^{(2)}$ and $y = r_B^{(2)}$. The equations for the KKT points can be derived based on the function $f_B^{(2)}$ and the constraints $0 \leq x, y \leq 1$, which can be written explicitly as four inequalities: $-x \leq 0$, $x - 1 \leq 0$, $-y \leq 0$, and $y - 1 \leq 0$. Correspondingly, four Lagrangian multipliers μ_1, μ_2, μ_3 , and μ_4 are defined. By Eq. (4a) and routine calculations, the equations for the KKT points are:

$$\begin{aligned}
 -2\alpha x + 2y(1 - \nu_A) - 2r_A^{(2)}(1 - \nu_A) + \mu_1 - \mu_2 &= 0; \\
 -4y + 2x(1 - \nu_A) + 2 \left(2r_A^{(2)} - 0.5(1 - \nu_A) \right) + \mu_3 - \mu_4 &= 0; \tag{16} \\
 x\mu_1 = (x - 1)\mu_2 = y\mu_3 = (y - 1)\mu_4 &= 0; \\
 0 \leq x \leq 1, 0 \leq y \leq 1; \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0. &
 \end{aligned}$$

To find all of the KKT points, we must consider the following nine cases: (i) $x = y = 0$; (ii) $x = 1$ and $y = 0$; (iii) $x = 0$ and $y = 1$; (iv) $x = y = 1$; (v) $0 < x < 1$ and $y = 0$; (vi) $0 < x < 1$ and $y = 1$; (vii) $x = 0$ and $0 < y < 1$; (viii) $x = 1$ and $0 < y < 1$; and (ix) $0 < x, y < 1$. For each case, we find $\{x, y, \mu_1, \mu_2, \mu_3, \mu_4\}$ satisfying the above equations. In the end, only cases (i) and (vii) give feasible KKT points. The results are presented in Table 1, with the last column displaying reasons for why some cases do not have KKT points. We also note that the function $f_B^{(2)}$ is not given for non-feasible KKT points in Table 1, since it is not used. Details are omitted. \square

We note that it is reasonable to assume $\alpha > \frac{(1-\nu_A)^2}{2}$. As ν_A is likely to be some value around $\frac{1}{2}$, assuming that Alice and Bob have new equal attribution to the project, the amount paid for the effort is greater than $\frac{1}{8}$ per round.

For the case with $r_A^{(2)} \geq \frac{1}{4}(1 - \nu_A)$, we have the following results.

Lemma 2. If $\alpha > \frac{(1-\nu_A)^2}{2}$ and $r_A^{(2)} \geq \frac{1}{4}(1 - \nu_A)$, then $(e_B^{(2)*}, r_B^{(2)*}) = (0, r_A^{(2)} - \frac{1}{4}(1 - \nu_A))$ and $\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)})] = \frac{1}{8}(1 - \nu_A)^2$.

Proof. The proof is similar to that of Lemma 1. If $\frac{1}{4}(1 - \nu_A) \leq r_A^{(2)}$, we find by comparing the values of $f_B^{(2)}$ of feasible KKT

Table 1
KKT points of $f_B^{(2)}$ for case 1.

Case	$(e_B^{(2)}, r_B^{(2)})$	$f_B^{(2)}$	KKT or not
(i)	(0, 0)	$(1 - \nu_A - 2r_A^{(2)})\mu_A^{(2)}$	Feasible KKT if $r_A^{(2)} \leq \frac{(1-\nu_A)}{4}$
(ii)	(1, 0)	$-\alpha - 2(r_A^{(2)})^2 - (1 - \nu_A)r_A^{(2)} < 0$	Not KKT ($\mu_3 < 0$)
(iii)	(0, 1)	$-(1 - r_A^{(2)})(3 - 2r_A^{(2)} - \nu_A) < 0$	Not KKT ($\mu_4 < 0$)
(iv)	(1, 1)	$(1 - r_A^{(2)})(2r_A^{(2)} - 1 - \nu_A) - \alpha < 0$	Not KKT ($\nu_A \geq 1$)
(v)	$(-\frac{1}{\alpha}r_A^{(2)}(1 - \nu_A), 0)$	-	Not feasible KKT ($x < 0$)
(vi)	$(\frac{1}{\alpha}(1 - r_A^{(2)})(1 - \nu_A), 1)$	$-(1 - r_A^{(2)})^2 \left(2 - \frac{(1-\nu_A)^2}{\alpha} + \frac{1-\nu_A}{1-r_A^{(2)}} \right) < 0$	Not KKT ($\mu_4 < 0$)
(vii)	$(0, r_A^{(2)} - \frac{1}{4}(1 - \nu_A))$	$\frac{1}{8}(1 - \nu_A)^2$	KKT if $r_A^{(2)} \geq \frac{(1-\nu_A)}{4}$
(viii)	$(1, r_A^{(2)} + \frac{1}{4}(1 - \nu_A))$	$-\alpha + \frac{1}{8}(1 - \nu_A)^2 < 0$	Not KKT ($\mu_2 < 0$)
(ix)	$e_B^{(2)} = \frac{4r_A^{(2)} - r_A^{(2)}(1 - \nu_A)^2 - (1 - \nu_A)}{2\alpha - (1 - \nu_A)^2}$, $r_B^{(2)} = \frac{\alpha(1 - \nu_A) - 2r_A^{(2)}(1 - \nu_A)^2 - 4\alpha r_A^{(2)}}{4\alpha - 2(1 - \nu_A)^2}$	-	Not feasible KKT ($x < 0$ if $r_A^{(2)} < (1 - \nu_A)/4$; $y < 0$ if $r_A^{(2)} > (1 - \nu_A)/4$)

points in Table 1 that $f_B^{(2)*}$ is obtained as $f_B^{(2)*} = \frac{1}{8}(1 - \nu_A)^2$ when $(e_B^{(2)*}, r_B^{(2)*}) = (0, r_A^{(2)} - \frac{1}{4}(1 - \nu_A))$. So Lemma 2 is true. □

Based on Lemmas 1 and 2, we now find the optimal action for Bob in the second round.

Theorem 1. If high efforts bring net costs and $\alpha > \frac{(1-\nu_A)^2}{2}$, then

$$(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (0, 0), & \text{if } r_A^{(2)} \leq \frac{1}{4}(1 - \nu_A); \\ (0, r_A^{(2)} - \frac{1}{4}(1 - \nu_A)), & \text{if } r_A^{(2)} \geq \frac{1}{4}(1 - \nu_A). \end{cases}$$

$$f_B^{(2)*} = \begin{cases} -2(r_A^{(2)})^2 + (1 - \nu_A)r_A^{(2)}, & \text{if } 0 \leq r_A^{(2)} \leq \frac{1}{4}(1 - \nu_A); \\ \frac{1}{8}(1 - \nu_A)^2, & \text{if } \frac{1}{4}(1 - \nu_A) \leq r_A^{(2)} \leq 1. \end{cases}$$

Proof. First, we show that the function $f_B^{(2)}$ is concave in $(e_B^{(2)}, r_B^{(2)})$. By routine calculations, we obtain the Hessian matrix of $f_B^{(2)}$ as:

$$\nabla^2 f_B^{(2)}(e_B^{(2)}, r_B^{(2)}) = \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} & \frac{\partial^2 f_B^{(2)}}{\partial e_B^{(2)} \partial r_B^{(2)}} \\ \frac{\partial^2 f_B^{(2)}}{\partial r_B^{(2)} \partial e_B^{(2)}} & \frac{\partial^2 f_B^{(2)}}{\partial (r_B^{(2)})^2} \end{bmatrix} = \begin{bmatrix} -2\alpha & 2(1 - \nu_A) \\ 2(1 - \nu_A) & -4 \end{bmatrix}.$$

Since the sum of the diagonal elements of the Hessian matrix is negative (i.e., $-2\alpha - 4 < 0$) and the determinant of the Hessian matrix is positive (i.e., $8\alpha - 4(1 - \nu_A)^2 > 0$), the two eigenvalues of the Hessian matrix must have the same sign, which has to be negative. Thus, the Hessian matrix is strictly negative definite, and therefore, the function $f_B^{(2)}$ is concave. Then, the optimal solution has to be the KKT point that maximizes $f_B^{(2)}$ or a boundary point of the feasible region. We find the optimal solution by considering two cases, each with a unique KKT point.

If $r_A^{(2)} \leq (1 - \nu_A)/4$, there is only one feasible KKT point: (0, 0). Although there is no other feasible KKT point to be considered for optimality, we also have to compare $f_B^{(2)}(0, 0)$ with all of the other feasible boundary solutions listed in Table 1. By routine calculations, it can be verified that either the other solutions are infeasible or their $f_B^{(2)}$ is smaller than that of (0, 0). Therefore, (0, 0) is the optimal solution for this case.

If $r_A^{(2)} > (1 - \nu_A)/4$, there is only one KKT point: $(0, r_A^{(2)} - (1 - \nu_A)/4)$. Similarly to the above case, it can be shown that $(0, r_A^{(2)} - (1 - \nu_A)/4)$ is the optimal solution for this case.

The corresponding $f_B^{(2)*}$ can be calculated accordingly, and is given in Table 1. □

Now, we consider Alice’s expected payoff score in round $i = 2$. We know from Theorem 1 that $e_B^{(2)*} = 0$, regardless of the value of $r_A^{(2)}$. After inserting $\tilde{e}_B^{(2)} = e_B^{(2)*} = 0$ into the expression $f_A^{(2)} = \mathbb{E}[\pi_A^{(2)}, r_B^{(1)}, a_A^{(2)}]$, where $a_A^{(2)} = (e_A^{(2)}, r_A^{(2)})$, we obtain:

$$f_A^{(2)} = -\alpha (e_A^{(2)})^2 + 2 (r_B^{(1)} + \nu_A e_A^{(2)}) (r_A^{(2)} - r_B^{(1)}) + 2 \left((r_B^{(1)})^2 - (r_A^{(2)})^2 \right).$$

Lemma 3. Assume that $\alpha > \frac{\nu_A^2}{2}$. Then, $f_A^{(2)}$ has a unique feasible KKT point $(0, \frac{1}{2}r_B^{(1)})$.

Proof. The proof is similar to that of Lemma 1. The system of equations used in the proof is as follows (note that $x = e_A^{(2)}$ and $y = r_A^{(2)}$):

$$\begin{aligned} -2\alpha x + 2\nu_A(y - r_B^{(1)}) + \mu_1 - \mu_2 &= 0; \\ -4y + 2(r_B^{(1)} + \nu_A x) + \mu_3 - \mu_4 &= 0; \\ x\mu_1 = (x - 1)\mu_2 = y\mu_3 = (y - 1)\mu_4 &= 0; \\ 0 \leq x \leq 1, 0 \leq y \leq 1; \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0. \end{aligned}$$

We find all of the feasible KKT points from all of the potential KKT points, which results in only one feasible KKT

Table 2
KKT points of $f_A^{(2)}$ in case 1.

Cases	$(e_A^{(2)}, r_A^{(2)})$	$f_A^{(2)}$	KKT or not
(i)	(0, 0)	0	Not KKT ($\mu_3 < 0$)
(ii)	(1, 0)	$-2\nu_A r_B^{(1)} - \alpha < 0$	Not KKT ($\mu_2 < 0$)
(iii)	(0, 1)	$-(1 - r_B^{(1)})^2 < 0$	Not KKT ($\mu_1 < 0$)
(iv)	(1, 1)	$2(1 - \nu_A)(r_B^{(1)} - 1) - \alpha < 0$	Not KKT ($\mu_4 < 0$)
(v)	$(-\frac{\nu_A r_B^{(1)}}{\alpha}, 0)$	-	Not feasible KKT ($e_A^{(2)} < 0$)
(vi)	$(\frac{\nu_A(1-r_B^{(1)})}{\alpha}, 1)$	$2(r_B^{(1)} - 1) + \frac{\nu_A^2(r_B^{(1)}-1)^2}{\alpha} < 0$	Not KKT ($\mu_4 < 0$)
(vii)	$(0, \frac{1}{2}r_B^{(1)})$	$\frac{1}{2}(r_B^{(1)})^2$	Feasible KKT
(viii)	$(1, \frac{1}{2}(r_B^{(1)} + \nu_A))$	$-\alpha + \frac{1}{2}\nu_A^2 - r_B^{(1)}\nu_A + \frac{1}{2}(r_B^{(1)})^2$	Not KKT ($\mu_2 < 0$)
(ix)	$e_A^{(2)} = \frac{r_B^{(1)}(\nu_A - 2)}{2\alpha - \nu_A^2} < 0$ $r_A^{(2)} = \frac{r_B^{(1)}(\alpha - \nu_A^2)}{2\alpha - \nu_A^2}$	-	Not feasible KKT ($e_A^{(2)} < 0$)

point. The results are shown in Table 2, but details are omitted. □

In Lemma 3 we introduce an additional bound with $\alpha > \frac{\nu_A^2}{2}$. As $\nu_A \in (0, 1)$, we know that the lower bound on α is at most $\frac{1}{2}$; however, Bob might not be involved in the project with ν_A so high. As such, if we consider more realistic values of $\nu_A \in [0.25, 0.75]$, the lower bound drops to at most 0.28125 or lower. This lower bound on α is not very restrictive, as $\alpha \in (0, 2]$.

Theorem 2. If high efforts bring net costs and $\alpha > \frac{\nu_A^2}{2}$, then Alice's optimal action set in round 2 is $(e_A^{(2)*}, r_A^{(2)*}) = (0, \frac{1}{2}r_B^{(1)})$, and her optimal expected payoff score in round 2 is $\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)})] = \frac{1}{2}(r_B^{(1)})^2$.

Proof. To find the optimal function value $f_A^{(2)*}$, similarly to Theorem 1, we need to check the Hessian of function $f_A^{(2)}$:

$$\nabla^2 f_A^{(2)}(e_A^{(2)}, r_A^{(2)}) = \begin{bmatrix} \frac{\partial^2 f_A^{(2)}}{\partial (e_A^{(2)})^2} & \frac{\partial^2 f_A^{(2)}}{\partial e_A^{(2)} \partial r_A^{(2)}} \\ \frac{\partial^2 f_A^{(2)}}{\partial r_A^{(2)} \partial e_A^{(2)}} & \frac{\partial^2 f_A^{(2)}}{\partial (r_A^{(2)})^2} \end{bmatrix} = \begin{bmatrix} -2\alpha & 2\nu_A \\ 2\nu_A & -4 \end{bmatrix}.$$

We find that $f_A^{(2)}$ is a concave function. By Lemma 3, there is only one candidate for our maximization, namely $(0, r_B^{(1)}/2)$. Therefore, $(0, r_B^{(1)}/2)$ is the optimal solution. □

Next, we consider Bob in round 1. From Theorems 1 and 2, we have $e_A^{(2)*} = e_B^{(2)*} = 0$. Knowing $\tilde{e}_A^{(2)} = \tilde{e}_B^{(2)} = 0$, we get the expression of $f_B^{(1)}$, Bob's expected cost in round 1 for action $a_B^{(1)} = (e_B^{(1)}, r_B^{(1)})$, as:

$$f_B^{(1)} = -\alpha \cdot (e_B^{(1)})^2 + 2 \left(r_A^{(1)} + (e_B^{(1)} - 0.5)(1 - \nu_A) \right) \left(r_B^{(1)} - r_A^{(1)} \right) + 2 \left((r_A^{(1)})^2 - (r_B^{(1)})^2 \right). \tag{17}$$

Lemma 4. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then

$$(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (0, 0), & \text{if } 0 \leq r_B^{(1)} \leq \frac{1}{2}(1 - \nu_A); \\ (0, r_A^{(2)} - \frac{1}{4}(1 - \nu_A)), & \text{if } \frac{1}{2}(1 - \nu_A) \leq r_B^{(1)} \leq 1, \end{cases}$$

$$f_B^{(2)*} = \begin{cases} -2(r_A^{(2)})^2 + (1 - \nu_A)r_A^{(2)}, & \text{if } 0 \leq r_B^{(1)} \leq \frac{1}{2}(1 - \nu_A); \\ \frac{1}{8}(\nu_A - 1)^2, & \text{if } \frac{1}{2}(1 - \nu_A) \leq r_B^{(1)} \leq 1. \end{cases}$$

Proof. According to Theorem 2, we know that $r_A^{(2)*} = \frac{1}{2}r_B^{(1)}$. Together with the statement of Theorem 1, we infer Lemma 4. □

We now move to Bob's optimal action in the first round. We find that Bob will exert low effort and report as 0 in round 1 regardless of Alice's report in round 1.

Theorem 3. If high efforts bring net costs, $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, and $0 < \nu_A < \frac{1}{3}$, then Bob's optimal action set in round 1 is $(e_B^{(1)*}, r_B^{(1)*}) = (0, 0)$, and his expected total payoff score from this round to the final round is $\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)})] = r_A^{(1)}(1 - \nu_A)$.

Proof. According to the definitions of $\mathbb{E}[\pi_B^{(1)}(r_A^{(1)}, a_B^{(1)})]$, $f_B^{(1)}$ and $f_B^{(2)*}$, we know that

$$\mathbb{E}[\pi_B^{(1)}(r_A^{(1)}, a_B^{(1)})] = f_B^{(1)} + \delta f_B^{(2)*},$$

and $f_B^{(2)*}$ is fixed for a given value of $r_A^{(2)*}$. We know from Lemma 4 that if $r_B^{(1)} \leq \frac{1}{2}(1 - \nu_A)$, then $(e_B^{(2)*}, r_B^{(2)*}) = (0, 0)$ and $f_B^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}(1 - \nu_A)r_B^{(1)}$; but if $r_B^{(1)} \geq \frac{1}{2}(1 - \nu_A)$, then $(e_B^{(2)*}, r_B^{(2)*}) = (0, r_A^{(2)} - \frac{1}{4}(1 - \nu_A))$ and $f_B^{(2)*} = \frac{1}{8}(1 - \nu_A)^2$.

We first consider the case in which $f_B^{(2)*}$ is a function of $r_B^{(1)}$ as $f_B^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}(1 - \nu_A)r_B^{(1)}$, if $r_B^{(1)} \leq \frac{1}{2}(1 - \nu_A)$.

Under this condition, $f_B^{(1)} + \delta f_B^{(2)*}$ is a concave function if $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$. Similarly to Theorems 1 and 2, we find and compare the KKT points of $f_B^{(1)} + \delta \cdot f_B^{(2)*}$. It turns out that $f_B^{(1)} + \delta f_B^{(2)*}$ is maximized at $(e_B^{(1)}, r_B^{(1)}) = (0, 0)$ with value $r_A^{(1)}(1 - \nu_A)$. Details are omitted. (Note: The proofs in the rest of the paper will all omit the details about KKT points.)

Second, we discuss the case in which $f_B^{(2)*}$ is a constant as $f_B^{(2)*} = \frac{1}{8}(\nu_A - 1)^2$ if $\frac{1}{2}(1 - \nu_A) \leq r_B^{(1)} \leq 1$. Under this condition, $f_B^{(1)} + \delta f_B^{(2)*}$ is also a concave function if $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$. Again, similarly to Theorems 1 and 2, we identify and compare all of the KKT points of the function in order to find the optimal solution. It turns out that the function is maximized at $(e_B^{(1)}, r_B^{(1)}) = (0, 0)$ with value $r_A^{(1)}(1 - \nu_A)$, under an additional condition: $0 < \nu_A < \frac{1}{3}$.

In conclusion, the optimal value of $\mathbb{E}[\pi_B^{(1)}(r_A^{(1)}, a_B^{(1)})]$ is obtained as $\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)})] = r_A^{(1)}(1 - \nu_A)$ at $(e_B^{(1)}, r_B^{(1)}) = (0, 0)$, under the given conditions. So Theorem 3 is true. □

Finally, we consider Alice in round 1. Theorems 1–3 have already proved that $e_B^{(1)*} = e_A^{(2)*} = e_B^{(2)*} = r_B^{(1)*} = 0$ and $f_A^{(2)*} = \frac{1}{2}(r_B^{(1)})^2 = 0$, no matter what effort Alice exerts and no matter what report she makes in round 1. In this situation $f_A^{(2)*} = 0$, and we get the expression $f_A^{(1)} + \delta f_A^{(2)*} = \mathbb{E}[\pi_A^{(1)}(r_B^{(0)}, a_A^{(1)})]$, where $a_A^{(1)} = (e_A^{(1)}, r_A^{(1)})$, as:

$$f_A^{(1)} + \delta f_A^{(2)*} = -\alpha(e_A^{(1)})^2 + 2\nu_A e_A^{(1)}(r_A^{(1)} - r_B^{(0)}) + 2((r_B^{(0)})^2 - (r_A^{(1)})^2) \tag{18}$$

We now move to Alice’s optimal action in the first round. We find that Alice will exert low effort and report as 0 in round 1 regardless of the initial market estimate.

Theorem 4. *If high efforts bring net costs and $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then Alice’s optimal action set in round 1 is $(e_A^{(1)*}, r_A^{(1)*}) = (0, 0)$, and her expected total payoff score between this round and the final round is $\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)})] = 2(r_B^{(0)})^2$.*

Proof. Similarly to the previous theorems, we can show that the function $f_A^{(1)} + \delta \cdot f_A^{(2)*}$ is concave, and can also find its KKT points. By comparing the function values of the feasible KKT points, we conclude immediately that the optimal value of $f_A^{(1)} + \delta \cdot f_A^{(2)*}$ is achieved as $2(r_B^{(0)})^2$ when $(e_A^{(1)*}, r_A^{(1)*}) = (0, 0)$. Details are omitted. Thus, Theorem 4 is true. □

Summarizing the results in Theorems 1–4, we infer the set of equilibrium strategies and payoffs for both agents over 2 rounds in Table 3 (under the conditions given in the theorems).

We can conclude immediately that if $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $0 < \nu_A < \frac{1}{3}$, then not only will all of the agents’ reports be 0, but so will all of their effort values. This suggests that even when agents are strategic and external incentives exist, the proposed market scoring rule is incentive-compatible regardless of the initial market estimate, as

Table 3
Equilibrium strategies and payoffs in case 1.

Round i	Player j	$a_j^{(i)*} = (e_j^{(i)*}, r_j^{(i)*})$	$\mathbb{E}[\pi_j^{(i)*}, \cdot]$
1	A	$a_A^{(1)*} = (0, 0)$	$2(r_B^{(0)})^2$
1	B	$a_B^{(1)*} = (0, 0)$	0
2	A	$a_A^{(2)*} = (0, 0)$	0
2	B	$a_B^{(2)*} = (0, 0)$	0

long as we set proper ranges for α and ν_A . The next section derives the symmetric result when the cost of high effort is negative.

4.3. Case 2: High efforts bring net rewards to agents

Case 2 assumes that the two agents’ efforts together decide the outcome of E and the ultimate value of security F , but that high efforts will bring negative net costs (equivalent to positive net rewards) to the agents who exert them. We further assume that the payoff function of the exerted effort is $\rho_e(e) = \alpha e^2 (\alpha > 0)$:

$$\rho_e(e_A^{(i)}) = \alpha(e_A^{(i)})^2, \tag{19a}$$

$$\rho_e(e_B^{(i)}) = \alpha(e_B^{(i)})^2. \tag{19b}$$

In order to enable us to compare cases 1 and 2, we set the same range for α as before, namely $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$. Here we set Alice’s impact on deciding the likelihood of E as $\nu_A \in (0, 1)$, and have the same definition of Alice’s (Bob’s) belief on the likelihood of E finally occurring held in each round as that of case 1. Likewise, we continue using the quadratic scoring rule in the prediction market. We also note that, since the proofs for case 2 are similar to those for case 1, some technical details will be omitted in this subsection.

In case 2, we still have $I_A^{(i)} = r_B^{(i-1)}$ and $I_B^{(i)} = r_A^{(i)}$ as in case 1. After inserting the reward functions of exerted efforts (Eq. (19)), the belief regarding the likelihood of E occurring (Eq. (7)) perceived by agents, and the reward functions in the prediction market (Eq. (11)) into the maximization equations (Eq. (8)) for Alice and Bob separately, we get the following maximization equations when $T = 2$.

For agent Alice:

$$\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)})] = \max_{(e_A^{(1)}, r_A^{(1)})} \left\{ \delta \mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)})] + \alpha(e_A^{(1)})^2 + 2 \left(\nu_A(e_A^{(1)} + \tilde{e}_A^{(2)}) + (1 - \nu_A) \sum_{n=1}^2 \tilde{e}_B^{(n)} \right) (r_A^{(1)} - r_B^{(0)}) + 2((r_B^{(0)})^2 - (r_A^{(1)})^2) \right\}, \tag{20a}$$

$$\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)})] = \max_{(e_A^{(2)}, r_A^{(2)})} \left\{ \alpha(e_A^{(2)})^2 + 2(r_B^{(1)} + \nu_A e_A^{(2)} + (1 - \nu_A)\tilde{e}_B^{(2)}) \times (r_A^{(2)} - r_B^{(1)}) + 2((r_B^{(1)})^2 - (r_A^{(2)})^2) \right\}. \tag{20b}$$

For agent Bob:

$$\mathbb{E} \left[\pi_B^{(1)*}(r_A^{(1)}) \right] = \max_{(e_B^{(1)}, r_B^{(1)})} \left\{ \delta \mathbb{E} \left[\pi_B^{(2)*}(\tilde{r}_A^{(2)}) \right] + \alpha (e_B^{(1)})^2 \right. \quad (21a)$$

$$+ 2 \left(r_A^{(1)} + \nu_A \tilde{e}_A^{(2)} + (1 - \nu_A) \right. \\ \left. \times (e_B^{(1)} + \tilde{e}_B^{(2)} - 0.5) \right) \\ \left. \times \left(r_B^{(1)} - r_A^{(1)} \right) + 2 \left((r_A^{(1)})^2 - (r_B^{(1)})^2 \right) \right\},$$

$$\mathbb{E} \left[\pi_B^{(2)*}(r_A^{(2)}) \right] = \max_{(e_B^{(2)}, r_B^{(2)})} \left\{ \alpha (e_B^{(2)})^2 \right. \\ + 2 \left(2r_A^{(2)} + (1 - \nu_A)(e_B^{(2)} - 0.5) \right) \\ \times \left(r_B^{(2)} - r_A^{(2)} \right) \\ \left. + 2 \left((r_A^{(2)})^2 - (r_B^{(2)})^2 \right) \right\}. \quad (21b)$$

Next, similarly to case 1, we identify all KKT points of the function and then find the optimal solutions.

In round 2, we consider Bob's function of expected payoff scores $f_B^{(2)} = \mathbb{E} \left[\pi_B^{(2)}(r_A^{(2)}, a_B^{(2)}) \right]$ for action $a_B^{(2)} = (e_B^{(2)}, r_B^{(2)})$:

$$f_B^{(2)} = \alpha (e_B^{(2)})^2 + 2 \left(2r_A^{(2)} + (1 - \nu_A)(e_B^{(2)} - 0.5) \right) \\ \times \left(r_B^{(2)} - r_A^{(2)} \right) + 2 \left((r_A^{(2)})^2 - (r_B^{(2)})^2 \right). \quad (22)$$

Lemma 5. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(2)} \leq \frac{3+\nu_A}{4}$, then $(e_B^{(2)*}, r_B^{(2)*}) = (1, r_A^{(2)} + \frac{1}{4}(1 - \nu_A))$ and $\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)})] = \alpha + \frac{1}{8}(1 - \nu_A)^2$.

Proof. Similarly to Lemma 1, we first find the Hessian of function $f_B^{(2)}$:

$$\nabla^2 f_B^{(2)}(e_B^{(2)}, r_B^{(2)}) = \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} & \frac{\partial^2 f_B^{(2)}}{\partial e_B^{(2)} \partial r_B^{(2)}} \\ \frac{\partial^2 f_B^{(2)}}{\partial r_B^{(2)} \partial e_B^{(2)}} & \frac{\partial^2 f_B^{(2)}}{\partial (r_B^{(2)})^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\alpha & 2(1 - \nu_A) \\ 2(1 - \nu_A) & -4 \end{bmatrix}.$$

The first principle minor of $\nabla^2 f_B^{(2)}(e_B^{(2)}, r_B^{(2)})$ is positive for $\det \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} \end{bmatrix} = 2\alpha > 0$. The second principle minor is negative because:

$$\det \begin{bmatrix} \frac{\partial^2 f_B^{(2)}}{\partial (e_B^{(2)})^2} & \frac{\partial^2 f_B^{(2)}}{\partial e_B^{(2)} \partial r_B^{(2)}} \\ \frac{\partial^2 f_B^{(2)}}{\partial r_B^{(2)} \partial e_B^{(2)}} & \frac{\partial^2 f_B^{(2)}}{\partial (r_B^{(2)})^2} \end{bmatrix} = -8\alpha - 4(1 - \nu_A)^2 < 0.$$

We find that the Hessian of $f_B^{(2)}$ is an indefinite matrix, so we need to find all KKT points and compare the corresponding values of $f_B^{(2)}$ for each point, since $f_B^{(2)*}$ must be obtained in one of these KKT points. All (feasible) KKT points of $f_B^{(2)}$ and their corresponding $f_B^{(2)}$ function values

are presented in Table 4. (Note: To simplify our analysis, unfeasible KKT points are not shown in the tables in this subsection.) We find that if $r_A^{(2)} \leq \frac{3+\nu_A}{4}$, $f_B^{(2)*}$ is attained as $\alpha + \frac{1}{8}(1 - \nu_A)^2$ when $(e_B^{(2)}, r_B^{(2)}) = (1, r_A^{(2)} + \frac{1}{4}(1 - \nu_A))$. So Lemma 5 is true. \square

Lemma 6. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(2)} \geq \frac{3+\nu_A}{4}$, then $(e_B^{(2)*}, r_B^{(2)*}) = (1, 1)$ and $\mathbb{E} \left[\pi_B^{(2)*}(r_A^{(2)}) \right] = -2(r_A^{(2)})^2 + (\nu_A + 3)r_A^{(2)} + \alpha - \nu_A - 1$.

Proof. Similarly to the proof for Lemma 5, if $r_A^{(2)} \geq \frac{3+\nu_A}{4}$, we find by comparing the values of $f_B^{(2)}$ for feasible KKT points that $f_B^{(2)*}$ is attained as $f_B^{(2)*} = -2(r_A^{(2)})^2 + (\nu_A + 3)r_A^{(2)} + \alpha - \nu_A - 1$ when $(e_B^{(2)*}, r_B^{(2)*}) = (1, 1)$. Details are omitted. So Lemma 6 is true. \square

Now, we are ready to determine Bob's round 2 decision.

Theorem 5. If high efforts bring net rewards and $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then

$$(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (1, r_A^{(2)} + \frac{1}{4}(1 - \nu_A)), & \text{if } 0 \leq r_A^{(2)} \leq \frac{3 + \nu_A}{4}; \\ (1, 1), & \text{if } \frac{3 + \nu_A}{4} \leq r_A^{(2)} \leq 1. \end{cases}$$

$$f_B^{(2)*} = \begin{cases} \alpha + \frac{1}{8}(1 - \nu_A)^2, & \text{if } 0 \leq r_A^{(2)} \leq \frac{3 + \nu_A}{4}; \\ -2(r_A^{(2)})^2 + (\nu_A + 3)r_A^{(2)} \\ + \alpha - \nu_A - 1, & \text{if } \frac{3 + \nu_A}{4} \leq r_A^{(2)} \leq 1. \end{cases}$$

Proof. With Lemmas 5 and 6, Theorem 5 is true. \square

Next, we consider Alice in round $i = 2$ when Bob's future effort is $\tilde{e}_B^{(2)}$. We know from Theorem 5 that $\tilde{e}_B^{(2)} = e_B^{(2)*} = 1$. After inserting $\tilde{e}_B^{(2)} = 1$ into $f_A^{(2)} = \mathbb{E} \left[\pi_A^{(2)}(r_B^{(1)}, a_A^{(2)}) \right]$, we get:

$$f_A^{(2)} = \alpha (e_A^{(2)})^2 + 2 \left(r_B^{(1)} + \nu_A \cdot e_A^{(2)} + 1 - \nu_A \right) \left(r_A^{(2)} - r_B^{(1)} \right) \\ + 2 \left((r_B^{(1)})^2 - 2(r_A^{(2)})^2 \right). \quad (23)$$

Regarding Alice's optimal actions in round 2, we find that Alice will exert high effort and report as 1 in round 2 no matter what Bob reports in round 1.

Theorem 6. If high efforts bring net rewards and $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, Alice's optimal action set in round 2 is $(e_A^{(2)*}, r_A^{(2)*}) = (1, \frac{1}{2}(r_B^{(1)} + 1))$, and the optimal expected payoff score of round 2 is $\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)})] = \alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$.

Proof. We find the value of $f_A^{(2)*}$ by finding all of the KKT points of $f_A^{(2)}$ and the corresponding function values (see Table 5). By routine calculations, we find that $f_A^{(2)}$ is maximized as $f_A^{(2)*} = \alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$ when $(e_A^{(2)}, r_A^{(2)}) = (1, \frac{1}{2}(r_B^{(1)} + 1))$. Thus, Theorem 6 is true. \square

Table 4
KKT points of $f_B^{(2)}$ in case 2.

$e_B^{(2)}$	$r_B^{(2)}$	$f_B^{(2)}$
0	$r_A^{(2)} - \frac{1}{4}(1 - \nu_A)$	$\frac{1}{8}(1 - \nu_A)^2$
1	$r_A^{(2)} + \frac{1}{4}(1 - \nu_A)$	$\alpha + \frac{1}{8}(1 - \nu_A)^2$
$\frac{r_A^{(2)}(1 - \nu_A)}{\alpha}$	0	$-2(r_A^{(2)})^2 + r_A^{(2)}(1 - \nu_A) - \frac{(r_A^{(2)})^2(1 - \nu_A)^2}{\alpha}$
0	0	$-2(r_A^{(2)})^2 + (1 - \nu_A)r_A^{(2)}$
0	1	$-2(r_A^{(2)})^2 + (5 - \nu_A)r_A^{(2)} + \nu_A - 3$
1	0	$-2(r_A^{(2)})^2 + (\nu_A - 1)r_A^{(2)} + \alpha$
1	1	$-2(r_A^{(2)})^2 + (\nu_A + 3)r_A^{(2)} + \alpha - \nu_A - 1$

Table 5
KKT points of $f_A^{(2)}$ in case 2.

$e_A^{(2)}$	$r_A^{(2)}$	$f_A^{(2)}$
0	$\frac{1}{2}r_B^{(1)} + \frac{1}{2}(1 - \nu_A)$	$\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}(1 - \nu_A)^2 - r_B^{(1)}(1 - \nu_A)$
1	$\frac{1}{2}(r_B^{(1)} + 1)$	$\alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$
$\frac{\nu_A r_B^{(1)}}{\alpha}$	0	$-2r_B^{(1)}(1 - \nu_A) - \frac{(\nu_A r_B^{(1)})^2}{\alpha}$
0	0	$-2r_B^{(1)}(1 - \nu_A)$
0	1	$-2\nu_A(1 - r_B^{(1)})$
1	0	$-2r_B^{(1)} + \alpha$
1	1	α

We now consider Bob in round 1. From Theorems 5 and 6, we know that $e_A^{(2)*} = e_B^{(2)*} = 1$. Knowing $\tilde{e}_A^{(2)} = \tilde{e}_B^{(2)} = 1$, we get the expression of $f_B^{(1)}$ as:

$$f_B^{(1)} = \alpha \cdot (e_B^{(1)})^2 + 2 \left(r_A^{(1)} + \nu_A + (1 - \nu_A)(e_B^{(1)} + 0.5) \right) \times (r_B^{(1)} - r_A^{(1)}) + 2 \left((r_A^{(1)})^2 - 2(r_B^{(1)})^2 \right). \tag{24}$$

Next, we need three lemmas for the maximum value of $\mathbb{E} \left[\pi_B^{(1)}(r_A^{(2)}, a_B^{(1)}) \right] = f_B^{(1)} + \delta f_B^{(2)*}$, where $a_B^{(1)} = (e_B^{(1)}, r_B^{(1)})$.

Lemma 7. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then

$$(e_B^{(2)*}, r_B^{(2)*}) = \begin{cases} (1, r_A^{(2)} + \frac{1}{4}(1 - \nu_A)), & \text{if } 0 \leq r_B^{(1)} \leq \frac{1}{2}(1 + \nu_A); \\ (1, 1), & \text{if } \frac{1}{2}(1 + \nu_A) \leq r_B^{(1)} \leq 1. \end{cases}$$

$$f_B^{(2)*} = \begin{cases} \alpha + \frac{1}{8}(\nu_A - 1)^2, & \text{if } 0 \leq r_B^{(1)} \leq \frac{1}{2}(1 + \nu_A); \\ -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}r_B^{(1)} + \frac{1}{2}r_B^{(1)}\nu_A \\ -\frac{1}{2}\nu_A + \alpha, & \text{if } \frac{1}{2}(1 + \nu_A) \leq r_B^{(1)} \leq 1. \end{cases}$$

Proof. By Theorem 6, we know that $r_A^{(2)*} = \frac{1}{2}(r_B^{(1)} + 1)$. Together with the statement of Theorem 5, we infer Lemma 7. □

Lemma 8. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(1)} \leq \frac{\nu_A+1}{2}$, then $(e_B^{(1)*}, r_B^{(1)*}) = (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3 - \nu_A))$ and

$$\mathbb{E} \left[\pi_B^{(1)*}(r_A^{(1)}) \right] = \frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^2 - \frac{3}{4}\nu_A + \frac{9}{8} + \delta \left(\alpha + \frac{1}{8}(1 - \nu_A)^2 \right).$$

Proof. We know from Lemma 7 that $f_B^{(2)*}$ is obtained as $f_B^{(2)*} = \alpha + \frac{1}{8}(1 - \nu_A)^2$ when $r_B^{(1)} \leq \frac{1}{2}(1 + \nu_A)$; and as $f_B^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}r_B^{(1)} + \frac{1}{2}r_B^{(1)}\nu_A - \frac{1}{2}\nu_A + \alpha$ when $r_B^{(1)} \geq \frac{1}{2}(1 + \nu_A)$.

We first discuss the situation where $f_B^{(2)*}$ is a constant as $f_B^{(2)*} = \alpha + \frac{1}{8}(1 - \nu_A)^2$ when $r_B^{(1)} \leq \frac{1}{2}(1 + \nu_A)$. We begin by finding all of the KKT points of $f_B^{(1)} + \delta \cdot f_B^{(2)*}$. We find that when $r_A^{(1)} \leq \frac{1+\nu_A}{2}$, $f_B^{(1)} + \delta \cdot f_B^{(2)*}$ is maximized as $\frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^2 - \frac{3}{4}\nu_A + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$ at $(e_B^{(1)}, r_B^{(1)}) = (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3 - \nu_A))$.

Second, we discuss the situation where $f_B^{(2)*}$ is a function of $r_B^{(1)}$ as $f_B^{(2)*} = -\frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2}r_B^{(1)} + \frac{1}{2}r_B^{(1)}\nu_A - \frac{1}{2}\nu_A + \alpha$, if $r_B^{(1)} \geq \frac{1}{2}(1 + \nu_A)$. Using the same method, we find that $f_B^{(1)} + \delta \cdot f_B^{(2)*}$ is maximized as $(1 - \nu_A)(1 - r_A^{(1)}) + \alpha(1 + \delta)$ at $(e_B^{(1)}, r_B^{(1)}) = (1, 1)$.

By routine calculations, we know that value of the first expression of $f_B^{(1)} + \delta \cdot f_B^{(2)*}$ is larger than that of the second expression. So $\mathbb{E} \left[\pi_B^{(1)*}(r_A^{(1)}) \right]$ is achieved as $\frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^2 - \frac{3}{4}\nu_A + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$ at $(e_B^{(1)*}, r_B^{(1)*}) = (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3 - \nu_A))$, and therefore Lemma 8 is true. □

Lemma 9. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(1)} \geq \frac{\nu_A+1}{2}$, then $(e_B^{(1)*}, r_B^{(1)*}) = (1, 1)$ and

$$\mathbb{E} \left[\pi_B^{(1)*}(r_A^{(2)}) \right] = (1 - \nu_A)(1 - r_A^{(1)}) + \alpha(1 + \delta) + \delta\nu_A \left(\frac{1}{2} - \nu_A \right).$$

Proof. The proof is similar to that of Lemma 8. Details are omitted. □

Given the two cases for Bob's equilibrium decisions in round 1, we may now determine Bob's round 1 decisions as follows.

Theorem 7. If high efforts bring net rewards and $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then

$$(e_B^{(1)*}, r_B^{(1)*}) = \begin{cases} (1, \frac{1}{2}r_A^{(1)} + \frac{1}{4}(3 - \nu_A)), & \text{if } 0 \leq r_A^{(1)} \leq \frac{\nu_A + 1}{2}; \\ (1, 1), & \text{if } \frac{\nu_A + 1}{2} \leq r_A^{(1)} \leq 1. \end{cases}$$

$$\mathbb{E} \left[\pi_B^{(1)*}(r_A^{(1)}) \right] = \begin{cases} \frac{1}{2}r_A^{(1)}(r_A^{(1)} - 3 + \nu_A) + \alpha + \frac{1}{8}\nu_A^2, \\ -\frac{3}{4}\nu_A + \frac{9}{8} + \delta \left(\alpha + \frac{1}{8}(1 - \nu_A)^2 \right), & \text{if } 0 \leq r_A^{(1)} \leq \frac{\nu_A + 1}{2}; \\ (1 - \nu_A)(1 - r_A^{(1)}) + \alpha \\ + \delta \left(\alpha + \nu_A \left(\frac{1}{2} - \nu_A \right) \right), & \text{if } \frac{\nu_A + 1}{2} \leq r_A^{(1)} \leq 1. \end{cases}$$

Table 6
Equilibrium strategies and payoffs in case 2.

Round, i	Player, j	$a_j^{(i)*} = (e_j^{(i)*}, r_j^{(i)*})$	$E[\pi_j^{(i)*}(\cdot)]$
1	A	$a_A^{(1)*} = (1, 1)$	$2(r_B^{(0)} - 1)^2 + (1 + \delta)\alpha$
1	B	$a_B^{(1)*} = (1, 1)$	$\alpha(1 + \delta) + \delta v_A(\frac{1}{2} - v_A)$
2	A	$a_A^{(2)*} = (1, 1)$	α
2	B	$a_B^{(2)*} = (1, 1)$	α

Proof. With Lemmas 8 and 9, Theorem 7 is true. \square

Finally, we consider Alice in round 1. We have already proved in Theorems 5–7 that $e_B^{(1)*} = e_A^{(2)*} = e_B^{(2)*} = 1$ and $\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)})] = \alpha + \frac{1}{2}(r_B^{(1)})^2 + \frac{1}{2} - r_B^{(1)}$. After inserting $\tilde{e}_B^{(1)} = \tilde{e}_B^{(2)} = \tilde{e}_B^{(2)} = 1$ and $\mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)})]$ into $\mathbb{E}[\pi_A^{(1)}(r_B^{(0)}, a_A^{(1)})]$, we obtain

$$\begin{aligned} \mathbb{E}[\pi_A^{(1)}(r_B^{(1)}, a_A^{(2)})] &= f_A^{(1)} + \delta f_A^{(2)*} \\ &= \delta \left(\alpha + \frac{1}{2}(1)^2 + \frac{1}{2} - 1 \right) + \alpha(e_A^{(1)})^2 \\ &\quad + 2 \left(v_A e_A^{(1)} + 2 - v_A \right) (r_A^{(1)} - r_B^{(0)}) \\ &\quad + 2 \left((r_B^{(0)})^2 - 2(r_A^{(1)})^2 \right) \tag{25} \\ &= \alpha \left(\delta + (e_A^{(1)})^2 \right) \\ &\quad + 2 \left(v_A e_A^{(1)} + 2 - v_A \right) (r_A^{(1)} - r_B^{(0)}) \\ &\quad + 2 \left((r_B^{(0)})^2 - (r_A^{(1)})^2 \right). \end{aligned}$$

We now move on to Alice’s optimal actions in the first round. We find that Alice will exert high effort and report as 1 in round 1 no matter what the initial market estimate is.

Theorem 8. If $\alpha > \max\{\frac{(1-v_A)^2}{2}, \frac{v_A^2}{2}\}$, then Alice’s optimal action set in round 1 is $(e_A^{(1)*}, r_A^{(1)*}) = (1, 1)$, and her expected total payoff score from this round to the final round is $\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)})] = 2(r_B^{(0)} - 1)^2 + (1 + \delta)\alpha$.

Proof. Similar to previous theorems, we first identify all of the KKT points of $f_A^{(1)} + \delta f_A^{(2)*}$. We then find that the optimal value of $f_A^{(1)} + \delta \cdot f_A^{(2)*}$ is achieved as $2(r_B^{(0)} - 1)^2 + (1 + \delta)\alpha$ when $(e_A^{(1)*}, r_A^{(1)*}) = (1, 1)$. Details are omitted. Thus, Theorem 8 is true. \square

Summarizing the results in Theorems 5–8, we infer the set of optimal equilibrium strategies $a_j^{(i)*} = (e_j^{(i)*}, r_j^{(i)*})$ and their corresponding payoffs for both agents over the two rounds in Table 6 (under the conditions specified in the theorems).

We can conclude immediately that if $\alpha > \max\{\frac{(1-v_A)^2}{2}, \frac{v_A^2}{2}\}$, then not only will all of the agents’ reports be 1, but so will all of their effort values. The results in Sections 4.2 and 4.3 generalize to the four-round setting, but it is still not clear whether they generalize for any value of T .

We carried out numerical simulations, via simulated annealing, for $T > 4$, and found that, if we start with random values of $r_j^{(i)}$ and $e_j^{(i)}$, the final Pareto dominant

solution for player $j \in \{A, B\}$ over all rounds was one, with all players either reporting truthfully and exerting high effort (referred to as the *truthful solution*), or mostly exerting high effort and being truthful but with some deviation in effort and reports, called the *mixed solution*. The most interesting aspect of our numerical simulations is that, for the solutions we found, the truthful solution always had a higher payoff for Alice than the mixed solution, while Bob had a higher payoff in the mixed solution than in the truthful solution. In addition, the values of the payoff functions tended to differ only in the hundredths or thousandths place of the reward function values across the two solutions. Given our findings, we conclude that the results of the numerical simulation are inconclusive, because we started with a random set of actions for all players across all rounds, and the Pareto dominant solution that we find may be infeasible, as Alice is the first mover in the game. As the first mover, Alice might take action to avoid ever reaching the initial state of the game (where we started the simulation), since she is rational and would receive a higher payoff in the truthful solution than the mixed solution. Thus, in order for a simulation to make sense, it must take into account Alice’s first-mover advantage.

4.4. Case 3: External incentives do not exist

Case 3 assumes that the two agents’ efforts together decide the outcome of E and the ultimate value of the security F , but that their efforts will bring no payoffs to the agents who exert them ($\rho_e(e) = 0$). Previous work has shown that when external incentives do not exist outside of the prediction market, non-myopic agents have an incentive to bluff in the prediction market (Chen et al., 2010; Dimitrov & Sami, 2008). Case 3 shows that this result also applies to our model. We assume $v_A = \frac{1}{2}$ (Alice and Bob have the same input when deciding the probability of E occurring) and $r_B^{(0)} = \frac{3}{4}$ (the initial market estimate is set to $\frac{3}{4}$). Under the above assumptions, we then have Alice’s and Bob’s Bellman equations in the two-round setting as follows.

For agent Alice (with $r_B^{(0)} = \frac{3}{4}$):

$$\begin{aligned} \mathbb{E}[\pi_A^{(1)*}(r_B^{(0)})] &= \max_{(e_A^{(1)}, r_A^{(1)})} \left\{ \delta \mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)})] \right. \tag{26a} \\ &\quad + 2 \left(v_A(e_A^{(1)} + \tilde{e}_A^{(2)}) + (1 - v_A)(\tilde{e}_B^{(1)} + \tilde{e}_B^{(2)}) \right) \\ &\quad \times (r_A^{(1)} - r_B^{(0)}) \\ &\quad \left. + 2 \left((r_B^{(0)})^2 - (r_A^{(1)})^2 \right) \right\}, \end{aligned}$$

$$\begin{aligned} \mathbb{E}[\pi_A^{(2)*}(r_B^{(1)})] &= \max_{(e_A^{(2)}, r_A^{(2)})} \left\{ 2 \left(r_B^{(1)} + v_A \cdot \tilde{e}_A^{(2)} + (1 - v_A)\tilde{e}_B^{(2)} \right) \right. \tag{26b} \\ &\quad \left. \times (r_A^{(2)} - r_B^{(1)}) + 2 \left((r_B^{(1)})^2 - (r_A^{(2)})^2 \right) \right\}. \end{aligned}$$

For agent Bob:

$$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)})] = \max_{(e_B^{(1)}, r_B^{(1)})} \left\{ \delta \mathbb{E}[\pi_B^{(2)*}(\tilde{r}_A^{(2)})] \right\} \tag{27a}$$

$$\begin{aligned} &+ 2 \left(r_A^{(1)} + v_A \cdot \tilde{e}_A^{(2)} + (1 - v_A) \right) \tag{27b} \\ &\times \left(e_B^{(1)} + \tilde{e}_B^{(2)} - 0.5 \right) \\ &\times \left(r_B^{(1)} - r_A^{(1)} \right) + \left((r_A^{(1)})^2 - (r_B^{(1)})^2 \right), \end{aligned}$$

Table 7
Equilibrium strategies in case 3.

Round, i	Player, j	$a_j^{(i)*} = (e_j^{(i)*}, r_j^{(i)*})$	$E[\pi_j^{(i)*}(\cdot)]$
1	A	$a_A^{(1)*} = (0, \frac{1}{2})$	$\frac{1}{8}$
1	B	$a_B^{(1)*} = (1, \frac{7}{8})$	$\frac{1}{32}$
2	A	$a_A^{(2)*} = (0, \frac{11}{16})$	$\frac{9}{128} + \frac{1}{8}\delta$
2	B	$a_B^{(2)*} = (1, \frac{13}{16})$	$\frac{1}{32}(1 + \delta)$

$$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)})] = \max_{(e_B^{(2)}, r_B^{(2)})} \left\{ 2(r_A^{(2)} + (1 - \nu_A)(e_B^{(2)} - 0.5)) \right. \\ \left. \times (r_B^{(2)} - r_A^{(2)}) + 2((r_A^{(2)})^2 - (r_B^{(2)})^2) \right\}. \quad (27c)$$

Using backwards induction, as was explained in detail for cases 1 and 2, we infer the set of equilibrium strategies $(e_j^{(i)*}, r_j^{(i)*})$ and their corresponding payoffs $E[\pi_j^{(i)*}(\cdot)]$ for both agents of 2 rounds in Table 7, under the initial condition $r_B^{(0)} = \frac{3}{4}$. Technical details are omitted.

Thus, in case 3 where external incentives do not exist, we observe directly from the optimal actions that agents do not necessarily exert high efforts on E , and also do not report truthfully in the prediction market.

4.5. Case 4: High efforts bring net rewards to agents in a four-round setting

In this case, similarly to Case 2, we assume that high efforts will bring negative net costs (equivalent to positive net rewards) to agents who exert them. Here, though, we extend the number of rounds from two to four. In order to compare case 4 properly with the previous cases, we assume the payoff function of exerted effort to be $\rho_e(e) = \alpha \cdot e^2 (\alpha > 0)$, as we did for case 2 in Eq. (19), and we again set the range for α to $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$. We set Alice’s impact on deciding the likelihood of E as $\nu_A \in (0, 1)$, and have the same definition of Alice’s (Bob’s) belief on the likelihood of E finally occurring held in each round as that of case 1. We also continue to use the quadratic scoring rule in the prediction market. In case 4, we still have $I_A^{(i)} = r_B^{(i-1)}$ and $I_B^{(i)} = r_A^{(i)}$ as in cases 1 and 2.

After inserting the reward functions of exerted efforts (Eq. (19)), the agents’ perceived belief regarding the likelihood of E occurring (Eq. (7)), and the reward functions in the prediction market (Eq. (11)) into the maximization equations (Eq. (8)) for Alice and Bob separately, we get the following maximization equations when $T = 4$.

For agent Alice:

$$\mathbb{E}[\pi_A^{(1)*}(r_B^{(0)})] = \max_{(e_A^{(1)}, r_A^{(1)})} \left\{ \delta \mathbb{E}[\pi_A^{(2)*}(\tilde{r}_B^{(1)})] + \alpha(e_A^{(1)})^2 \right. \\ \left. + \left(\nu_A(e_A^{(1)} + \sum_{n=2}^4 \tilde{e}_A^{(n)}) + (1 - \nu_A) \sum_{n=1}^4 \tilde{e}_B^{(n)} \right) \right. \\ \left. \times (r_A^{(1)} - r_B^{(0)}) + 2((r_B^{(0)})^2 - (r_A^{(1)})^2) \right\}, \quad (28a)$$

$$\mathbb{E}[\pi_A^{(2)*}(r_B^{(1)})] = \max_{(e_A^{(2)}, r_A^{(2)})} \left\{ \delta \mathbb{E}[\pi_A^{(3)*}(\tilde{r}_B^{(2)})] + \alpha(e_A^{(2)})^2 \right. \\ \left. + \left(\nu_A(e_A^{(2)} + \sum_{n=3}^4 \tilde{e}_A^{(n)}) + (1 - \nu_A) \sum_{n=1}^4 \tilde{e}_B^{(n)} \right) \right. \\ \left. \times (r_A^{(2)} - r_B^{(1)}) + 2((r_B^{(1)})^2 - (r_A^{(2)})^2) \right\}, \quad (28b)$$

$$+ \left(r_B^{(1)} + \nu_A \left(e_A^{(2)} + \sum_{n=3}^4 \tilde{e}_A^{(n)} \right) \right. \\ \left. + (1 - \nu_A) \sum_{n=2}^4 \tilde{e}_B^{(n)} \right) (r_A^{(2)} - r_B^{(1)}) \\ + 2((r_B^{(1)})^2 - (r_A^{(2)})^2) \Big\},$$

$$\mathbb{E}[\pi_A^{(3)*}(r_B^{(2)})] = \max_{(e_A^{(3)}, r_A^{(3)})} \left\{ \delta \mathbb{E}[\pi_A^{(4)*}(\tilde{r}_B^{(3)})] + \alpha(e_A^{(3)})^2 \right. \\ \left. + \left(2r_B^{(2)} + \nu_A (e_A^{(3)} + \tilde{e}_A^{(4)}) \right) \right. \\ \left. + (1 - \nu_A) \sum_{n=3}^4 \tilde{e}_B^{(n)} \right) (r_A^{(3)} - r_B^{(2)}) \\ + 2((r_B^{(2)})^2 - (r_A^{(3)})^2) \Big\}, \quad (28c)$$

$$\mathbb{E}[\pi_A^{(4)*}(r_B^{(3)})] = \max_{(e_A^{(4)}, r_A^{(4)})} \left\{ \alpha(e_A^{(4)})^2 \right. \\ \left. + \left(3r_B^{(3)} + \nu_A e_A^{(4)} + (1 - \nu_A) \tilde{e}_B^{(4)} \right) \right. \\ \left. \times (r_A^{(4)} - r_B^{(3)}) + 2((r_B^{(3)})^2 - (r_A^{(4)})^2) \right\}, \quad (28d)$$

For agent Bob:

$$\mathbb{E}[\pi_B^{(1)*}(r_A^{(1)})] = \max_{(e_B^{(1)}, r_B^{(1)})} \left\{ \delta \mathbb{E}[\pi_B^{(2)*}(\tilde{r}_A^{(2)})] + \alpha(e_B^{(1)})^2 \right. \\ \left. + \left(r_A^{(1)} + \nu_A \sum_{n=2}^4 \tilde{e}_A^{(n)} + (1 - \nu_A) \right) \right. \\ \left. \times \left(e_B^{(1)} + \sum_{n=2}^4 \tilde{e}_B^{(n)} - 0.5 \right) \right. \\ \left. \times (r_B^{(1)} - r_A^{(1)}) + 2((r_A^{(1)})^2 - (r_B^{(1)})^2) \right\}, \quad (29a)$$

$$\mathbb{E}[\pi_B^{(2)*}(r_A^{(2)})] = \max_{(e_B^{(2)}, r_B^{(2)})} \left\{ \delta \mathbb{E}[\pi_B^{(3)*}(\tilde{r}_A^{(3)})] + \alpha(e_B^{(2)})^2 \right. \\ \left. + \left(2r_A^{(2)} + \nu_A \sum_{n=3}^4 \tilde{e}_A^{(n)} + (1 - \nu_A) \right) \right. \\ \left. \times \left(e_B^{(2)} + \sum_{n=3}^4 \tilde{e}_B^{(n)} - 0.5 \right) \right. \\ \left. \times (r_B^{(2)} - r_A^{(2)}) + 2((r_A^{(2)})^2 - (r_B^{(2)})^2) \right\}, \quad (29b)$$

$$\mathbb{E}[\pi_B^{(3)*}(r_A^{(3)})] = \max_{(e_B^{(3)}, r_B^{(3)})} \left\{ \delta \mathbb{E}[\pi_B^{(4)*}(\tilde{r}_A^{(4)})] + \alpha(e_B^{(3)})^2 \right. \\ \left. + \left(3r_A^{(3)} + \nu_A \tilde{e}_A^{(4)} + (1 - \nu_A) \right) \right. \\ \left. \times \left(e_B^{(3)} + \tilde{e}_B^{(4)} - 0.5 \right) \right. \\ \left. \times (r_B^{(3)} - r_A^{(3)}) + 2((r_A^{(3)})^2 - (r_B^{(3)})^2) \right\}, \quad (29c)$$

$$\mathbb{E}[\pi_B^{(4)*}(r_A^{(4)})] = \max_{(e_B^{(4)}, r_B^{(4)})} \left\{ \alpha(e_B^{(4)})^2 \right. \\ \left. + \left(3r_A^{(4)} + \nu_A \tilde{e}_A^{(4)} + (1 - \nu_A) \right) \right. \\ \left. \times \left(e_B^{(4)} + \tilde{e}_B^{(4)} - 0.5 \right) \right. \\ \left. \times (r_B^{(4)} - r_A^{(4)}) + 2((r_A^{(4)})^2 - (r_B^{(4)})^2) \right\}. \quad (29d)$$

$$\begin{aligned}
 &+ \left(4r_A^{(4)} + (1 - \nu_A)(e_B^{(4)} - 0.5) \right) \\
 &\times (r_B^{(4)} - r_A^{(4)}) \\
 &+ 2 \left((r_A^{(4)})^2 - (r_B^{(4)})^2 \right) \}.
 \end{aligned}$$

Lemma 10. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(4)} \leq \frac{7+\nu_A}{8}$, then $(e_B^{(4)*}, r_B^{(4)*}) = (1, r_A^{(4)} + \frac{1}{8}(1 - \nu_A))$ and $\mathbb{E}[\pi_B^{(4)*}(r_A^{(4)})] = \alpha + \frac{1}{8}(1 - \nu_A)^2$.

Proof. When $i = 4$, there are no future rounds. Consider Bob's function of expected payoff scores in round 4 as $f_B^{(4)}$:

$$\begin{aligned}
 f_B^{(4)} &= \mathbb{E}[\pi_B^{(4)}(r_A^{(4)}, a_B^{(4)})] \\
 &= \alpha(e_B^{(4)})^2 + \left(4r_A^{(4)} + (1 - \nu_A)(e_B^{(4)} - 0.5) \right) \\
 &\quad \times (r_B^{(4)} - r_A^{(4)}) \\
 &\quad + 2 \left((r_A^{(4)})^2 - (r_B^{(4)})^2 \right).
 \end{aligned} \tag{30}$$

Then, we obtain

$$\begin{aligned}
 \nabla^2 f_B^{(4)}(e_B^{(4)}, r_B^{(4)}) &= \begin{bmatrix} \frac{\partial^2 f_B^{(4)}}{\partial (e_B^{(4)})^2} & \frac{\partial^2 f_B^{(4)}}{\partial e_B^{(4)} \partial r_B^{(4)}} \\ \frac{\partial^2 f_B^{(4)}}{\partial r_B^{(4)} \partial e_B^{(4)}} & \frac{\partial^2 f_B^{(4)}}{\partial (r_B^{(4)})^2} \end{bmatrix} \\
 &= \begin{bmatrix} 2\alpha & 1 - \nu_A \\ 1 - \nu_A & -4 \end{bmatrix}.
 \end{aligned}$$

The first principle minor of $\nabla^2 f_B^{(4)}(e_B^{(4)}, r_B^{(4)})$ is positive because $\det \begin{bmatrix} \frac{\partial^2 f_B^{(4)}}{\partial (e_B^{(4)})^2} \end{bmatrix} = 2\alpha > 0$. The second principle minor is negative because:

$$\det \begin{bmatrix} \frac{\partial^2 f_B^{(4)}}{\partial (e_B^{(4)})^2} & \frac{\partial^2 f_B^{(4)}}{\partial e_B^{(4)} \partial r_B^{(4)}} \\ \frac{\partial^2 f_B^{(4)}}{\partial r_B^{(4)} \partial e_B^{(4)}} & \frac{\partial^2 f_B^{(4)}}{\partial (r_B^{(4)})^2} \end{bmatrix} = -8\alpha - (1 - \nu_A)^2 < 0.$$

We find that the Hessian of $f_B^{(4)}$ is an indefinite matrix, so we still need to find its KKT points and compare the corresponding values of $f_B^{(4)}$, as the maximal value $f_B^{(4)*}$ must be attained in one of these KKT points. Table 8 reports the KKT points of $f_B^{(4)}$ and the corresponding function values. However, to simplify our analysis, the table does not show unfeasible KKT points. We find that if $r_A^{(4)} \leq \frac{7+\nu_A}{8}$, $f_B^{(4)*}$ is attained as $\alpha + \frac{1}{8}(1 - \nu_A)^2$ when $(e_B^{(4)}, r_B^{(4)}) = (1, r_A^{(4)} + \frac{1}{8}(1 - \nu_A))$. So Lemma 10 is true. □

Lemma 11. If $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$ and $r_A^{(4)} \geq \frac{7+\nu_A}{8}$, then $(e_B^{(4)*}, r_B^{(4)*}) = (1, 1)$ and $\mathbb{E}[\pi_B^{(4)*}(r_A^{(4)})] = \alpha - 2(1 - r_A^{(4)})^2 + \frac{(1-\nu_A)(1-r_A^{(4)})}{2}$.

Proof. Similarly to the proof for Lemma 10, if $r_A^{(4)} \geq \frac{7+\nu_A}{8}$, we find by comparing the values of $f_B^{(4)}$ of feasible KKT points in Table 8 that $f_B^{(4)*}$ is attained as $f_B^{(4)*} = \alpha -$

$2(1 - r_A^{(4)})^2 + \frac{(1-\nu_A)(1-r_A^{(4)})}{2}$ when $(e_B^{(4)*}, r_B^{(4)*}) = (1, 1)$. So Lemma 11 is true. □

Given the two cases for Bob's equilibrium decisions in round 4, we may now determine Bob's round 4 decisions:

Theorem 9. If high efforts bring net rewards and $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, then

$$(e_B^{(4)*}, r_B^{(4)*}) = \begin{cases} (1, r_A^{(4)} + \frac{1}{8}(1 - \nu_A)) & \text{if } 0 \leq r_A^{(4)} \leq \frac{7 + \nu_A}{8} \\ (1, 1) & \text{if } \frac{7 + \nu_A}{8} \leq r_A^{(4)} \leq 1 \end{cases}$$

$$f_B^{(4)*} = \begin{cases} \alpha + \frac{1}{8}(\nu_A - 1)^2 & \text{if } 0 \leq r_A^{(4)} \leq \frac{7 + \nu_A}{8} \\ \alpha - 2(1 - r_A^{(4)})^2 + \frac{(1 - \nu_A)(1 - r_A^{(4)})}{2} & \text{if } \frac{7 + \nu_A}{8} \leq r_A^{(4)} \leq 1 \end{cases}$$

Proof. With Lemmas 10 and 11, Theorem 9 is true. □

We now move to Alice's optimal actions in round 4. From Theorem 9 we know that Bob will always exert high effort in round 4. In other words, Alice will assume $\tilde{e}_B^{(4)} = 1$ when calculating her maximal expected payoff in round 4. We therefore find that Alice will exert high effort and report as 1 in round 4 no matter what Bob reports in round 3.

Theorem 10. If high efforts bring net rewards and $\alpha > \max\{\frac{(1-\nu_A)^2}{2}, \frac{\nu_A^2}{2}\}$, Alice's optimal action set in round 4 is $(e_A^{(4)*}, r_A^{(4)*}) = (1, \frac{3r_B^{(3)}+1}{4})$, and the optimal expected payoff score of round 4 is $\mathbb{E}[\pi_A^{(4)*}(r_B^{(3)})] = \alpha + \frac{1}{8}(r_B^{(3)} - 1)^2$.

Proof. Consider Alice's function of expected payoff scores in round 4 as $f_A^{(4)}$:

$$\begin{aligned}
 f_A^{(4)} &= \mathbb{E}[\pi_A^{(4)}(r_B^{(3)}, a_A^{(4)})] \\
 &= \alpha(e_A^{(4)})^2 + \left(3 \cdot r_B^{(3)} + \nu_A e_A^{(4)} + (1 - \nu_A)\tilde{e}_B^{(4)} \right) (r_A^{(4)} - r_B^{(3)}) \\
 &\quad + 2 \left((r_B^{(3)})^2 - (r_A^{(4)})^2 \right).
 \end{aligned} \tag{31}$$

Furthermore, inserting $\tilde{e}_B^{(4)} = 1$ into the equation above gives:

$$\begin{aligned}
 f_A^{(4)} &= \alpha(e_A^{(4)})^2 + \left(3 \cdot r_B^{(3)} + \nu_A \cdot e_A^{(4)} + (1 - \nu_A) \right) (r_A^{(4)} - r_B^{(3)}) \\
 &\quad + 2 \left((r_B^{(3)})^2 - (r_A^{(4)})^2 \right).
 \end{aligned} \tag{32}$$

To find the value of $f_A^{(4)*}$, we need to check the Hessian of function $f_A^{(4)}$:

$$\begin{aligned}
 \nabla^2 f_A^{(4)}(e_A^{(4)}, r_A^{(4)}) &= \begin{bmatrix} \frac{\partial^2 f_A^{(4)}}{\partial (e_A^{(4)})^2} & \frac{\partial^2 f_A^{(4)}}{\partial e_A^{(4)} \partial r_A^{(4)}} \\ \frac{\partial^2 f_A^{(4)}}{\partial r_A^{(4)} \partial e_A^{(4)}} & \frac{\partial^2 f_A^{(4)}}{\partial (r_A^{(4)})^2} \end{bmatrix} \\
 &= \begin{bmatrix} 2\alpha & \nu_A \\ \nu_A & -4 \end{bmatrix}.
 \end{aligned}$$

Table 8
KKT points of $f_B^{(4)}$ in case 4.

$e_B^{(4)}$	$r_B^{(4)}$	$f_B^{(4)}$
0	$r_A^{(4)} + \frac{1}{8}(\nu_A - 1)$	$\frac{1}{32}(1 - \nu_A)^2$
1	$r_A^{(4)} + \frac{1}{8}(1 - \nu_A)$	$\alpha + \frac{1}{8}(1 - \nu_A)^2$
$\frac{r_A^{(4)}(1 - \nu_A)}{2\alpha}$	0	$-2(r_A^{(4)})^2 + \frac{r_A^{(4)}}{2}(1 - \nu_A) + \frac{(1 - \nu_A)^2}{4\alpha}(1 - \frac{r_A^{(4)}}{2})$
0	0	$-2(r_A^{(4)})^2 + \frac{r_A^{(4)}}{2}(1 - \nu_A)$
0	1	$-2(r_A^{(4)})^2 + 4r_A^{(4)} - 2 - \frac{(1 - \nu_A)(1 - r_A^{(4)})}{2} \leq 0$
1	0	$-2(r_A^{(4)})^2 - \frac{(1 - \nu_A)r_A^{(4)}}{2} + \alpha \leq \alpha$
1	1	$\alpha - 2(1 - r_A^{(4)})^2 + \frac{(1 - \nu_A)(1 - r_A^{(4)})}{2}$

Table 9
KKT points of $f_A^{(4)}$ in case 4.

$e_A^{(4)}$	$r_A^{(4)}$	$f_A^{(4)}$
0	$\frac{3r_B^{(3)} + 1 - \nu_A}{4}$	$\frac{1}{8}(r_B^{(3)} + \nu_A - 1)^2$
1	$\frac{3r_B^{(3)} + 1}{4}$	$\alpha + \frac{1}{8}(r_B^{(3)} - 1)^2$
$\frac{\nu_A r_B^{(B)}}{2\alpha}$	0	$r_B^{(3)}(\nu_A - r_B^{(3)} - 1) - \frac{(\nu_A r_B^{(3)})^2}{4\alpha}$
0	0	$r_B^{(3)}(\nu_A - 1 - r_B^{(3)})$
0	1	$(1 - r_B^{(3)})(r_B^{(3)} - 1 - \nu_A)$
1	0	$\alpha - r_B^{(3)}(1 + r_B^{(3)})$
1	1	$\alpha - (1 - r_B^{(3)})^2$

We find that the Hessian of $f_A^{(4)}$ is also an indefinite matrix, because the value of its first principle minor is positive as $2\alpha > 0$, and that of its second principle minor is negative as $-8\alpha - \nu_A^2 < 0$. Table 9 reports the KKT points of $f_A^{(4)}$ and the corresponding function values. We find that $f_A^{(4)}$ is maximized as $f_A^{(4)*} = \alpha + \frac{1}{8}(r_B^{(3)} - 1)^2$ when $(e_A^{(4)}, r_A^{(4)}) = (1, \frac{3r_B^{(3)} + 1}{4})$. Thus, Theorem 10 is true. □

Consider Bob in round 3. From Theorems 9 and 10 we have $e_A^{(4)*} = e_B^{(4)*} = 1$. Knowing $\tilde{e}^{(4)} = \tilde{e}_B^{(4)} = 1$, we get the expression of Bob's expected payoff score in round 3 as $f_B^{(3)}$:

$$f_B^{(3)} = \alpha \cdot (e_B^{(3)})^2 + (3r_A^{(3)} - 0.5(1 - \nu_A) + \nu_A + (1 - \nu_A) \cdot (e_B^{(3)} + 1)) \times (r_B^{(3)} - r_A^{(3)}) + 2((r_A^{(3)})^2 - (r_B^{(3)})^2) \tag{33}$$

Lemma 12. *If high efforts bring net rewards and $\alpha > \max\{\frac{1 - \nu_A}{2}, \frac{\nu_A}{2}\}$, then*

$$(e_B^{(4)*}, r_B^{(4)*}) = \begin{cases} (1, \frac{3r_B^{(3)} + 1}{4} + \frac{1}{8}(1 - \nu_A)), & \text{if } 0 \leq r_B^{(3)} \leq \frac{5 + \nu_A}{6} \\ (1, 1), & \text{if } \frac{5 + \nu_A}{6} \leq r_B^{(3)} \leq 1 \end{cases}$$

Table 10
KKT points of $f_B^{(3)} + \delta \cdot f_B^{(4)*}$ in case 4 when $r_B^{(3)} \leq \frac{5 + \nu_A}{6}$.

$e_B^{(3)}$	$r_B^{(3)}$	$f_B^{(3)} + \delta \cdot f_B^{(4)*}$
0	$\frac{3}{4}r_A^{(3)} + \frac{1}{8}(\nu_A + 1)$	$\frac{1}{8}r_A^{(3)}(r_A^{(3)} - 1 - \nu_A) + \frac{1}{32}(\nu_A + 1)^2 + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$
1	$\frac{3}{4}r_A^{(3)} + \frac{3 - \nu_A}{8}$	$\alpha + \frac{r_A^{(3)}}{8}(r_A^{(3)} + \nu_A - 3) + \frac{(\nu_A - 3)^2}{32} + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$
$\frac{r_A^{(3)}(1 - \nu_A)}{2\alpha}$	0	$-(r_A^{(3)})^2 - \frac{r_A^{(3)}(\nu_A + 1)}{2} - \frac{(r_A^{(3)})^2(1 - \nu_A)^2}{4\alpha} + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$
0	0	$-r_A^{(3)}(r_A^{(3)} + \frac{\nu_A + 1}{2}) + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$
1	0	$\alpha - r_A^{(3)}(r_A^{(3)} + \frac{3 - \nu_A}{2}) + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$

$$f_B^{(4)*} = \begin{cases} \alpha + \frac{1}{8}(1 - \nu_A)^2 & \text{if } 0 \leq r_B^{(3)} \leq \frac{5 + \nu_A}{6} \\ \alpha + \frac{3(1 - r_B^{(3)})(3r_B^{(3)} - \nu_A - 2)}{8} & \text{if } \frac{5 + \nu_A}{6} \leq r_B^{(3)} \leq 1. \end{cases}$$

Proof. According to Theorem 10, we know that $r_A^{(4)*} = \frac{3r_B^{(3)} + 1}{4}$. Taking this along with the statement of Theorem 9, we infer Lemma 12. □

Lemma 13. *If $\alpha > \max\{\frac{1 - \nu_A}{2}, \frac{\nu_A}{2}\}$ and $r_A^{(3)} \leq \frac{7\nu_A + 11}{18}$, then $(e_B^{(3)*}, r_B^{(3)*}) = (1, \frac{3}{4}r_A^{(3)} + \frac{3 - \nu_A}{8})$ and*

$$\mathbb{E}[\pi_B^{(3)*}(r_A^{(3)})] = \alpha + \frac{r_A^{(3)}}{8}(r_A^{(3)} + \nu_A - 3) + \frac{(\nu_A - 3)^2}{32} + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2).$$

Proof. We know from Lemma 12 that $f_B^{(4)*}$ is obtained as $f_B^{(4)*} = \alpha + \frac{1}{8}(1 - \nu_A)^2$ when $r_B^{(3)} \leq \frac{5 + \nu_A}{6}$, and obtained as $f_B^{(4)*} = \alpha + \frac{3(1 - r_B^{(3)})(3r_B^{(3)} - \nu_A - 2)}{8}$ when $r_B^{(3)} \geq \frac{5 + \nu_A}{6}$.

We begin by discussing the situation where $f_B^{(4)*}$ is a constant as $f_B^{(4)*} = \alpha + \frac{1}{8}(\nu_A - 1)^2$ when $r_B^{(3)} \leq \frac{5 + \nu_A}{6}$. Similarly, we denote the KKT points of $f_B^{(3)} + \delta \cdot f_B^{(4)*}$ in Table 10. We find that when $r_A^{(3)} \leq \frac{7\nu_A + 11}{18}$, $f_B^{(3)} + \delta \cdot f_B^{(4)*}$ is maximized as $\alpha + \frac{r_A^{(3)}}{8}(r_A^{(3)} + \nu_A - 3) + \frac{(\nu_A - 3)^2}{32} + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$, which we denote as $\mathbb{E}[\pi_B^{(3)*}(r_A^{(3)}, a_B^{(3)})]$, when $(e_B^{(3)}, r_B^{(3)}) = (1, \frac{3}{4}r_A^{(3)} + \frac{3 - \nu_A}{8})$.

Next, we discuss the situation where $f_B^{(4)*}$ is a function of $r_B^{(3)}$ as $f_B^{(4)*} = \alpha + \frac{3(1 - r_B^{(3)})(3r_B^{(3)} - \nu_A - 2)}{8}$, if $r_B^{(3)} \geq \frac{5 + \nu_A}{6}$. Using the same method, we find that $f_B^{(3)} + \delta \cdot f_B^{(4)*}$ is maximized as $\alpha + \frac{r_A^{(3)}}{2}(3 - 2r_A^{(3)}) + \frac{\nu_A(r_A^{(3)} - 1) - 1}{2} + \alpha(1 + \delta)$ at $(e_B^{(3)}, r_B^{(3)}) = (1, 1)$.

By routine calculations, we know that the first expression of $f_B^{(3)} + \delta \cdot f_B^{(4)*}$ is larger than the value of the second expression. Thus, $\mathbb{E}[\pi_B^{(3)*}(r_A^{(3)})]$ is achieved as $\alpha + \frac{r_A^{(3)}}{8}(r_A^{(3)} + \nu_A - 3) + \frac{(\nu_A - 3)^2}{32} + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(\nu_A - 1)^2)$ at $(e_B^{(3)*}, r_B^{(3)*}) = (1, \frac{3}{4}r_A^{(3)} + \frac{3 - \nu_A}{8})$. So Lemma 13 is true. □

Lemma 14. If $\alpha > \max\{\frac{(1-v_A)^2}{2}, \frac{v_A^2}{2}\}$ and $r_A^{(3)} \geq \frac{7v_A+11}{18}$, then $(e_B^{(3)*}, r_B^{(3)*}) = (1, 1)$ and

$$\mathbb{E} \left[\pi_B^{(3)*}(r_A^{(3)}) \right] = \alpha + \frac{r_A^{(3)}}{2}(3 - 2r_A^{(3)}) + \frac{v_A(r_A^{(3)} - 1) - 1}{2} + \alpha(1 + \delta).$$

Proof. The proof is similar to that of Lemma 13, so the details are omitted. \square

Given the two cases for Bob’s equilibrium decisions in round 3, we may now determine Bob’s round 3 decisions:

Theorem 11. If high efforts bring net rewards and $\alpha > \max\{\frac{(1-v_A)^2}{2}, \frac{v_A^2}{2}\}$, then

$$(e_B^{(3)*}, r_B^{(3)*}) = \begin{cases} (1, \frac{3}{4}r_A^{(3)} + \frac{3-v_A}{8}), & \text{if } 0 \leq r_A^{(3)} \leq \frac{7v_A+11}{18} \\ (1, 1), & \text{if } \frac{7v_A+11}{18} \leq r_A^{(3)} \leq 1 \end{cases}$$

$$f_B^{(3)*} = \begin{cases} \alpha + \frac{r_A^{(3)}}{8}(r_A^{(3)} + v_A - 3) + \frac{(v_A - 3)^2}{32} + \frac{9}{8} + \delta(\alpha + \frac{1}{8}(v_A - 1)^2), & \text{if } 0 \leq r_A^{(3)} \leq \frac{7v_A+11}{18} \\ \alpha + \frac{r_A^{(3)}}{2}(3 - 2r_A^{(3)}) + \frac{v_A(r_A^{(3)} - 1) - 1}{2} + \alpha(1 + \delta), & \text{if } \frac{7v_A+11}{18} \leq r_A^{(3)} \leq 1. \end{cases}$$

Proof. With Lemmas 13 and 14, Theorem 11 is true. \square

Consider Alice in round 3. From Theorems 9–11 we have $e_B^{(3)*} = e_A^{(4)*} = e_B^{(4)*} = 1$. Knowing $\tilde{e}_B^{(3)} = \tilde{e}_A^{(4)} = \tilde{e}_B^{(4)} = 1$, we get the expression of Alice’s expected payoff score in round 3 as $f_A^{(3)}$:

$$f_A^{(3)} = \alpha \left(e_A^{(3)} \right)^2 + \left(2r_B^{(2)} + v_A \left(e_A^{(3)} + 1 \right) + 2 \left(1 - v_A \right) \right) \times \left(r_A^{(3)} - r_B^{(2)} \right) - 2 \left(r_A^{(3)} \right)^2 + 2 \left(r_B^{(2)} \right)^2 \tag{34}$$

$$= \alpha \left(e_A^{(3)} \right)^2 + \left(2r_B^{(2)} + v_A e_A^{(3)} + 2 - v_A \right) \left(r_A^{(3)} - r_B^{(2)} \right) - 2 \left(r_A^{(3)} \right)^2 + 2 \left(r_B^{(2)} \right)^2.$$

Theorem 12. If high efforts bring net rewards and $\alpha > \max\{\frac{(1-v_A)^2}{2}, \frac{v_A^2}{2}\}$, then

$$(e_A^{(3)*}, r_A^{(3)*}) = \begin{cases} (1, \frac{9r_B^{(3)}+3}{16} + \frac{3-v_A}{8}), & \text{if } 0 \leq r_A^{(3)} \leq \frac{7v_A+11}{18} \\ (1, 1), & \text{if } \frac{7v_A+11}{18} \leq r_A^{(3)} \leq 1 \end{cases}$$

$$f_A^{(3)*} = \begin{cases} \alpha + \frac{1}{8} \left(\frac{3}{4}r_A^{(3)} + \frac{v_A-5}{8} \right)^2 & \text{if } 0 \leq r_A^{(3)} \leq \frac{7v_A+11}{18} \\ \alpha & \text{if } \frac{7v_A+11}{18} \leq r_A^{(3)} \leq 1. \end{cases}$$

Proof. According to Theorem 11, we know that:

$$r_B^{(3)*} = \begin{cases} \frac{3}{4}r_A^{(3)} + \frac{3-v_A}{8}, & \text{if } 0 \leq r_A^{(3)} \leq \frac{7v_A+11}{18} \\ 1, & \text{if } \frac{7v_A+11}{18} \leq r_A^{(3)} \leq 1. \end{cases}$$

Together with the statement of Theorem 10, we infer Theorem 12. \square

Finishing the proof algebraically is quite involved and lengthy, and does not provide any further insights. As such, we will finish the four-round case using a combinatorial argument. So far, we know that, regardless of an agent’s report, the actions of both Alice and Bob in rounds 3 and 4 will be to exert high efforts. We now consider Bob’s actions in round 2. Being rational, he deduces that all subsequent effort levels of all other agents, including himself, will be high. Thus, from his perspective, at the end of round 2 he has no uncertainty regarding future effort levels, and all past effort levels are already fixed. As such, his only possible trade-off would be to exert low effort and lose the reward from high effort, but make it up within in the prediction market. If α is sufficiently high, then the reward within the prediction market will be outweighed by that from exerting high effort. Thus, assuming that α is sufficiently high, Bob will exert high effort and report truthfully.

Turning to Alice’s actions in round 2, we may apply this same combinatorial argument again. Similarly, we may apply the same arguments to first Bob and then Alice in round 1. In turn, we deduce that, for sufficiently high values of α , all agents will exert high effort in all rounds and report their beliefs truthfully.

5. Discussion and conclusion

We have shown that the cost actions that agents take which determine the outcome of a prediction-market-traded event have a great influence on the agents’ prediction market and external behaviors. First in a two-round setting and later in a four-round setting, we find that when agents are forward-looking and want to maximize their total expected payoffs from both exerting effort towards realizing the traded event and trading in the prediction market, asymmetric action costs result in agents avoiding taking the costliest action. This observation implies that if a market maker rewards her preferred action more than a less-preferred action, agents will take the desired action even when a prediction market is present. We find that the value of the net reward for each desirable action should be larger than a certain amount, which is determined by the value of v_A in our two-round setting.³ It makes sense that the reward for desirable actions must be sufficiently large, as this avoids the “playing for peanuts” results (Harinck, Van Dijk, Van Beest, & Mersmann, 2007; Weber & Chapman, 2005). Perhaps unexpectedly, when given sufficiently high external rewards, agents will always report truthfully during each prediction market round, even though their

³ We did not determine the exact threshold for α explicitly in the four-round setting. Just as in the two-round setting, though, this threshold exists and is a function of v_A .

actions are influenced by external costs. This observation shows that the quadratic market scoring rule is incentive-compatible even when external incentives are present, so long as action costs are costly and the preferred action is rewarded adequately.

In the past, decision and policy makers have expressed concern that the existence of a prediction market will inspire undesirable actions for agents who are trading in the prediction market, especially when these agents have a direct impact on deciding the likelihood of the predicted event, as is the case in corporate settings. However, previous research has not taken into account the potential payoffs (either net costs or net rewards) to the agents who take such actions. We base our research on the assumption that forward-looking agents will wish to maximize their total expected payoffs, not only from the prediction market, but also from their actions related to the traded event. Our finding enables market makers who care about the result of the traded event to give agents an incentive to take desirable actions. More importantly, our results address the concern that firms may have regarding deploying a prediction market within their organization, as we have shown that the existence of such a market will not induce participants to change their behavior outside the market, assuming that they are compensated appropriately. A market maker can also gain accurate and true information about agents' actions from the same agents' reports in the prediction market, as the regular market scoring rule is incentive-compatible.

Here, we set a range for the expected payoff scores of exerted efforts, the absolute value of which should be larger than a certain value in each round, in order to provide an incentive for the desired actions and truthful reports. We do not discuss whether or not the prediction market will still be incentive-compatible when this range is violated. However, we do show that when efforts are not costly actions (the payoff scores for exerted efforts are zero), agents will bluff in the prediction market, a well-known existing result. More importantly, we set the number of rounds (T) here to either two or four, which guarantees our assumption that agents are forward-looking. However, we still need to extend our model to a finite round setting, with T being a large number, in order to determine whether our conclusion still holds. In addition, the result that we have presented considered only the quadratic market scoring rule. By the nature of our result, the logarithmic market scoring rule cannot be used to verify our result. However, it is not clear whether there are other scoring rules that may be used, with the particular question being raised: what members of the large class of scoring rules introduced by [Gneiting and Raftery \(2007\)](#) restore incentive compatibility when external incentives are present? The results of our paper should not be applied blindly, but we hope that this study will encourage laboratory and field experiments that can determine whether incentive compatibility is restored when the cost of external actions is considered. We view the results presented here as a first step in incorporating the costs of external actions into the prediction markets that are used within corporations today.

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Appendix. Notation

- α constant that represents the cost (reward) for efforts; we assume $0 < \alpha \leq 2$.
- i number of rounds, $i = 1, \dots, T$.
- $r_A^{(i)}$ the reported prediction probability of E occurring in round i if the report has already been made by Alice, $r_A^{(i)} \in [0, 1]$, $\bar{r}_A^{(i)} = 1 - r_A^{(i)}$.
- $r_B^{(i)}$ the reported prediction probability of E occurring in round i if the report has already been made by Bob, $r_B^{(i)} \in [0, 1]$, $\bar{r}_B^{(i)} = 1 - r_B^{(i)}$.
- $\bar{r}_A^{(i)}$ the reported prediction probability of E occurring in round i if the report has not yet been made by Alice.
- $\bar{r}_B^{(i)}$ the reported prediction probability of E occurring in round i if the report has not yet been made by Bob.
- $r_B^{(0)}$ the beginning market estimate set in the prediction market by the market maker.
- $e_A^{(i)}$ value of Alice's effort in round i if the effort action has already happened: $e_A^{(1)} = 1$ if Alice devotes high effort in round i , otherwise 0, $\bar{e}_A^{(i)} = 1 - e_A^{(i)}$.
- $e_B^{(i)}$ value of Bob's effort in round i if the effort action has already happened: $e_B^{(1)} = 1$ if Bob devotes high effort in round i , otherwise 0, $\bar{e}_B^{(i)} = 1 - e_B^{(i)}$.
- $\bar{e}_A^{(i)}$ value of Alice's future effort in round i if the effort action has not yet taken place.
- $\bar{e}_B^{(i)}$ value of Bob's future effort in round i if the effort action has not yet taken place.
- $\mathbb{E}_A(\bar{e}_B^{(i)})$ the expectation of Bob's future effort value of round i perceived by Alice when she reports in round i .
- $a_A^{(i)}$ the action set that Alice takes in round i , $a_A^{(i)} = (e_A^{(i)}, r_A^{(i)})$.
- $a_A^{(i)*}$ the optimal action set that Alice takes in round i , $a_A^{(i)*} = (e_A^{(i)*}, r_A^{(i)*})$.
- a_A the policy that Alice takes for all rounds, $a_A := (a_A^{(1)}, \dots, a_A^{(T)})$.
- a_A^* the optimal policy for Alice, $a_A^* := (a_A^{(1)*}, \dots, a_A^{(T)*})$.
- $\mathcal{A}_A^{(i)}$ the action set that Alice has in round i , $a_A^{(i)} \in \mathcal{A}_A^{(i)}$.
- \mathcal{A}_A the whole action set that Alice has, $\mathcal{A}_A = \mathcal{A}_A^{(1)} \times \dots \times \mathcal{A}_A^{(i)}$ and $a_A \in \mathcal{A}_A$.

- $a_B^{(i)}$ the action set that Bob takes in round i , $a_B^{(i)} = (e_B^{(i)}, r_B^{(i)})$.
- $a_B^{(i)*}$ the optimal action set that Bob takes in round i , $a_B^{(i)*} = (e_B^{(i)*}, r_B^{(i)*})$.
- a_B the policy that Bob takes for all rounds, $a_B := (a_B^{(1)}, \dots, a_B^{(T)})$.
- a_B^* the optimal policy for Bob, $a_B^* := (a_B^{(1)*}, \dots, a_B^{(T)*})$.
- $\mathcal{A}_B^{(i)}$ the action set that Bob has in round i , $a_B^{(i)} \in \mathcal{A}_B^{(i)}$.
- \mathcal{A}_B the whole action set that Bob has, $\mathcal{A}_B = \mathcal{A}_B^{(1)} \times \dots \times \mathcal{A}_B^{(T)}$ and $a_B \in \mathcal{A}_B$.
- $h_A^{(i)}$ number of high efforts Alice has exerted in total from rounds 1 to i , $h_A^{(0)} = 0$.
- $h_B^{(i)}$ number of high efforts Bob has exerted in total from rounds 1 to i , $h_B^{(0)} = 0$.
- ν_A Alice's impact on deciding the likelihood of project E occurring, $0 < \nu_A < 1$.
- $p_A^{(i)}$ Alice's belief on the likelihood of E occurring in round i after she observes Bob's most recent report and before she takes any action in that round, taking into account actions in future rounds.
- $p_B^{(i)}$ Bob's belief on the likelihood of E occurring in round i after he observes Alice's most recent report and before he takes any action in that round, taking into account actions in future rounds.
- $\rho_e^{(i)}(\cdot)$ function of payoff scores of efforts devoted in round i .
- $\rho_s^{(i)}(\cdot)$ function of payoff scores earned from moving the probability in the prediction market in round i if E occurs (succeeds) after T rounds.
- $\rho_f^{(i)}(\cdot)$ function of payoff scores earned from moving the probability in the prediction market in round i if E does not occur (fails) after T rounds.
- δ discounting factor on the future profits, $0 < \delta < 1$.
- $\pi_A^{(i)}$ payoff scores earned by Alice from rounds i to T , including scores earned in this round and discounting future scores.
- $\pi_B^{(i)}$ payoff scores earned by Bob from rounds i to T , including scores earned in this round and discounting future scores.
- $I_A^{(i)}$ the system state that Alice has in round i after she observes Bob's most recent report in the prediction market but before she takes any actions in that round.
- $\mathbb{E}[\pi_A^{(i)}(I_A^{(i)}, a_A^{(i)})]$ the expected value of $\pi_A^{(i)}$ if the current state is $I_A^{(i)}$ and the current round's action set is $a_A^{(i)}$.
- $\mathbb{E}[\pi_A^{(i)*}(I_A^{(i)}, a_A^{(i)})]$ the optimal expected value of $\pi_A^{(i)}$ given the current state $I_B^{(i)}$.
- $I_B^{(i)}$ the system state that Bob has in round i after he observes Alice's most recent report in the prediction market but before he takes any action in that round.

- $\mathbb{E}[\pi_B^{(i)}(I_B^{(i)}, a_B^{(i)})]$ the expected value of $\pi_B^{(i)}$ given if the current state is $I_B^{(i)}$ and the current round's action set is $a_B^{(i)}$.
- $\mathbb{E}[\pi_B^{(i)*}(I_B^{(i)}, a_B^{(i)})]$ the optimal expected value of $\pi_B^{(i)}$ given the current state $I_B^{(i)}$.

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